

A Multinomial Approximation for American Option Prices in Lévy Process Models¹

Ross A. Maller¹ David H. Solomon² Alex Szimayer³

¹The Australian National University (e-mail: Ross.Maller@anu.edu.au)

²University of Chicago (e-mail: dsolomon@chicagogsb.edu)

³The University of Western Australia (e-mail: aszimaye@ecel.uwa.edu.au)

Abstract. This paper gives a tree based method for pricing American options in models where the stock price follows a general exponential Lévy process. A multinomial model for approximating the stock price process, which can be viewed as generalising the binomial model of Cox, Ross and Rubinstein (1979) for geometric Brownian motion, is developed. Under mild conditions, it is proved that the stock price process and the prices of American-type options on the stock, calculated from the multinomial model, converge to the corresponding prices under the continuous time Lévy process model. Explicit illustrations are given for the variance gamma model and the normal inverse Gaussian process when the option is an American put, but the procedure is applicable to a much wider class of derivatives including some path-dependent options. Our approach overcomes some practical difficulties that have previously been encountered when the Lévy process has infinite activity.

Key words: American Options, Lévy Process, Multinomial Approximation

¹The authors take pleasure in thanking participants of the Oberwolfach “Statistics in Finance” Workshop of January 2004 and the Bachelier Finance conference in Chicago, July 2004, as well as an anonymous referee, for helpful comments and references. We are grateful also to Professors Peter Carr and Liuren Wu for providing their Matlab code for computing closed form European option prices for variance gamma and normal inverse Gaussian models. This research was partially supported by ARC grant DP0210572 and University of Western Australia Research Grant RA/1/485.

1 Introduction

This paper examines the valuation of American options in models where the stock price follows an exponential Lévy process. The general Lévy process setup includes many of the popular option pricing models as special cases, including geometric Brownian motion (the basis of the Black-Scholes model), the jump diffusion model, and infinite activity models such as the variance gamma model.

In these kinds of models, derivative pricing has previously been approached in three main ways. Since the seminal paper of Black and Scholes (1973) using a partial differential equation (PDE) approach, partial-integro differential equations (PIDEs) have been widely used; in the Lévy context, see Albanese and Kuznetsov (2003), Matache et al. (2005), Cont and Voltchkova (2005) and Hirta and Madan (2004) for examples of their use. An alternative approach is to compute directly the risk-neutral expectation of the option pay-off in some way. For example, Fourier inversion (see Madan et al. (1998) for the variance gamma) or Monte Carlo methods (see Fu et al. (2001) and Glasserman (2003) for overviews) can be used. A third methodology employs tree based methods (or related lattice methods). These date back to Cox et al. (1979), who proposed a binomial model for approximating the continuous time Black-Scholes model and the respective option prices. Amin (1993) and Mulinacci (1996) extended this approach to the finite activity setting of the jump diffusion models. (Finite activity refers to the underlying Lévy process having only finitely many discontinuities per unit time, almost surely.) More generally, for the infinite activity Lévy case, Këllezi and Webber (2004) recently advocated a lattice method based on transition probabilities, but due to reported computational difficulties restricted themselves to Bermudan options which can only be exercised at a designated number of pre-specified times.

So far, however, no general, workable, model in the class of tree or lattice based methods has been proposed to cater for the valuation of American options on an arbitrary Lévy process. Here we put forward such a model, using a recombining multinomial tree which can be viewed as a generalisation of the Cox et al. (1979) binomial model. Under mild conditions, it is proved that American option prices calculated from the multinomial model converge to the corresponding prices calculated from the continuous time Lévy process model. As special cases, explicit illustrations are given for an American put priced under variance gamma and normal inverse Gaussian models, but the procedure is applicable to a much wider class of derivatives including path-dependent options such as lookbacks and barrier options; see Section 6 for some further discussion

of this. The flexible framework we develop allows for a particular Lévy process to be approximated in various ways in discrete time, so as potentially to improve convergence to the continuous time setup, and our approach overcomes the above-mentioned difficulties with infinite activity Lévy processes.

2 Overview, and Stochastic Setup

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space on which a Lévy process $L = (L_t)_{t \geq 0}$ with càdlàg paths is defined, and let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the right-continuous filtration generated by L . Assume that \mathcal{F}_0 contains all \mathbb{P} -null sets and that $\mathcal{F}_\infty = \mathcal{F}$. The process is characterised by its Lévy triplet (γ, σ, Π) , where γ is a shift constant, $\sigma \geq 0$ is the diffusion coefficient and $\Pi(\cdot)$ is the Lévy measure. See Bertoin (1996) and Sato (1999) for background and properties of Lévy processes. In particular, we have $\int (x^2 \wedge 1) \Pi(dx) < \infty$, implying $\lim_{x \downarrow 0} x^2 (\Pi((x, \infty)) + \Pi((-\infty, -x))) = 0$, an important factor in our analysis.

In Section 3, we show how the Lévy process can be approximated by interpolating a sequence of discrete time, finite state space, processes, which for computational convenience, and practical necessity, are required to constitute a recombining lattice. The weak convergence of the discrete time processes to the given continuous time process is proved using classical results concerning convergence of row sums of triangular arrays to infinitely divisible distributions, and associated functional limit theorems. The scheme is set up so that no boundedness assumptions whatsoever – moment assumptions, or bounded activity – are required for this part of the paper.

In Section 4, the stock price process $S = (S_t)_{t \geq 0}$ is introduced and assumed to follow an exponential Lévy process:

$$S_t = S_0 e^{L_t}, \quad t \geq 0, \quad (2.1)$$

where $S_0 \in \mathbb{R}^+ = (0, \infty)$ is an initial stock price, taken as a random variable (rv) independent of $(L_t)_{t \geq 0}$. