Efficient and Accurate Quantification of Uncertainty for Multiphase Flow With the Probabilistic Collocation Method

Heng Li and Dongxiao Zhang, SPE, University of Southern California

Summary
In this study, we explore an efficient and accurate method for uncertainty quantification of petroleum reservoir simulations. The essence of the approach is the combination of Karhunen-Loeve (KL) expansion and probabilistic collocation method. Monte Carlo (MC) simulation is the most common and straightforward approach for uncertainty quantification. It generates a large number of realizations of the underlying reservoir. Solving the multiple realizations leads to a large computational effort, especially for large-scale problems. We present an accurate and efficient alternative. In this approach, the KL expansion, such as permeability and porosity, are represented by the KL expansion and the resulting random fields (e.g., fluid saturations and pressures) or variables (e.g., hydrocarbon production) are expressed by the polynomial chaos expansions. The probabilistic collocation method (PCM) is used to determine the coefficients of the polynomial chaos expansions by solving for the fluid saturation and pressure fields via the original partial differential equations for selected sets of collocation points. This approach is nonintrusive because it results in independent deterministic differential equations, which, similar to the MC method, can be implemented with existing codes or simulators. However, the required number of simulations in the PCM is much less than that in the MC method. The approach is demonstrated with black-oil problems in heterogeneous reservoirs with the commercial Eclipse simulator. The accuracy, efficiency, and compatibility of this approach are compared against MC simulations. This study reveals that, while its computational efforts are greatly reduced compared to the MC method, the PCM is able to estimate accurately the statistical moments and probability density functions of the fluid saturations (and pressures) and the hydrocarbon production.

Introduction
Quantification of uncertainty associated with petroleum reservoir simulations has recently attracted a significant amount of attention. The uncertainty can result from the combination of the formation heterogeneity and the incomplete knowledge of its properties. Traditional reservoir simulations treat the reservoir deterministic, thus resulting in deterministic predictions. However, taking uncertainty into consideration calls for a stochastic description of the reservoir properties and, hence, stochastic approaches to reservoir simulations.

MC simulation is the most common approach for uncertainty quantification in the industry. In the MC method, a large number of realizations of the random inputs are generated and solved to obtain a set of model outputs, which can be further analyzed statistically. The direct sampling MC method is straightforward and easy to implement. However, its main disadvantage is the requirement of large computational efforts because of the large number of model simulations needed to obtain statistically accurate results. On the other hand, some other stochastic approaches have been developed in the context of subsurface hydrology in the last two decades (Dagan 1989; Gelhar 1993; Cushman 1997; Neuman 1997; Zhang 2002; Rubin 2003) that could be applied to the petroleum industry.

Zhang and Tchelepi (1999) and Zhang et al. (2000) applied the Lagrangian statistical moment approach to uncertainty quantification in the water/oil two-phase flow system. This moment-equation-based method derives a system of deterministic differential equations governing the statistical moments (usually the first two) of the random variables with the method of perturbation or some type of closure approximation. However, the computational cost for the (conventional) moment equation method is still high for large-scale problems. And there are two major assumptions and limitations in the moment method: (1) The moment method is based on first-order perturbation approximations, which limits its application to relatively small variances in log permeability; (2) the total flow is assumed at steady state.

Another alternative approach, the polynomial chaos expansion (PCE) method, pioneered by Ghanem and Spanos (1991), was applied to single-phase and multiphase flow in porous media (Ghanem 1998; Ghanem and Dham 1998). This technique includes representing the random variables using polynomial chaos basis and deriving appropriate discretized equations for the expansion coefficients using the Galerkin technique. The polynomial chaos expansion allows high order approximation of random variables and possesses the property of fast convergence for particular types of input random variables. However, the deterministic coefficients of the polynomial chaos expansion are governed by a set of coupled equations, which are difficult to solve when the number of coefficients is large (owing to a large random dimensionality or a high polynomial order).

In this study, an efficient approach, the PCM, is employed to quantify uncertainty associated with multiphase flow in random porous media, in which the permeability or porosity is treated as a stochastic process (random field). Sarma et al. (2005) made use of the PCM and KL expansion for optimizing reservoir production in the presence of an uncertain permeability field. Li and Zhang (2007) explored the PCM for single-phase flow in random heterogeneous porous media by combining the KL expansion and polynomial chaos expansion and showed its superiority compared to other stochastic approaches. In the present study, applications of PCM and KL expansion for uncertainty quantification for multiphase flow in oil reservoirs are performed with a commercial reservoir simulator. With the PCM, the statistical properties of hydrocarbon production and the flow-related quantities of interest, such as fluid saturation and pressure, can be estimated.

Mathematical Formulation
Governing Equations. The three-phase black-oil model is expressed by the following continuity equations (Chen et al. 2006):

\[
\frac{\partial}{\partial t} \left( \phi \frac{S_i}{B_i} \right) - \nabla \left[ \frac{k(x)k_{r_l}}{\mu \phi_i} (\nabla p_i - \rho_i \phi \nabla \psi) \right] = q_i, \quad i = w, o
\]

\[
\frac{\partial}{\partial t} \left[ \frac{S_i}{B_i} \right] - \nabla \left[ \frac{k(x)k_{r_l} - k_{r_l_0} - k_{r_l_0}}{\mu \phi_i} (\nabla p_i - \rho_i \phi \nabla \psi) \right] + \frac{k(x)k_{r_l}}{\mu \phi_i} (\nabla p_i - \rho_i \phi \nabla \psi) \right] = q_i,
\]

Copyright © 2009 Society of Petroleum Engineers

Original SPE manuscript received for review 24 December 2007. Revised manuscript received for review 15 October 2008. Paper (SPE 114802) peer approved 3 December 2008.
where and denote the position and time, respectively; and denote the three phases (i.e., water, oil, and gas, respectively); is the porosity of the media; and are the formation volume factor and density of phase , respectively; is the solubility of gas in oil; is the source or sink term; and are the saturation and pressure of fluid , respectively; is the absolute (intrinsic) permeability; is the relative permeability of fluid , which is a function of ; is the viscosity of fluid ; is the depth; and is the gravitational acceleration.

Eqs. 1 and 2 are coupled with

and

where which is a function of is the capillary pressure between water and oil, and which is a function of is the capillary pressure between gas and oil. Based on the governing equations (Eqs. 1 through 5) subject to certain initial and boundary conditions, one can solve for the fluid saturation and pressure. In this study, the properties of the heterogeneous porous media, such as permeability, are treated as random functions, thus, the governing equations become stochastic partial differential equations whose solutions are no longer deterministic values but probability distributions or related statistical moments.

KL Expansion. Let [e.g., ] be a random space function, where and (a probability space). One may write

where and is the fluctuation. The spatial structure of the random field may be described by the covariance function

Because the covariance is bounded, symmetric, and positive-definite, it may be decomposed as (Ghanem and Spanos 1991)

where and are eigenvalues and deterministic eigenfunctions, respectively, and can be solved from the following Fredholm equation:

Then, the random process can be expressed as

where are orthogonal Gaussian random variables with zero mean and unit variance. The expansion in Eq. 8 is called the KL expansion. The KL expansion, which is a spectral expansion, is optimal with mean square convergence when the underlying process is Gaussian (Ghanem and Spanos 1991). Because the KL expansion only preserves the two-point covariance of the random field, it is only suitable for Gaussian random fields. For non-Gaussian fields, the kernel principal component analysis was applied for parameterization of the non-Gaussian random fields (Sarma et al. 2008).

Although, in general, the eigenvalue problem (Eq. 7) has to be solved numerically, analytical or semianalytical solutions exist under certain conditions. For a 1D stochastic process with a covariance function

where and are the variance and the correlation length of the process, respectively, the eigenvalues and their corresponding eigenfunctions can be expressed as (Zhang and Lu 2004)

and

where are the positive roots of the characteristic equation,

For problems in multidimensions, if we assume that the covariance function is separable, for example

where is a cubic domain . The eigenvalues and eigenfunctions can be obtained by combining those in each dimension. For a nonseparable covariance in a domain of arbitrary shape, the eigenvalue problem of Eq. 7 has to be solved numerically. Furthermore, the KL expansion is not limited to statistically homogeneous random fields (Lu and Zhang 2007). When a finite number of realizations of the reservoir description are available, the eigenvalues and eigenfunctions may be obtained with the approach of kernel principal component analysis (Sarma et al. 2008). From Eq. 8, one can get

where is the domain size, which indicates that the total variance is decomposed by an infinite series of eigenvalues (Eq. 11) has an infinite number of positive roots. If the roots are sorted in an increasing order, the related eigenvalues are nonincreasing, which allows us to truncate the KL expansion with a finite number of terms. The rate of decay of determines the number of terms that need to be retained in the KL expansion, which determines the random dimensionality of the problem.

PCE. The fluid saturation and pressure are dependent on medium properties, such as permeability and porosity. While the covariance of the dependent random processes are yet to be found, the KL expansion cannot be used to represent their random structures. Instead, the PCE, introduced by Wiener (1938) and now widely used in many fields, can be used to effectively express the dependent random fields. For example, we express the output random fields and with the polynomial chaos expansions,

and

where the coefficients are deterministic functions of and and are orthogonal polynomial chaos of order with respect to the random variables . For independent standard Gaussian
random variables \((\xi_1, \ldots, \xi_d)\), \(\Gamma_d(\xi_1, \ldots, \xi_d)\) are the multidimensional Hermite polynomials of degree \(d\) expressed as

\[
\Gamma_d(\xi_1, \ldots, \xi_d) = (-1)^d e^{\frac{1}{2} \xi^T \xi} \frac{\partial^d}{\partial \xi_1 \cdots \partial \xi_d} \left[ e^{-\frac{1}{2} \xi^T \xi} \right] \quad \text{for} \quad d = 1, 2, \ldots, N
\]

where \(\xi\) is a vector denoting \((\xi_1, \ldots, \xi_d)^T\). The Hermite polynomials form the best orthogonal basis for Gaussian random variables in the \(L_2\) sense (Ghanem and Spanos 1991). In case of other random distributions, generalized polynomial chaos expansions (Xiu and Karniadakis 2002) can be used to represent the random fields as in Eqs. 12 and 13.

Eqs. 12 and 13 can be expressed as

\[
S_j(\mathbf{x}, t) = \sum_{j=1}^{N} c_j(\mathbf{x}, t) \Psi_j(\xi). \quad \text{where} \quad \Psi_j(\xi) = \frac{1}{\sqrt{\pi \sigma_j^2}} e^{-\frac{1}{2} \left( \frac{\xi - \mu_j}{\sigma_j} \right)^2}
\]

and

\[
p_j(\mathbf{x}, t) = \sum_{j=1}^{N} d_j(\mathbf{x}, t) \Psi_j(\xi), \quad \text{where} \quad d_j = \frac{1}{\sqrt{\pi \sigma_j^2}} e^{-\frac{1}{2} \left( \frac{\xi - \mu_j}{\sigma_j} \right)^2}
\]

where \(c_j(\mathbf{x}, t)\) and \(d_j(\mathbf{x}, t)\) are deterministic coefficients and \(\Psi_j(\xi)\) are polynomials of random vector \(\xi\). There are \(P = (N \times d) / (N \times d')\) terms in each of the polynomials chaos expansions, where \(N\) is the random dimensionality and \(d\) is the degree of polynomial chaos expansion. The \(P\) terms of coefficients have to be determined.

The Galerkin method is used usually to solve for the coefficients in polynomial chaos expansions (Ghanem 1998; Xiu and Karniadakis 2002; Mathelin et al. 2005) and leads to a set of coupled equations governing those coefficients. For the multiphase flow problem, where the governing equations are nonlinear partial differential equations, the approach becomes computationally demanding. An alternative approach is PCM, which is introduced by Tatang et al. (1997) and applied for uncertainty analysis in some fields with independent uncertain parameters. The KL expansion and the PCM are successfully incorporated to solve subsurface flow problems where uncertain parameters are correlated and random fields exist (Sarma et al. 2005; Li and Zhang 2007).

**PCM.** In this section, we introduce the concept of PCM. For a stochastic differential equation

\[
L y(\mathbf{x}, t, \theta) = f(\mathbf{x}, t), \quad \text{where} \quad \mathbf{y}(\mathbf{x}, t, \theta) = y(\mathbf{x}, t, \theta), \text{and} \quad f(\mathbf{x}, t) \text{is the source term. The operator L involves} \text{differences in space and time and can be nonlinear. If} \quad \mathbf{y}(\mathbf{x}, t, \theta) \quad \text{is approximated by the polynomial chaos expansion and the approximation is denoted as} \quad \hat{\mathbf{y}}(\mathbf{x}, t, \theta),
\]

\[
\hat{\mathbf{y}}(\mathbf{x}, t, \theta) = \sum_{j=1}^{N} c_j(\mathbf{x}, t) \Psi_j(\xi(\theta)). \quad \text{where} \quad \xi(\theta) = \xi_1, \ldots, \xi_d\quad \text{are the collocation points}
\]

Define the residual \(R\) as

\[
R(c_j, \xi) = L \hat{\mathbf{y}} - f. \quad \text{where} \quad \hat{\mathbf{y}} = \sum_{j=1}^{N} c_j(\mathbf{x}, t) \Psi_j(\xi(\theta)). \quad \text{where} \quad \Psi_j(\xi(\theta)) = \frac{1}{\sqrt{\pi \sigma_j^2}} e^{-\frac{1}{2} \left( \frac{\xi(\theta) - \mu_j(\theta)}{\sigma_j(\theta)} \right)^2}
\]

The weighted residual method in the random space is expressed as

\[
\int \left[ R(c_j, \xi) \right] w_j(\xi) d\xi = 0, \quad \text{where} \quad \int \left[ R(c_j, \xi) \right] w_j(\xi) d\xi = 0 \quad \text{and} \quad p(\xi) = \text{the joint probability density function of} \xi
\]

where \(w_j(\xi)\) is the weighting function, \(j = 1, \ldots, P\), and \(p(\xi)\) is the joint probability density function of \(\xi\).

In the probabilistic collocation method, the weighting function is chosen as the Dirac delta function

\[
w_j(\xi) = \delta(\xi - \xi_j), \quad \text{where} \quad \xi_j \text{is a particular set of the random vector} \xi \text{selected with certain algorithm. The elements in} \xi \text{are called the collocation points. Then, Eq. 20 becomes,}
\]

\[
R(c_j, \xi) = 0, \quad \text{where} \quad \xi_j \text{is a particular set of the random vector} \xi \text{selected with certain algorithm. The elements in} \xi \text{are called the collocation points. Then, Eq. 20 becomes,}
\]

\[
\int \left[ R(c_j, \xi) \right] w_j(\xi) d\xi = 0, \quad \text{where} \quad \int \left[ R(c_j, \xi) \right] w_j(\xi) d\xi = 0 \quad \text{and} \quad p(\xi) = \text{the joint probability density function of} \xi
\]

which results in a set of independent equations, evaluated at the given sets of collocation points, \(\xi_j\) where \(j = 1, 2, \ldots, P\). The choice of Dirac delta function as the weighing function leads to independent, deterministic simulations. Other weighting functions result in coupled equations for those coefficients of the polynomial chaos expansion. It is seen that \(P\) sets of collocation points are needed, and, thus, Eq. 22 has to be solved for \(P\) times to obtain the \(P\) coefficients \(\{c_j\}\), where \(i = 1, 2, \ldots, P\). The collocation points at a given order of polynomial chaos expansion can be selected from the roots of the next higher order orthogonal polynomial for each uncertain parameter (Li and Zhang 2007).

By choosing different sets of collocation points for the input random variables and solving independent partial differential equations, the coefficients of the polynomial chaos expansions can be evaluated by solving a linear system of equations on the basis of Eq. 18. Once obtaining the coefficients of the polynomial chaos expansions, one could easily evaluate the statistical quantities, such as various moments and the probability density function, of the outputs based on expansions of Eq. 18 by certain sampling methods, such as the MC method. Because the expansion is reduced to a polynomial form and it does not involve solving equations, it can be evaluated efficiently. Alternatively, the statistical moments of the outputs may be directly derived from the polynomial chaos expansions. For example, the mean and variance of \(y(\mathbf{x}, t)\) are derived from Eq. 18 as

\[
\langle y(\mathbf{x}, t) \rangle = c(\mathbf{x}, t) \quad \text{and} \quad \sigma_y^2 = \sum_{j=2}^{N} \left[ c_j(\mathbf{x}, t) \right]^2 \langle \Psi_j^2 \rangle
\]

Higher order moments can be obtained similarly.

The number \((P)\) of terms in the polynomial chaos expansions, hence the number of simulations, strongly depends on the random dimensionality \((N)\) of the input random fields and the order \((d)\) of the polynomial chaos expansions of the output random fields. The solution accuracy enhances with the increase of \(N\) and \(d\), but the computational effort also increases rapidly. Although, for each problem, the sufficient \(N\) or \(d\) may be numerically examined by comparing with the next level PCM, such a procedure could be computationally demanding. A posterior error estimator may help in determining the proper \(N\) and \(d\) that balance the accuracy and the efficiency of the PCM. However, such an error estimator is beyond the scope of this work and is a subject of our ongoing research.

This PCM is a nonintrusive method because it results in independent deterministic differential equations, which, similar to the MC method, can be implemented with existing codes or simulators. In the present study, we apply the PCM to the multiphase flows where nonlinear partial differential equations exist.

**Case Studies**

The PCM is demonstrated for multiphase flow with the commercial Eclipse black-oil simulator. Different scenarios of 1D, (oil and water) two-phase flows are used to analyze the effect of various factors, such as type of variability, variance, correlation length, viscosity ratio, and relative permeability type. On the basis of the Ninth SPE Comparative Solution Project, a 3D three-phase reservoir model with a moderate grid size is designed for reservoir performance forecast.
1D Cases. At first, a 1D (water and oil) system is considered. The reservoir depth is 4,000 ft, and the length \( L = 1,000 \) ft, uniformly divided by 40 gridblocks. Assume constant water injection rate of 100 STB/D for the injection well at the inlet, and constant bottomhole pressure of 3,000 psi for the production well at the outlet. The capillary pressure is neglected for simplicity. The following Corey-type relative permeability functions are considered:

\[
k_{sw} = (S^*)^m, \quad k_{ro} = (1 - S^*)^m \tag{25}
\]

and

\[
S^* = \frac{S_w - S_{wc}}{1 - S_{sw} - S_{ro}}, \quad (26)
\]

where \( S_{wc} \) and \( S_{sw} \) are the connate water saturation and residual oil saturation, respectively. Because the PCM makes a direct use of existing simulators, it is not restrictive to particular types of relative permeability curves or tabulations. The following illustrative examples employ the linear Corey-type relative permeability functions or tabulations. The following expressions are then solved from the

\[
\frac{\partial}{\partial t} \left( \exp \left( \langle \beta(x) \rangle + \sum_{j=1}^{N} \sqrt{\lambda_j} f_j(x) \xi_j \right) S_i \right)
\]

where \( \lambda_j \) and \( \eta \) are the variance and the correlation length of the process, respectively. \( \beta(x) \) is represented by the KL expansion

\[
\beta(x) = \langle \beta(x) \rangle + \sum_{j=1}^{N} \sqrt{\lambda}_j f_j(x) \xi_j \tag{29}
\]

Substituting Eq. 29 into Eq. 1 yields,

\[
-\nabla \left( \frac{k(x)k_i}{\mu_i B_i} \left( \nabla p_i - \rho_i g \nabla z \right) \right) = q_i, \quad i = w, o. \tag{30}
\]

With the PCM, we only need to choose certain sets of collocation points for random variables \( \xi_j = (\xi_{j1}, \xi_{j2}, \ldots, \xi_{jP}) \), where \( j = 1, 2, \ldots, P \). For each set \( \xi_j \), the water saturation and pressure fields are solved using the reservoir simulator in a manner similar to MC simulations. The coefficients \( c_i \) and \( d_i \) in Eqs. 15 and 16 are then solved from the \( P \) sets of saturation and pressure fields, respectively. Statistical properties of the output random fields can then be evaluated with these coefficients using Eqs. 23 and 24.

Here, we assume \( \langle \beta(x) \rangle = -0.9613 \), corresponding to the mean porosity of approximately 0.4, \( \sigma_p \) as 0.1, and \( \eta L = 2/5 \). The eigenvalue and eigenfunction \( \lambda_j \) and \( f_j(x) = 1, 2, \ldots \) can be determined by solving Eqs. 9 and 10. The eigenvalues are monotonically decreasing as illustrated in Fig. 1a, for cases with different correlation lengths \( \eta = 1/5 \) and 2/5. Fig. 1b shows the sum of eigenvalues as a function of number of terms included. Because of the rapid decay, only the first six terms are retained in the KL expansion for \( \eta = 2/5 \). That is, the random dimensionality of \( \{ \xi_j \} \) is \( N = 6 \). One can see from Fig. 1b that approximately 90% of the energy of the process is preserved in this case. For the second-order PCM, the total number of collocation sets is \( P = 28 \). More precisely, the PCM involves 28 sets of collocation points, each of which requires one run of the simulator.

Different cases with different viscosity ratios of oil to water, defined as \( m = \mu_o/\mu_w \), are performed. Figs. 2 and 3 depict the mean values of the water saturation and pressure and the associated standard deviations at 100 days for different viscosity ratios \( m = 0.5 \) and 2. The results obtained from the second-order PCM and MC simulations with 1,000 realizations are presented together for comparison. Excellent agreements are achieved in all the cases, both for water saturation and pressure. However, the computational cost of the PCM is much less than the MC. The PCM needs to perform just 28 deterministic simulations while the MC requires 1,000 simulations. It is observed from the figures that the standard deviations of pressure are small compared to their mean values. This indicates that the random heterogeneity of porosity has a small effect on the pressure variations for this particular setup of porosity variability and boundary conditions. And the viscosity ratio affects distributions of the mean and standard variation of the water saturation: The spreading of the mean saturation is larger (indicating larger transition zone) when oil is more viscous than water, and the standard deviation profile of water saturation has a higher peak and is more compact when oil is less viscous. This is consistent with the finding of Zhang and Tchelepi (1999).

As shown in Fig. 1, the rate of decay in the eigenvalues is dependent on the correlation length \( \eta \) relative to the domain length \( L \). When the correlation length is smaller, the convergence of the eigenvalues in the KL expansion becomes slower. To further test the effect of correlation length on the PCM, we perform another case with smaller correlation length \( \eta L = 1/5 \), keeping other conditions the same as in the previous case of viscosity ratio \( m = 2 \). In this case, to keep the number of terms retained in the
Fig. 2—Mean and standard deviation of water saturation and pressure obtained from second-order PCM and the MC (with 1,000 realizations) for the case with standard deviation of log porosity $\sigma_{\ln \phi} = 0.1$ and viscosity ratio $m = 0.5$: (a) and (b) for water saturation; (c) and (d) for pressure.

Fig. 3—Mean and standard deviation of water saturation and pressure obtained from second-order PCM and MC (with 1,000 realizations) for the case with standard deviation of log porosity $\sigma_{\ln \phi} = 0.1$ and viscosity ratio $m = 2$: (a) and (b) for water saturation; (c) and (d) for pressure.
KL expansion as six, approximately 80% of the total energy of the random process is preserved, as seen from Fig. 1b. The mean and standard deviation of water saturation (and pressure) at 100 days obtained with the second-order PCM and the MC are presented in Fig. 4. It can be seen that the PCM results are still in perfect agreement with the MC results.

Once the dependent random field such as the water saturation is approximated by the polynomial chaos expansion (Eq. 18), the probability density functions (pdfs) of the random field can be simulated with certain sampling methods. By using the case with \( \eta/L = 2/5 \) and viscosity ratio \( m = 2 \) as an example, we calculate the pdfs of water saturation and pressure at several locations, with the PCM and direct sampling MC simulations. Figs. 5a and 5c show the pdfs of water saturation at 100 days at two locations (\( x = 150 \) ft and 350 ft, respectively). And the pdfs of the pressure at the two locations are shown in Figs. 5b and 5d, respectively. It can be seen that the PCM results agree with the MC results very well. With the direct sampling MC simulations, 1,000 realizations are solved with the simulator and then the results are statistically evaluated to obtain the pdfs. However, with the PCM, one needs to run only 28 deterministic simulations to construct the polynomial expansions, on the basis of which 10,000 samplings are then drawn to approximate the pdfs. Because the dependent random field is explicitly expressed in a polynomial form, the computational time is significantly reduced compared to running the reservoir simulator with the MC method.

Random Permeability Field. Next, we consider the permeability \( k(x) \) to be a random field while keeping the porosity uniform. The covariance of \( Y(x) = \ln k(x) \) satisfies the exponential form

\[
C_y(x_i, x_j) = \sigma_y^2 \exp(-|x_i - x_j| / \eta),
\]

where \( \sigma_y^2 \) and \( \eta \) are the variance of \( Y(x) \) and correlation length of the process, respectively.

Substituting the KL expansion Eq. 8 into Eq. 1 yields,

\[
\frac{\partial}{\partial t} \left( \phi \frac{S_w}{B_i} \right) - \nabla \cdot \left[ \frac{\exp \left( Y(x) \right) + \sum_{n=1}^{N} \sqrt{\lambda_n f_n(x)} \right] k_n \right] \left( \nabla p_i - \rho_w \nabla \phi \right) = q, \quad t = w.o. \tag{32}
\]

The PCM can be implemented as described in the previous section. By choosing \( P \) sets of collocation points for random variables \( \xi \), we can obtain the \( P \) number of water saturation and pressure fields with the reservoir simulator and then compute the coefficients \( c_i \) and \( d_i \) in Eqs. 15 and 16, from which statistical properties of the output random fields are evaluated.

Assume \( Y(x) = 5.0 \) (in mD), corresponding to the mean permeability of approximately 148 mD. Consider the same reservoir conditions as in the previous case of correlation length \( \eta/L = 2/5 \) and viscosity ratio \( m = 1 \). Different spatial variabilities of permeability (i.e., \( \sigma_y^2 = 0.25, 1.0, \) and 4.0) are explored. The mean (and standard deviation) of water saturation and pressure at 100 days, obtained from the second-order and fourth-order PCM and MC simulations, are presented in Figs. 6 through 8. The number of terms retained in the KL expansion is six, thus the second-order PCM needs 28 sets of collocation points and the fourth-order PCM requires 210 sets of collocation points. For the cases of \( \sigma_y^2 = 0.25 \), both the second-order and fourth-order PCM agree quite well with MC simulations of 1,000 realizations, both for the water saturation and pressure. When the spatial variability of the permeability becomes larger (i.e., in the cases of \( \sigma_y^2 = 1.0 \) and 4.0), so does the uncertainty in water saturation and pressure. MC simulations with up to 10,000 realizations are performed for comparison. It is seen from Figs. 7 and 8 that the standard deviation of water saturation as well as the mean and standard deviation of pressure from the MC with 1,000 realizations are different from those with 10,000.
Fig. 5—Probability density functions of water saturation $S_w$ and pressure at two different locations ($x = 150$ ft and $350$ ft), obtained from second-order PCM and MC (with 1,000 realizations) for the case with standard deviation of log porosity $\sigma_{\ln \phi} = 0.1$, correlation length $\nu/L = 2/5$, and viscosity ratio $m = 2$: (a) and (c) for water saturation; (b) and (d) for pressure.

