guessing facets

polytope structure and improved LP decoder

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outline

- LP decoding
- Results on the polytope structure
- Improving decoding: guessing facets

LP decoding

- ML decoding can be written as a linear program:
- For a code ***C*** define ***Poly(C)*** the convex hull of codewords.
- ML decoding: minimize negative log likelihood:

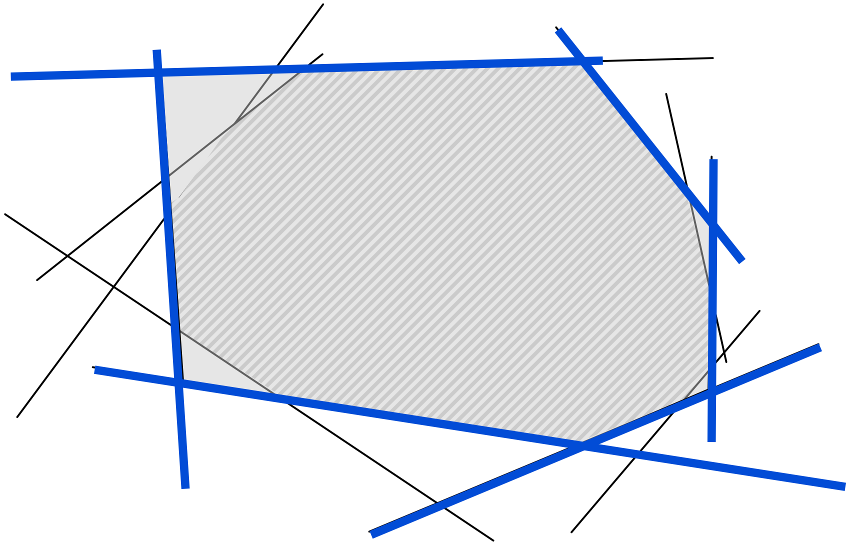
$$\gamma_i = \log\left(\frac{\Pr(r/u=0)}{\Pr(r/u=1)}\right)$$

- ML decoding can be written as

$$\min \gamma^T x$$
$$x \in Poly(C)$$

Relaxed polytope

- Unfortunately, ***Poly(C)*** cannot be described efficiently (ML decoding is NP-hard)
- However suggests a way to approximate: Relax the polytope:



How to relax

- Every check c_j in the code defines a local codeword polytope LCP(c_j):

$$P = \bigcap_{\forall j} LCP(c_j)$$

$$\sum_{i\in(N(j)\setminus S)} x_i + \sum_{i\in S}(1-x_i) \ge 1$$

Forbidden Set Inequalities

$$0 \le x_i \le 1$$

Box Inequalities

Feldman et al, Vontobel et al, Wainwright et al.

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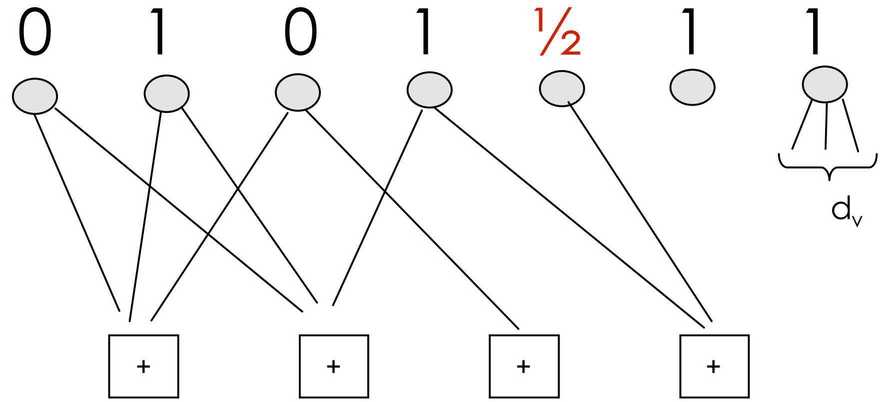
Polytope structure I
fractional support (?)

