The primary aim of this lecture is to present many of the foundations used by Suslin and Voevodsky in their discussion of algebraic singular homology [4]. We view this as an excellent introduction to the roles played by rigidity and homotopy invariance. Throughout this lecture, $k$ will be a fixed field and $(\text{Sch}/k)$ will denote the category of schemes of finite type over an algebraically closed field $k$.

2.1. Presheaves with transfer. The concept of presheaf with transfers is an abstraction of the technique Suslin introduced in [3] to determine the mod-$n$ $K$-theory of algebraically closed fields as discussed in the previous lecture. There are various slightly different definitions of this concept and the one we present here is the one introduced by Suslin and Voevodsky in [4].

**Definition 2.1.** A contravariant functor on $(\text{Sch}/k)^{\text{op}}$, $F : (\text{Sch}/k)^{\text{op}} \to Ab$, is said to be a presheaf with transfers if $F$ is provided with a homomorphism for every finite surjective $p : Y \to S$ with $Y$ irreducible, reduced and $S$ irreducible, regular

$$Tr_{Y/S} : F(Y) \to F(S)$$

satisfying the following conditions:

- If $p$ is an isomorphism, then $Tr_{Y/S} \circ p^* = id$.
- If $V \subset S$ is a closed, irreducible, regular subscheme and if $p^{-1}(V) = \sum n_i W_i$ (where the multiplicity $n_i$ is the usual intersection multiplicity of $p^{-1}(Y)$ along the component $W_i$), then the following diagram commutes

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{Tr_{Y/S}} & F(S) \\
\downarrow & & \downarrow \\
\oplus F(W_i) & \xrightarrow{\sum n_i Tr_{W_i}} & F(V)
\end{array}
$$

Observe that we do not require $Y$ to be regular. Indeed, the way these transfers are employed is as follows. We are given a scheme $X$ over $S$ and some closed, irreducible subscheme $Y \subset X$ which is finite and surjective over $S$. Then the contravariant functoriality of $F$ together with the transfer $Tr_{Y/S}$ determines a transfer map

$$Tr_{Y/S} : F(X) \to F(Y) \to F(S).$$

In other words, if $C_0(X/S)$ denotes the free abelian group on the closed irreducible subschemes $Y \subset X$ which are finite and surjective over $S$, then $F$ is provided with a pairing

$$F(X) \otimes C_0(X/S) \to F(S).$$

Let $C_1(X/S)$ denote the free abelian group on the closed irreducible subschemes $W \subset X \times A^1$ which are finite and surjective over $S \times A^1$, and let $H_0(X/S)$ denote the cokernel of the difference of the two maps $C_1(X/S) \to C_0(X/S)$ given by evaluating at 0,1.

We now see the role of homotopy invariance. We can view the following proposition as a generalization of the rigidity used in the previous lecture (by using an
awkward notational shift, so that $F = K'_2(X \times -; \mathbb{Z}/n)$, $X$ in the proposition below is replaced by a smooth projective curve $C$, and $S$ by Spec $k$.

**Proposition 2.2.** Let $F$ be a homotopy invariant presheaf with transfers. Then the pairing (2.1.2) factors as

\[(2.2.1)\quad F(X) \otimes H_0(X/S) \rightarrow F(S).\]

We say that $X/S$ is a smooth relative curve with good compactification if $X$ is a smooth, affine, irreducible scheme of relative dimension 1 over $S$ which embeds as an open subset of some $\overline{X}/S$ with $\overline{X}$ normal, $\overline{X} \rightarrow S$ proper, and $\overline{X} - X$ admits an affine open neighborhood in $\overline{X}$.

We remind the reader that if $W \subset X$ is a closed subscheme, then the relative Picard group $\text{Pic}(X, W)$ is the abelian group of isomorphism classes of pairs $(L, \phi)$, where $L$ is a line bundle on $X$ and $\phi : L|_W \sim \rightarrow \mathcal{O}_Y$ is a trivialization of $L$ restricted to $W$.

**Proposition 2.3.** Assume that $S$ is a normal, affine scheme and that $X/S$ is a smooth relative curve with good compactification. Then $H_0(X/S)$ equals the relative Picard group $\text{Pic}(\overline{X}, W)$, where $X \subset \overline{X}$ is a good compactification with complement $W = \overline{X} - X$.