Fig. 6—Mean and standard deviation of water saturation and pressure obtained from PCM (second and fourth order) and MC (with 1,000 realizations) for variance of log permeability $\sigma_Y^2 = 0.25$: (a) and (b) for water saturation; (c) and (d) for pressure.
Fig. 7—Mean and standard deviation of water saturation and pressure obtained from PCM (second and fourth order) and MC (with 1,000 and 10,000 realizations) for $\sigma_Y^2 = 1.0$: (a) and (b) for water saturation; (c) and (d) for pressure.

Fig. 8—Mean and standard deviation of water saturation and pressure obtained from PCM (second and fourth order) and MC (with 1,000 and 10,000 realizations) for $\sigma_Y^2 = 4.0$: (a) and (b) for water saturation; (c) and (d) for pressure.
realizations. The convergence of MC results shall be investigated in more details. In general, it is easier to accurately estimate the mean and standard deviation of saturation than those of the pressure as the log permeability variability increases. At \( \sigma^2 = 1.0 \), corresponding to the permeability coefficient of variation being 131\%, there is a good agreement between the second-order PCM and the MC approach for the mean and standard deviation of saturation while there exist large deviations for the pressure statistics. It is found that the fourth-order PCM gives better agreements with the MC of 10,000 realizations than does the second-order PCM. Even at \( \sigma^2 = 4.0 \), corresponding to the permeability coefficient of variation being 732\%, the fourth-order PCM results in reasonable approximations for the saturation and pressure statistics.

It is seen that the low-order PCM approximations require a smaller number of simulations but give reasonable results compared to the high-resolution MC method. One apparent advantage of the MC method is flexibility in that it directly uses the existing simulators and can be performed with any number of simulations. In practice, only a small number of realizations are solved owing to the large computational effort required for each of them. The salient question is whether the MC approach gives consistent and accurate results if a small number of simulations are performed. For the case of \( \sigma^2 = 4.0 \), we evaluated the mean and standard deviation of water saturation and pressure using 40 different sets of MC simulations for the saturation and pressure statistics. The PCM and MC results agree well with each other.

It shows that, after using 1,000 realizations, the mean values start to converge while the standard deviations have fluctuations even with 5,000 realizations. With the direct sampling MC simulations resorting to the law of large numbers, normally a large number of realizations are needed to achieve the statistically accurate results. Both the MC method and the PCM involve sampling. The difference is that, in the direct sampling MC method, the realizations are equally probable, whereas, in the PCM, a structural expression (i.e., the polynomial chaos expansion) of the output random field is generated at first and then the collocation technique is adopted according to the Gaussian quadrature rule (Villadsen and Michelsen 1978). Because of the fast convergence of the polynomial chaos expansion and the high accuracy of Gaussian quadrature, the PCM can yield quite robust results with much fewer samplings than the direct sampling MC method.

It should be noted that the statistical properties of other flow-related quantities of interest can also be quantified with the PCM. For example, the total (cumulative) oil production from the well can be treated as a random function of time and represented with the polynomial chaos expansion. When the expansion coefficients are obtained with the \( P \) sets of simulations, the mean and standard deviation of the total oil production can be determined, from which the confidence interval can be estimated. For the cases of \( \sigma^2 = 1.0 \) and 4.0, Figs. 11 and 12, respectively, exhibit the confidence interval of total well oil production as a function of time, obtained from the fourth-order PCM and the MC with 10,000 realizations. The confidence intervals are constructed by the mean plus or minus one standard deviation. The PCM and MC results agree well with each other.

![Fig. 9—Mean and standard deviation of water saturation and pressure obtained from 40 sets of MC (with 210 realizations) for \( \sigma^2 = 4.0 \). (a) and (b) for water saturation; (c) and (d) for pressure.](image-url)
other and can be used as the measures of the production uncertainty.

Fig. 11 shows that, when \( Y_2 = 1.0 \), the uncertainty of the total oil production is quite small, while Fig. 12 indicates a larger uncertainty of the total oil production when \( Y_2 = 4.0 \).

Relative Permeability Type. To further investigate the effect of relative permeability functions of the fluids on the statistical quantities of the two-phase flow, two more cases with different types of relative permeability functions are performed in this section. Let \( n = m = 2 \) in Eqs. 25 and 26. Consequently, the following Corey-type relative permeability functions are obtained with both connate water and residual oil saturation equal to zero:

\[
k_{Sw} = S_w^2, \quad \text{and} \quad k_{So} = (1 - S_o)^2.
\]

Fig. 10—Mean and standard deviation of water saturation and pressure (at \( x = 275 \text{ ft} \)) obtained from MC simulations vs. the number of realizations for \( Y_2^2 = 4.0 \): (a) and (b) for water saturation; (c) and (d) for pressure.