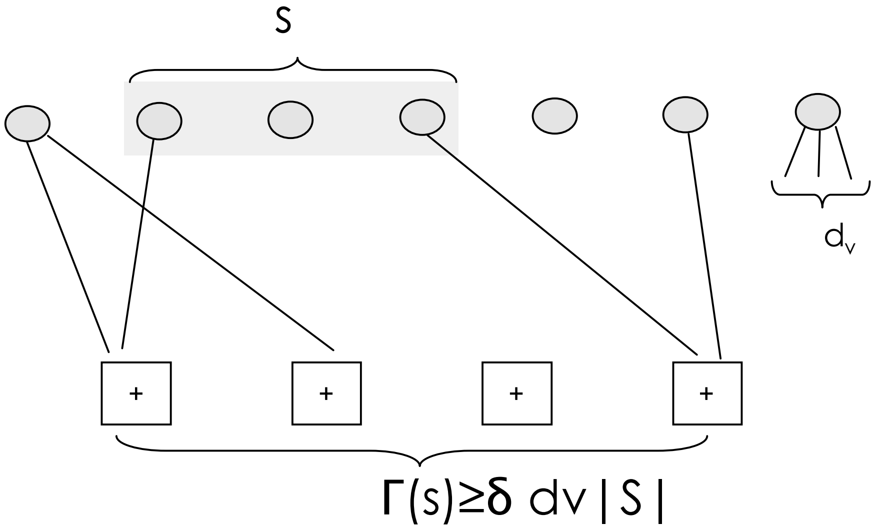


How small structures can be responsible for decoding failure ?

Can be arbitrarily small (for some codes)

(λ,δ)-expander graphs

The Tanner graph of a code is an **(λ,δ)-expander**:



If for every set *S*, such that $|S| \leq \lambda n$,

$$\Gamma(S) \geq \delta d_v |S|$$

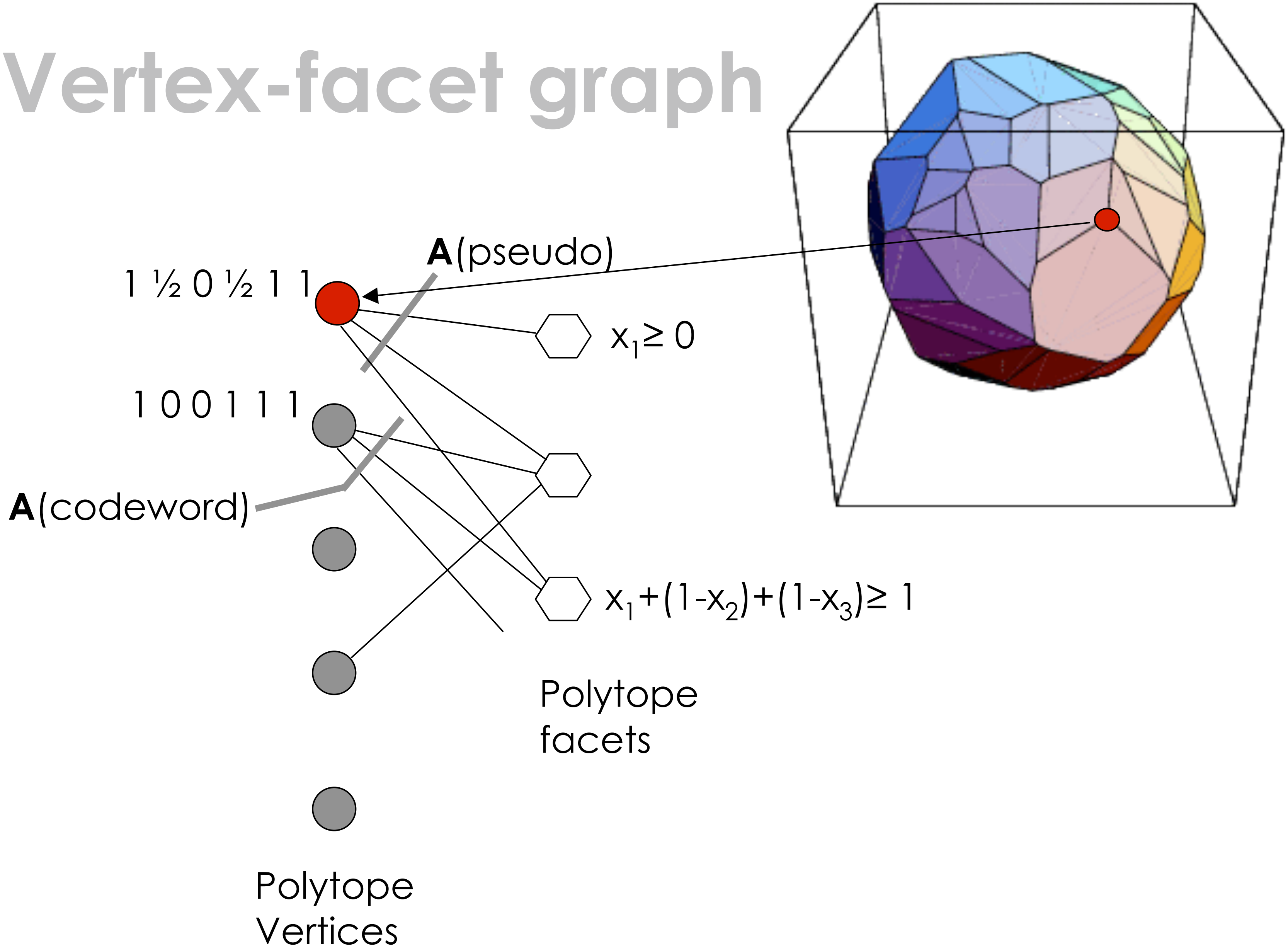
Polytope structure I (fractional support)

