CS 670 (Fall 2018) — Assignment 3
Due: 10/01/2018

(1) [10 points]
You are given a directed graph $G$ with non-negative edge costs $c_e$, and a root node $r \in V$, as well as a set $X \subseteq V$ of target nodes. The goal is to find a set $F$ of edges of minimum total cost, such that $F$ contains a directed path from $r$ to each node $v \in X$. (Let's say that at least one such path always exists.)

Phrase this problem as a decision problem and prove that it is NP-complete.

(2) [10 points]
In class, we saw that the INDEPENDENT SET problem is NP-hard. Here, we will look at a variant that we call EXTRA-INDEPENDENT SET. In the EXTRA-INDEPENDENT SET problem, we are still given an undirected graph $G$, but now, we're also given a target distance $d$. Not only do we want to select no two neighbors in $G$, but no pair of nodes that is at distance $d$ or less. (So INDEPENDENT SET is the special case where $d = 1$.) The goal is still to select as many nodes as possible, subject to this constraint. Notice that there are two subtly different versions of the problem: in one version, $d$ is part of the input, i.e., the algorithm is given the graph $G$ and the number $d$. In the other version, $d$ is fixed in advance to some value, e.g., $d = 3$, and the input is just the graph $G$.

(a) Phrase both versions of the problem as decision problems.
(b) Show that the version with $d$ as part of the input is NP-complete. (Note: this is meant as a super-easy warmup.)
(c) Prove that the version with $d$ fixed is also NP-complete, for whatever value you fix $d$ to. That is, for any number $y$, prove that the version with $d = y$ fixed is NP-complete.

(3) [10 points]
Problem 8.27 from the textbook.

(4) [10 points]
You are given a directed graph $G$ (without edge weights) and a source $s$ and sink $t$, as well as a bound $b$. You are to decide if there are two vertex-disjoint paths from $s$ to $t$ (i.e., they share no vertex besides $s, t$), each using at most $b$ edges. Prove that this problem is NP-complete. Note that without the length bound of $b$, this problem can be solved in polynomial time, as we will see in about a week in class.

(Hint: we recommend a gadget reduction from 3SAT, and roughly using one path for variable choices and one for clauses (though of course, the choices will have to interact). This problem is not easy.)

(5) [0 points]
**Chocolate Problem (2 chocolate bars):** Imagine that you are a private company owning a toll road. You want to decide on tolls to charge for the different segments of your road so as to maximize the company profit.

You have $m$ ramps to get on/off your toll road, numbered $1, 2, \ldots, m$. This divides your toll road into segments $T_j = [j, j + 1)$. You are given a list of $n$ drivers. For each driver $i$, you are told the starting and ending ramp of their trip $s_i < t_i$, as well as a budget $b_i$. To drive from $s_i$ to $t_i$, the driver has to drive along all the intermediate segments $T_{s_i}, T_{s_i+1}, \ldots, T_{t_i-1}$.

You get to choose a price for each segment $T_j$. If the sum of prices on the trip is affordable to the driver (at most $b_i$), then they will drive on your toll road and pay you the total for the segments. If the sum exceeds $b_i$, driver $i$ will take surface streets for the entire trip, and not pay you anything. Your goal is to choose prices for the segments that maximize your revenue from the drivers under this model.

Phrase this problem as a decision problem, and prove that it is NP-complete.