Proposition 2.3 enables us to investigate the behaviour of $H_0(X/S) \otimes \mathbb{Z}/n$ using etale cohomology. In particular, the proper base change theorem in etale cohomology enables us to conclude with the hypotheses of the preceding theorem that

$$H_0(X/S) \otimes \mathbb{Z}/n \subset H_0(X_0, S_0)$$

where $S_0 \subset S$ is a closed regular subscheme and $X_0 = X \times_S S_0$.

The preceding propositions easily enable Suslin and Voevodsky to prove the following rigidity theorem.

**Theorem 2.4.** Let $F$ be a homotopy invariant presheaf with transfers satisfying the condition that $nF = 0$ for some positive integer $n$ invertible in $k$, let $X$ denote the henselization of a smooth variety over $k$ at some closed point, and let $X/S$ denote a smooth relative curve with a good compactification.

If $g_1, g_2 : S \rightarrow X$ are two sections of $X/S$ which agree on the closed point of $S$, then $g_1^* = g_2^* : F(X) \rightarrow F(S)$.

This enables Suslin and Voevodsky to prove the following theorem which essentially tells us that a presheaf with transfers as in the preceding theorem is essentially locally constant for the etale topology.

**Theorem 2.5.** Let $F$ be a homotopy invariant presheaf with transfers satisfying the condition that $nF = 0$ for some positive integer $n$ invertible in $k$, and let $S$ denote the henselization of a smooth variety at a closed point. Then the restriction map

$$F(S) \rightarrow F(\text{Spec} k)$$

is an isomorphism.

**2.2. Grothendieck topologies.** In this section, we shall give a brief introduction to the etale topology and to two other Grothendieck topologies introduced by Voevodsky and used by Suslin and Voevodsky.
**Definition 2.6.** A site $\mathcal{C}/X$ is a subcategory of the category of schemes over a fixed scheme closed under fiber products and is equipped with a distinguished class of morphisms which is required to be closed under composition, base change and which includes all isomorphisms. One selects as coverings of an object $Y \in \mathcal{C}/X$ families of distinguished morphism $\{g_i : V_i \to Y\}$ with the property that $Y = \cup_i g_i(V_i)$; one requires various closure properties of coverings: all isomorphisms are coverings, the pull-back of a covering of $Y$ by a morphism $Y' \to Y$ in $\mathcal{C}/X$ should be a covering of $Y'$.

The data of a site together with coverings consisting of distinguished morphisms is a Grothendieck topology on $X$.

The most “classical” Grothendieck topology (other than the Zariski topology, which of course is a topology in the usual sense as well as a Grothendieck topology) is the etale topology. The reader is refereed to the book by J. Milne [2] for considerable foundational detail on this important construction. The etale topology plays a significant role in our understanding of $K$-theory mod-$n$, and the Suslin-Voevodsky theorem provides a more naive means of determining the etale cohomology with $\mathbb{Z}/n$ coefficients of varieties over $k$.

**Definition 2.7.** A map $U \to X$ is said to be etale if it is flat, unramified, and locally of finite type. The small etale site $\mathcal{C}_{et}$ is the category of schemes étale over $X$ and whose distinguished morphisms are étale morphisms and whose coverings are all collections $\{g_i : V_i \to Y\}$ of étale morphisms with the property that $Y = \cup_i g_i(V_i)$. The big etale site $\mathcal{C}_{et}$ is the category all schemes locally of finite type over $X$ with distinguished morphisms and coverings as in $\mathcal{C}_{et}$.

**Example 2.8.** The following morphisms are examples of étale morphisms.

- $U \to X$ a Zariski open immersion.
- $X \to X$ a finite covering space
- Spec $R \to$ Spec $F$, where $F$ is a field and $R$ is a finite separable $F$-algebra (i.e., $\overline{F} \otimes_F R$ splits as a product of copies of $\overline{F}$, the algebraic closure of $F$).
- $V \to X$ a morphism of complex algebraic varieties with the property that $V^{an} \to X^{an}$ is a local homeomorphism.
- If $R$ is a domain, $g(t), h(t) \in R[t]$, then the map Spec $R[t]/[g(t)] R(t)^{-1} \to \text{Spec} R$ is etale provided that $g'(t)$ is invertible in $R[t]/[g(t)] R(t)^{-1}$ (i.e., the zero locus of $h(t)$ contains the common zeros of $g(t), g'(t)$).

One of the many insights of Grothendieck is that one can formulate sheaf theory and sheaf cohomology on a site with a Grothendieck topology with essentially no change from the usual sheaf theory for sheaves on a topological space.

**Definition 2.9.** Let $\mathcal{C}/X$ be a site provided with a Grothendieck topology and let $\mathcal{A}$ denote the category of sets, groups, abelian groups, rings, or modules over a given ring. Then a presheaf on $\mathcal{C}/X$ (i.e., a contravariant functor $\mathcal{F} : (\mathcal{C}/X)^{op} \to \mathcal{A}$) is said to be a sheaf if for all $Y \in \mathcal{C}/X$ and all coverings $\{V_i \to Y\}$ the following sequence is exact:

$$\mathcal{F}(Y) \to \prod_i \mathcal{F}(V_i) \to \prod_{i,j} \mathcal{F}(V_i \times_X V_j).$$

If $\mathcal{A}$ is an abelian category with enough injectives, then the category (topos) of sheaves for the Grothendieck topology on $\mathcal{C}/X$ with values in $\mathcal{A}$ is an abelian category with enough injectives, permitting us to apply standard homological algebra...
in defining the sheaf cohomology
\[ H^*(C/X, \mathcal{F}) \equiv R^i \Gamma(X, -)(\mathcal{F}). \]

One important property of étale cohomology is that \( H^*(X_{et}, \mathbb{Z}/n) \approx H^*_\text{sing}(X^{an}, \mathbb{Z}/n) \) for any quasi-projective complex algebraic variety. We mention another in the following example.

**Example 2.10.** Let \( F \) be a field with separable closure \( \bar{F} \). A sheaf of sets \( \mathcal{F} \) on \( F_{\text{et}} \) consists of a set \( S \) together with a “continuous” action of \( \text{Gal}(\bar{F}/F) \) on this set. In other words, an action of the discrete group \( \lim_{\leftarrow} F \) in defining the sheaf cohomology \( H^*(\bar{F}/F, \mathbb{Z}/n) \) is quasi-finite, then the Zariski topology is the quotient of of the \( \text{Gal}(\bar{F}/F) \) on \( S \) with the property that for each element \( t \in S \) there exists some (finite) Galois \( L_t/F \) with the property that the kernel of \( \lim_{\leftarrow} \text{Gal}(L/F) \rightarrow \text{Gal}(L_t/F) \) fixes \( t \).

In particular, if \( \mathcal{F} \) is an abelian sheaf on \( F_{\text{et}} \), then \( H^*(F_{\text{et}}, \mathcal{F}) \) is the Galois cohomology \( H^*_{\text{Gal}}(F, \mathcal{F}) \).

Recall that Theorem 2.5 involved the henslization \( S \) of a smooth variety at a closed point. Although such a henselization can be defined more algebraically, let us formulate this in terms of the étale topology.

**Definition 2.11.** Let \( X = \text{Spec } A \) be an affine variety and \( y \in \text{Spec } A \) be a closed point associated local ring \( \mathcal{O}_{X,y} = R \). The henselization \( R^h \) of \( R \) is the colimit of local étale morphisms \( R \rightarrow B \), \( R^h = \lim_{\rightarrow} \mathcal{O}_{X,y} = B \), where the map of local rings \( R \rightarrow B \) is required to induce an isomorphism on residue fields. The strict henselization \( R^{sh} \) of \( R \) is the colimit of all local étale morphisms \( R \rightarrow B \) without the condition that the induced map on residue fields is an isomorphism; thus, the residue field of \( R^{sh} \) is the separable closure of the residue field of \( R \).

If \( X = \text{Spec } R \) is the spectrum of a strict hensel local ring (i.e., \( R = R^{sh} \)), then the entire section functor \( \mathcal{F} \mapsto \mathcal{F}(X) \) is an equivalence from the category of sheaves on \( X_{\text{et}} \) with values in \( \mathcal{A} \) to the category \( \mathcal{A} \).

For a variety \( X \) over the algebraically closed field \( k \), a sequence of sheaves on \( X_{et} \) with values in an abelian category \( 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0 \) is exact if and only if its restriction to each of the (strict ) henselizations at closed points of \( X \) is exact.

Two other Grothendieck topologies play a role in the proof of the Suslin-Voevodsky theorem.

