Problem 1
(a) A problem is a set (of numbers, strings, ...).
(b) A program $P$ decides a problem $X$ if given an input $y$, $P$ returns 1 (or “Yes”) whenever $y \in X$, and 0 (or “No”) whenever $y \notin X$.
(c) A reduction from $X$ to $Y$ is a function $f$ such that for every input $x$: if $x \in X$, then $f(x) \in Y$, and if $x \notin X$, then $f(x) \notin Y$.
(d) When we write $X \leq_m Y$, we mean that there is a reduction $f$ from $X$ to $Y$, which is itself total and recursive, i.e., is computed by a program which always terminates.
(e) If we have proved $X \leq_m Y$, then we have proved that if $Y$ is decidable, so is $X$. Thus, either both are decidable, neither is decidable, or $X$ is decidable while $Y$ is not. We have ruled out that $Y$ is decidable and $X$ is not.

Problem 2
We will prove this by proving that HALT $\leq_m$ NONEMPTY. A reduction $f$ from HALT to NONEMPTY would need to take a program $P$ (input to HALT) and transform it to a program $f(P)$ (input to NONEMPTY), such that if $P(P) \downarrow$, then $f(P)$ terminates for at least one input $x$, and if $P(P) \uparrow$, then $f(P)$ does not terminate for any inputs $x$.

Here is what the reduction does: given a program $P$, it produces the following new program $f(P)$. $f(P)$ obtains an input $x$, but ignores it. It executes $P(x)$. If and when $P(x)$ terminates, it returns 0. First, we check that given the source code of $P$, we can clearly produce the source code of $f(P)$. If $P(P) \downarrow$, then $f(P)$ makes it through that part and terminates on each input. If $P(P) \uparrow$, then $f(P)$ will always get stuck in that part, regardless of the input, so it will never terminate. Thus, $f$ is a correct reduction, and NONEMPTY is undecidable.

Problem 3
Similar to Problem 2, we will reduce from HALT to VARIED. We will produce a reduction $f$, which takes a program $P$ (input to HALT) and transform it to a program $f(P)$ (input to VARIED), such that if $P(P) \downarrow$, then $f(P)$ outputs at least two different values, and if $P(P) \uparrow$, then $f(P)$ does not output at least two different values.

The reduction, given a program $P$, produces a program $f(P)$. That program gets as an input $x$. It first runs $P(P)$. If and when that terminates, it returns the value $x$ itself. Again, this can clearly be computed by a total program. Now, if $P(P) \downarrow$, then for each input $x$, $f(P)$ will output $x$. In particular, it will output infinitely many different values, so at least two. On the other hand, if $P(P) \uparrow$, then $f(P)$ will not terminate on any input. In particular, it cannot output two different values. Thus, $f$ is a correct reduction, and VARIED is undecidable.

Problem 4
We want to define a reduction from $X$ to $Y$. Given a program $P$, we want to produce a program $f(P)$, such that if $P(42) \downarrow$ and $P(y) \downarrow$ for all $y \neq 42$, then $f(P)$ terminates for infinitely many inputs, but if $P(42) \uparrow$ or $P(y) \uparrow$ for some $y \neq 42$, then $f(P)$ terminates for only finitely many inputs. This is a bit more tricky, as non-termination of $P$ should imply termination of $f(P)$.
The program $f(P)$, given input $x$, behaves as follows: first it runs $P(42)$. If and when $P(42)$ terminates, it runs each of $P(1), P(2), P(3), \ldots, P(x)$ for $x$ steps (except $P(42)$, which we don’t run). This will take $O(x^2)$ simulation steps ($x$ steps each for $x - 1$ programs). If any one of these runs of $P$ terminates in $x$ steps, then $f(P)$ deliberately enters an infinite loop. Otherwise, it returns some value, such as 0.

First, we should convince ourselves that $f$ can be computed. Notice that it just simulates $P$ for a limited number of steps, so all we need to do is augment $P$ by a counter or timer, and add a for loop. So $f$ is total and recursive.

Now, if $P \in X$, then $P(42) \downarrow$, and $P(y) \uparrow$ for all $y \neq 42$. So the first step, running $P(42)$, will terminate for all inputs $y$. For the second part, because $P(y) \uparrow$ for all $y$, none of the simulations will terminate within the allotted $x$ steps, for any $x$. So the second part will never enter the infinite loop. Thus, $f(P)$ will terminate on all inputs $x$, in particular on infinitely many.

On the other hand, if $P \notin X$, then either $P(42) \uparrow$, or $P(y) \downarrow$ for some $y \neq 42$. In the first case, running $P(42)$ will never terminate, so $f(P)$ will not terminate on any input. In particular, it will not terminate on infinitely many inputs. On the other hand, if $P(y) \downarrow$ for some $y \neq 42$, let $t$ be the number of steps that $P(y)$ takes to terminate for that particular $y$. Then, for each $x \geq \max(t, y)$, $f(P)$ will find that $P(y)$ terminates within $x$ steps, and will deliberately enter an infinite loop. Therefore, $f(P)$ will terminate at most on inputs $1, \ldots, \max(t, y)$, which is only finitely many. Thus, $f(P) \notin Y$.

This completes the proof that the reduction is correct, and $X \leq_m Y$. 

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