(1) [15 points]
(Notice: the solution to this problem can probably be found fairly easily online or in textbooks. Recall that using these resources to solve the problem is considered cheating, and could result in corresponding sanctions.)

This problem deals with the concept of matroids. Recall that a matroid is a set system \((E, \mathcal{I})\), where \(\mathcal{I} \subseteq 2^E\), with the following properties:

- \(\emptyset \in \mathcal{I}\).
- If \(S \in \mathcal{I}, S' \subseteq S\), then \(S' \in \mathcal{I}\).
- If \(S, S' \in \mathcal{I}, |S| < |S'|\), then there exist an \(e \in S' \setminus S\) such that \(S \cup \{e\} \in \mathcal{I}\).

(a) Prove that for any given undirected graph \(G\), the following is a matroid: \(E\) is the set of all edges, and \(\mathcal{I}\) is the set of all acyclic edge sets of \(G\).

(b) Prove that the following is a matroid: You have \(k\) disjoint sets \(S_1, S_2, \ldots, S_k\), and \(E = \bigcup_i S_i\). Furthermore, you are given \(k\) integers \(n_1, n_2, \ldots, n_k\), and one more integer \(n\). \(\mathcal{I}\) consists of all subsets \(S \subseteq E\) such that \(|S \cap S_i| \leq n_i\) for all \(i\), and \(|S| \leq n\).

(c) Now, give each element \(e\) of the matroid a weight \(w_e \geq 0\). (You can assume that all weights are distinct.) The goal is to find a set \(S \in \mathcal{I}\) maximizing \(\sum_{e \in S} w_e\). (Generalization of a maximum spanning tree.)

Consider the following natural adaptation of Kruskal’s algorithm: Sort all elements \(e\) by decreasing weights. Greedily (in the sorted order) add element \(e\) to the current set \(S\) if \(S \cup \{e\} \in \mathcal{I}\); otherwise, discard \(e\). Prove that this algorithm finds a maximum weight set in \(\mathcal{I}\).

(2) [10 points]
Problem 4.9 from the textbook.

(3) [10 points]
Problem 4.15 from the textbook. (I strongly recommend reading Section 4.1 and 4.2 to do this problem.)

(4) [10 points]
Problem 4.21 from the textbook.

(5) [0 points]
Chocolate Problem (1 chocolate bar): Problem 4.30 from the textbook.