**Definition 2.12.** An \( h \)-covering of a scheme \( Y \) is a finite family of morphisms of finite type \( g_i : V_i \rightarrow Y \) with the property that the induced map \( p : \coprod V_i \rightarrow Y \) is a universal topological epimorphism (i.e., for any \( Y' \rightarrow Y \), the pull-back \( p' : \coprod V_i \times_Y Y' \rightarrow Y' \) satisfies the property that \( Y' \) as a topological space with the Zariski topology is the quotient of of \( \coprod V_i \)). If each of these morphisms \( g_i : V_i \rightarrow Y \) is quasi-finite, then the \( h \)-covering \( \{ g_i : V_i \rightarrow Y \} \) is called a qfh-covering.

The \( h \) (respectively, qfh) topology on \( X \) is the site of schemes of finite type over \( X \) whose

2.3. **Dold-Thom Theorem and the Suslin complex.** A well known theorem of A. Dold and R. Thom enable one to express the singular homology of a cell complex in terms of the homotopy groups of its symmetric powers.

**Theorem 2.13.** (cf. [1]) Let \( T \) be a C.W. complex and let \( \coprod_{d \geq 0} S^d T \) denote the free abelian monoid on the points of \( T \), where \( S^d T \) is the given the quotient topology.
with respect to the natural surjection $T^{\times d} \to S^d$. Then

$H^i_{sing}(T) = \pi_i(Sing.(\prod_{d \geq 0} S^d T^+))$

where $Sing.(\prod_{d \geq 0} S^d T^+)$ is the simplicial abelian group given in degree $n$ as the group completion of the abelian monoid $\prod_{d \geq 0} Sing_n(S^d T)$. Alternatively, if $\mathbb{Z}(T)$ denotes the topological abelian group on the points of $T$ (topologized as a quotient of $(\prod_{d \geq 0} S^d T)^{\times 2}$), then

$H^i_{sing}(T) = \pi_i(\mathbb{Z}(T))$.

Two observations help to make this homotopy theoretic construction applicable to algebraic geometry. First, if $X$ is an algebraic variety over $k$, then for each $d > 0$ the $d$-fold symmetric power $S^d X$ of $X$ is also an algebraic variety. Second, the homotopy groups of a simplicial abelian group $A_\bullet$ are naturally identified with the homology groups of the chain complex given in dimension $n$ by $A_n$ and with differential the alternating sum of the face maps $d_i : A_n \to A_{n-1}$. This chain complex is easily seen to be quasi-isomorphic to the normalized chain complex associated to $A_\bullet$ which is given in dimension $n$ as the intersection of the kernels of the face maps $d_i : A_n \to A_{n-1}$ for $i > 0$ and whose differential is the restriction of $d_0$ to this kernel.

Recall the conventional notation of $\Delta^n \equiv \text{Spec } k[t_0, \ldots, t_n]/\sum_i t_i - 1$, the “$n$-simplex over $k$”. The natural face and degeneracy maps determine a cosimplicial object $\Delta_* \in (\text{Sch}/k)$. Suslin’s observation is that it is profitable to consider the algebraic-geometric analogue of the topological construction $T \mapsto Sing.(\prod_{d \geq 0} S^d T^+)$. 

**Definition 2.14.** Let $X \in (\text{Sch}/k)$. We define

$Sus_* (X) \equiv N(\prod_{d \geq 0} \text{Hom}_{(\text{Sch}/k)}(\Delta^*, S^d X^+)),$

the normalized chain complex associated to the simplicial abelian group given in dimension $n$ by $\text{Hom}_{(\text{Sch}/k)}(\Delta^n, \prod_{d \geq 0} S^d X)$.

2.4. **Statement of Suslin-Voevodsky theorem.** We conclude this lecture with the statement of the remarkable theorem of Suslin-Voevodsky.

**Theorem 2.15.** For $X \in (\text{Sch}/k)$ and positive integer $n$ invertible in the algebraically closed field $k$,

$H^*(X_{et}, \mathbb{Z}/n) = H^*(Sus_*(X), \mathbb{Z}/n)$.

In particular, if $k = \mathbb{C}$, then

$H^*_{sing}(X^{an}, \mathbb{Z}/n) = H^*(Sus_*(X), \mathbb{Z}/n)$.

**References**


Department of Mathematics, Northwestern University, Evanston, IL 60208

E-mail address: eric@math.nwu.edu