Fig. 11—Confidence intervals of the total well oil production at different time for \( Y_2^2 = 1.0 \), obtained from the fourth-order PCM and MC (with 10,000 realizations).
With the previous relative permeability functions, two cases with random log porosity and permeability, respectively, are conducted. The first one has the same conditions as the previous case shown in Fig. 3 with the standard deviation of log porosity \( \sigma_{\ln \phi} = 0.1 \), correlation length \( \eta L = 2/5 \), and viscosity ratio \( m = 2 \). Fig. 13 depicts the mean and standard deviation of water saturation and pressure obtained with the second-order PCM and the MC with 1,000 realizations. It can be seen that the results are still closely matched between the PCM and MC. It is also observed that the shape of the water saturation profile changes with different relative permeability functions and that the standard deviation profile of water saturation appears to be more compact and have a higher peak in the water front zone, in this case with the quadratic Corey-type relative permeability functions compared to the previous case of Fig. 3 with linear Corey-type relative permeability functions. The profile of pressure standard deviation shows a similar effect.

The other case has the same conditions as the previous case shown in Fig. 7 with the variance of log permeability \( \sigma_y^2 = 1.0 \), correlation length \( \eta L = 2/5 \), and viscosity ratio \( m = 1 \). The mean and standard deviation of water saturation and pressure obtained

Fig. 12—Confidence intervals of the total well oil production at different time for \( \sigma_y^2 = 4.0 \), obtained from the fourth-order PCM and MC (with 10,000 realizations).

Fig. 13—Mean and standard deviation of water saturation and pressure obtained from second-order PCM and MC (with 1,000 realizations) for the case with standard deviation of log porosity (i.e., \( \sigma_{\ln \phi} = 0.1 \)), viscosity ratio \( m = 2 \), and new relative permeability functions: (a) and (b) for water saturation; (c) and (d) for pressure.
from the PCM (with both the second and fourth order) and the MC (with both 1,000 and 10,000 realizations) are shown in Fig. 14. It is found that the standard deviation of pressure is similar to that in Fig. 7 with the linear relative permeability functions and that the peak value of the standard deviation of water saturation is one magnitude higher than that in Fig. 7. The second-order PCM yields reasonable results compared to the high-resolution MC results with 10,000 realizations, while the fourth-order PCM enhances the accuracy. The two cases performed in this section indicate that the type of relative permeability functions does affect the two-phase flow responses and, hence, their statistical quantities.

3D Case. In this section, we apply the PCM to a 3D three-phase model with moderate grid size, modified from the Ninth SPE Comparative Solution Project. The Ninth SPE Comparative Solution Project was originally organized to investigate the complications for black-oil reservoir simulation brought about by a high degree of heterogeneity in a geostatistically based permeability field (Killough 1995). The problem involves waterflooding with natural water encroachment from an aquifer at the bottom of the dipping reservoir. The model has dimension of $7,200 \times 7,500 \times 360$ ft, which is represented by a $24 \times 25 \times 15$ grid. Cell (1,1,1) is at a depth of 9,000 ft subsea, and the remaining cells dip in the $x$ direction at an angle of 10°. There is no dip in the $y$ direction. There is one water injection well in the corner grids ($i = 24, j = 25$), and 25 randomly distributed producers in the reservoir. The initial oil-phase pressure is 3,600 psia at a depth of 9,035 ft subsea, which is the saturation pressure of oil. The oil/water contact is at 9,950 ft subsea. There is no initial free gas in the reservoir. Fig. 15 depicts the configuration of the reservoir with one injection well (I1) and 25 production wells (P2 through P26). The initial oil saturation is shown in Fig. 15. More detailed descriptions of the reservoir can be found in the literature (Killough 1995). The Ninth SPE Comparative Solution Project considered just one realization of heterogeneous permeability field and performed deterministic simulation. In this study, we adopt the same reservoir and fluid descriptions; however, we impose a random permeability field to the reservoir and perform stochastic simulations to predict uncertainty in the reservoir. In our simulations, the bottomhole pressure of the injection well is controlled at 4,000 psi and all production wells are controlled at 1,000 psi.

The vertical permeability is assumed 1% of the horizontal permeability. We assume the log horizontal permeability $Y(x) = \ln(k(x))$ is a Gaussian random field with the separate exponential covariance function:

$$
\sigma_Y^2 = 1.0
$$

Fig. 14—Mean and standard deviation of water saturation and pressure obtained from PCM (second and fourth order) and MC (with 1,000 and 10,000 realizations) for the case with $\sigma_Y^2 = 1.0$, viscosity ratio $m = 1$, and new relative permeability functions: (a) and (b) for water saturation; (c) and (d) for pressure.
\[ C_r(x, y) = \sigma^2_y \exp \left(-\frac{1}{\eta_1} |x_1 - y_1| + \frac{1}{\eta_2} |x_2 - y_2| + \frac{1}{\eta_3} |x_3 - y_3| \right), \quad \ldots \ldots (35) \]

where \( \sigma^2_y \) is the variance of log permeability, and \( \eta_1, \eta_2, \eta_3 \) are the correlation lengths in \( x, y, \) and \( z \) direction, respectively.

Here, we assume the mean of the log horizontal permeability \( \langle Y \rangle = 4 [+ln \text{ md}] \), variance \( \sigma^2_Y = 1.0 \), and the correlation lengths are \( 1/4 \) of the domain sizes in each direction. \( Y(x) \) is represented by the KL expansion (Eq. 8). Because of the rapid decay of the eigenvalues, as shown in Fig. 16, only the first 20 terms are retained in the KL expansion for PCM. Thus, the total number of collocation points for the second-order PCM is 231. The formula for the PCM in 3D can be given similar to Eq. 32 as in 1D, and the implementation is the same. The PCM is performed with the commercial reservoir simulator Eclipse and compared to MC simulations.