For codes where Tanner graph is an **(λ,δ)-expander**:

Result 1: every pseudocodeword has **fractional support that grows linearly (λn or larger in size)**.

~~0 0 1 0 1 1 0 ½ 0 1 0 1 0 1 0 1 1 0 1~~ 0 1

Vertex-facet graph



1 ½ 0 ½ 1 1

A(pseudo)

$x_1 \geq 0$

1 0 0 1 1 1

A(codeword)

$x_1+(1-x_2)+(1-x_3)\geq 1$

Polytope facets

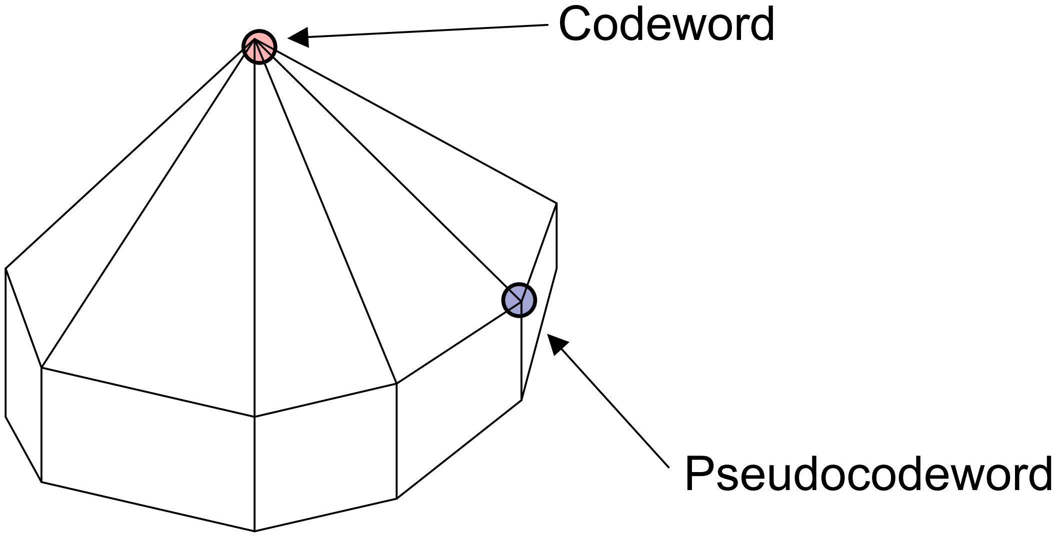
Polytope Vertices

Polytope structure

Theorem: For codes where Tanner graph is an **(λ,δ)-expander**, there exist constants **$c_1 > c_2$** , st:

$$\text{Á}(\text{codeword}) = c_1 n$$

$$\text{Á}(\text{pseudocodeword}) \leq c_2 n$$



Proof ingredients

- Polytope symmetry (Feldman et al)
- Fractional support

(expansion + a check cannot be adjacent a single fractional bit)

- Linear fractional support implies linear gap, from the box inequalities.

$$0 \le x_i \le 1 \qquad \sum_{i \in (N(j) \setminus S)} x_i + \sum_{i \in S} (1 - x_i) \ge 1$$

- Must also bound the number of active forbidden set inequalities. (techniques from Ziegler)

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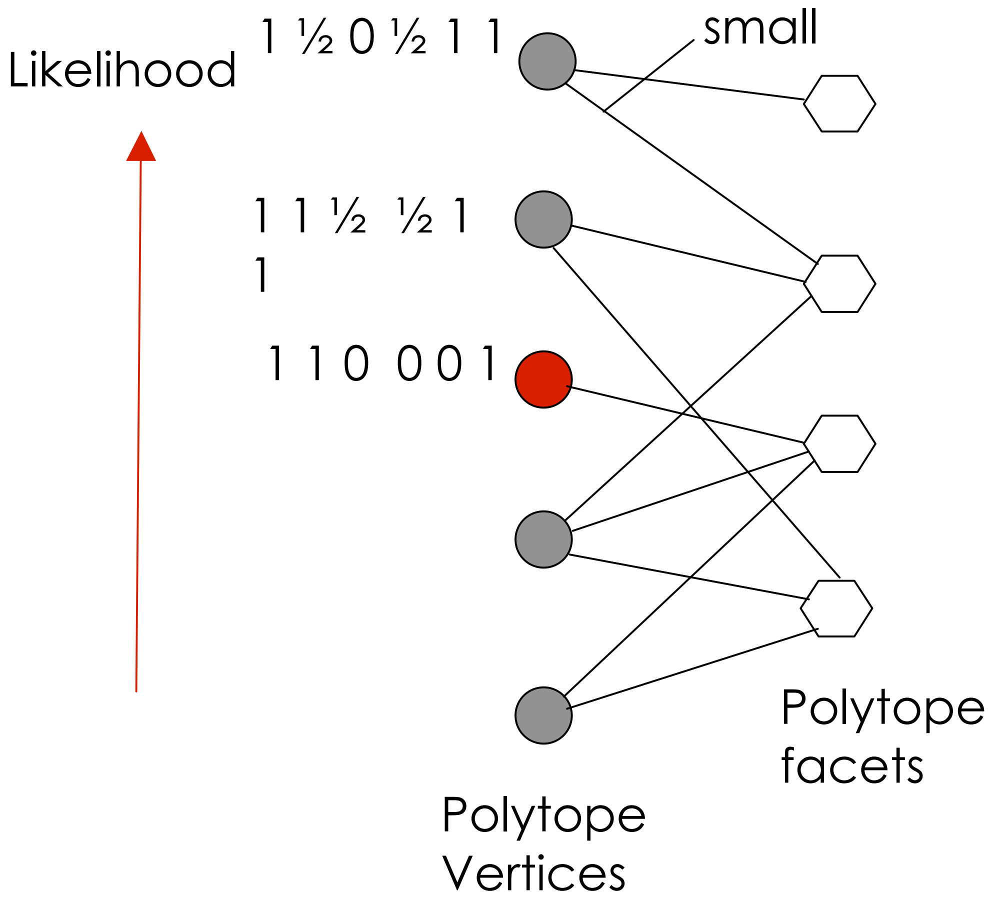
Facet guessing

- ML certificate: you know when you fail.
- Can one adaptively improve LP decoding by exploiting polytope structure ?
- Intersect polytope with facets ,

  to eliminate pseudocodewords with high likelihood.

Geometry of LP decoding failure

Can we go down this list?



Likelihood

1 ½ 0 ½ 1 1

small

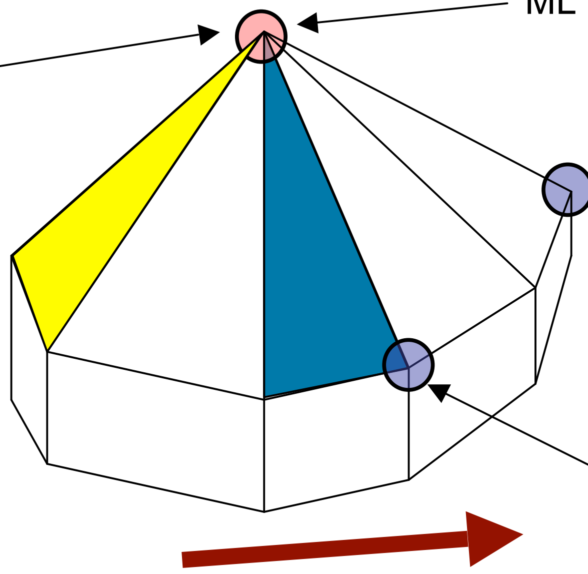
1 1 ½ ½ 1 1

1 1 0 0 0 1

Polytope Vertices

Polytope facets

Geometry of facet guessing



ML Codeword

When yellow facet is made active,

LP decoder outputs the ML codeword

LP decoder outputs this Pseudocodeword

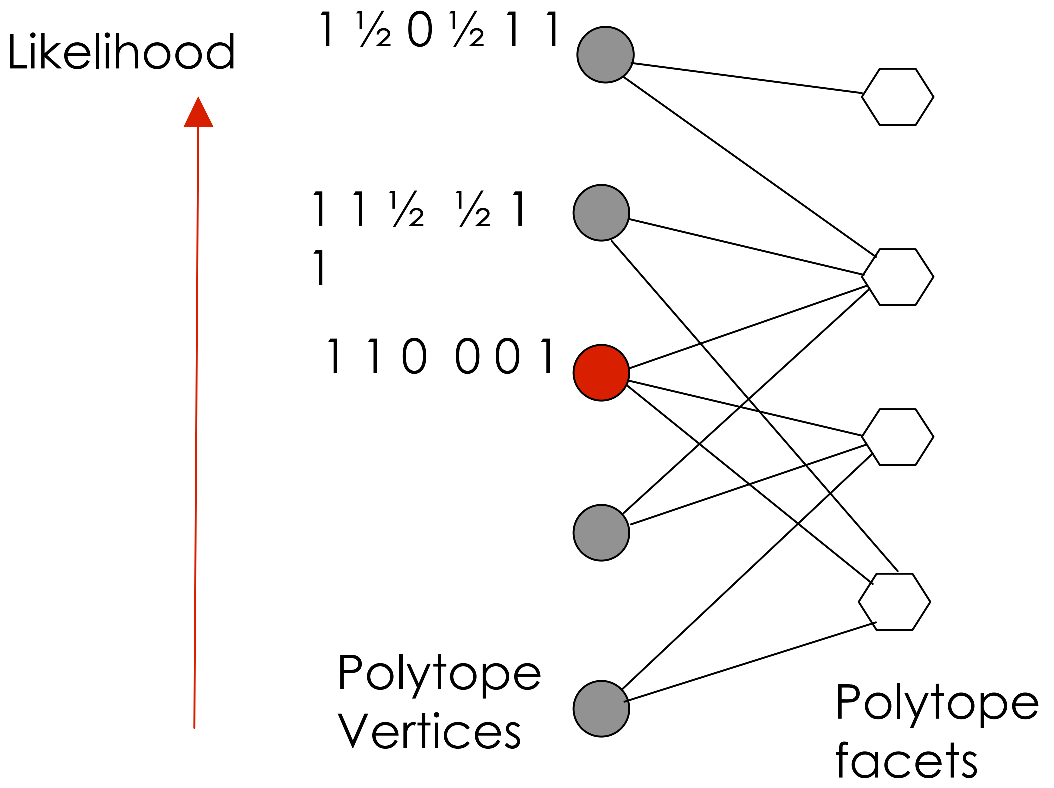
When blue facet is made active,

LP decoder outputs this Pseudocodeword

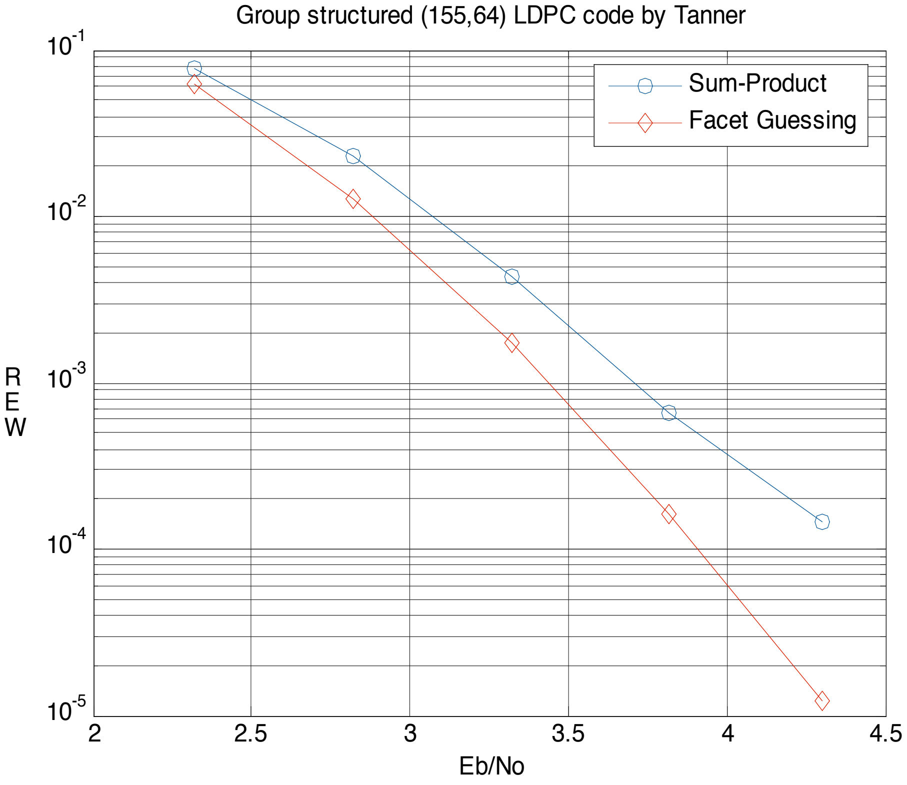
Channel direction

Provably improves LP decoding

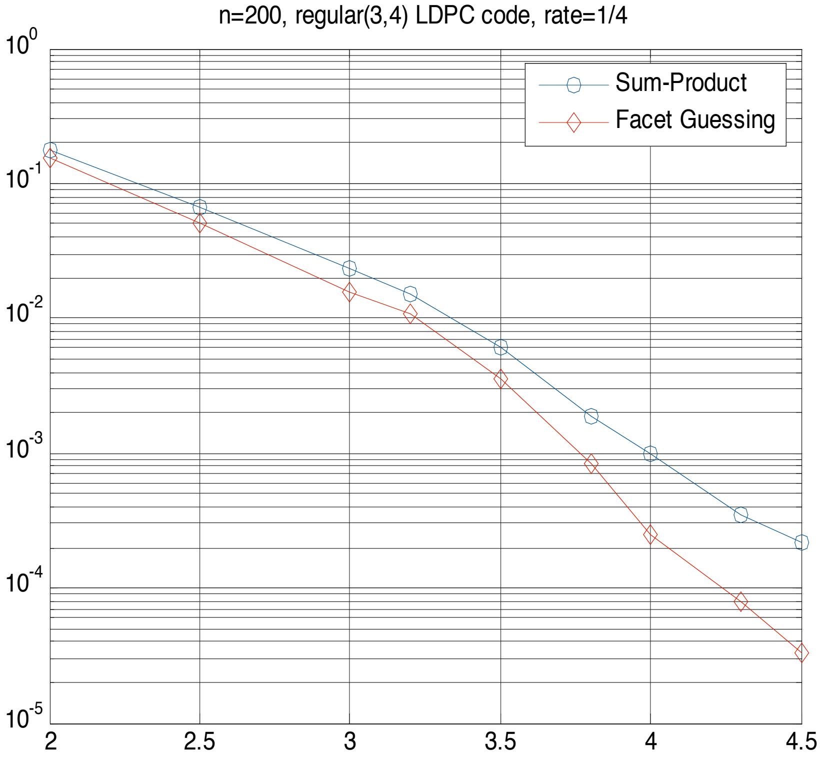
Theorem: if there are **c_1/ c_2** (or less) pseudocodewords above ML, **guessing a constant number of facets** succeeds with high probability



Experimental results.



Group structured (155,64) LDPC code by Tanner



n=200, regular(3,4) LDPC code, rate=1/4

Conclusions

- How many pseudocodewords can we eliminate?
- In this paper we establish:

constant number with same complexity as LP (whp)

- Expansion implies nice polytope structure
- Facet guessing decoder outperforms LP decoding and sum-product.