One realization of the horizontal permeability field is shown in Fig. 17, indicating the high heterogeneity of the reservoir. Fig. 18 shows the contour plots of the mean (and standard deviation) of oil saturation at the end of simulation (600 days) obtained from the PCM and MC. And Fig. 19 is for gas saturation. It reveals that the statistics of both oil and gas saturation achieve good agreement between the PCM and MC. The spatial distributions of the statistical properties (such as the mean and standard deviation) can be regarded as important visual measures of the uncertainty rising from the randomness of the heterogeneous permeability. The PCM is more efficient than the MC, because the PCM is involved with 231 independent simulations and the MC requires 1,000.

Besides the flow-related quantities, such as fluid saturation, the output variables could be any other reservoir responses generated from the reservoir simulation. For instance, if the field oil and gas production are the quantities of interest, we can obtain the statistical moments (e.g., mean and variance) of them, on the basis of which the confidence intervals can then be constructed. Figs. 20a and 20b exhibit the confidence intervals (using mean plus or minus the standard deviation) of the cumulative field oil and gas production, respectively, both normalized by the reservoir pore volume. It can be seen that the results obtained from the PCM agree well with those from the MC. It indicates that, for such a three-phase model with high nonlinearity and spatial variability, the PCM can quantify uncertainty of the reservoir performances.
economically, with a smaller number of simulations compared to the MC simulations.

Conclusions
A stochastic uncertainty quantification approach has been developed for multiphase flow in random heterogeneous media with the aid of PCM. In this approach, the underlying random fields, such as permeability and porosity, are represented by KL expansions, and the dependent (output) random fields (e.g., fluid saturation and pressure) or variables (e.g., hydrocarbon production) are expressed by polynomial chaos expansions. The PCM is used to determine the coefficients in the polynomial chaos expansion and evaluate the statistical properties of the random outputs by running the deterministic reservoir simulation models independently. The commercial Eclipse reservoir simulator is incorporated for uncertainty quantification in a straightforward manner. Applications of the PCM to different scenarios of multiphase flow are performed in one and three physical dimensions. The 1D examples are used to illustrate the approach and study its sensitivity to various factors, such as type of variability, variance, correlation length, viscosity ratio, and relative permeability type. A 3D, three-phase reservoir model is modified from the Ninth SPE Comparative Solution Project for the purpose of reservoir performance forecast. The statistical properties of the flow-related quantities of interest, such as the fluid saturation, pressure, and the cumulative oil (and gas) production, are predicted and compared with those from MC simulations. This study leads to the following conclusions:

- The PCM is found to be a feasible approach for accurately and efficiently quantifying uncertainty associated with reservoir simulations at the field scale. With a smaller computational effort, the PCM can achieve a good agreement with the MC results from a large number of realizations.
- This PCM approach is nonintrusive because it results in independent deterministic differential equations, which, similar to the MC method, can be implemented with existing codes or simulators. Both the direct sampling MC method and the PCM involve sampling. The MC method resorts to the law of large numbers by generating a large number of equally probable realizations and, thus, is computationally demanding; the PCM builds upon an optimal approximation evaluated at selected sets of collocation points and, hence, reduces the computational efforts significantly.
- The second-order PCM has been found to be consistently efficient in different cases with moderate correlation lengths of the underlying random porosity field and in cases with mild spatial variability of the random permeability field. When the variance of random log permeability becomes large (e.g., with the coefficient of variation being larger than 253%), a high order of polynomial chaos may be needed to achieve higher accuracy.
- For a given computational effort, the PCM gives more robust results, which do not change with the choice of the realizations.
- The statistical moments (e.g., mean and standard deviation) and probability density functions of water saturation and pressure are important measures of uncertainty arising from the randomness of the heterogeneous porosity or permeability. Confidence intervals of the cumulative oil (and gas) production can be constructed to show the uncertainty in production. The viscosity ratio of oil to water and the relative permeability type affect distributions of the mean and standard variation of the water saturation: the spreading of the mean water saturation is larger (indicating larger transition zone) when oil is more viscous than water, and the standard deviation profile of water saturation has a higher peak and is more compact when oil is less viscous; with the quadratic Corey-type relative permeability functions in contrast to the linear relative permeability functions, the spreading of mean water saturation is smaller, while the standard deviation profile of water saturation has a higher peak and is more compact.

Acknowledgments
This work is partially funded by American Chemical Society through its Petroleum Research Fund Grant No. 42109-AC9, by the National Science Foundation through Grant DMS-0801425, and by the University of Oklahoma Consortium on Ensemble Methods.

References


Heng Li is a PhD student at the University of Southern California majoring in petroleum engineering. His research interests include reservoir simulation, stochastic uncertainty quantification, and history matching. He holds an MS degree in fluid mechanics from Peking University, China. Dongxiao Zhang is the Marshall Professor at the Petroleum Engineering Program at the University of Southern California, where he has been a faculty member since 2007. He holds a joint appointment with the Sonny Astani Department of Civil and Environmental Engineering at USC. From 1996 to 2003, he was a technical staff member and a team leader at the Los Alamos National Laboratory. His research interests include stochastic uncertainty quantification of reservoir simulations, pore-scale processes, and geological carbon sequestration. Zhang holds MS and PhD degrees in hydrology from the University of Arizona.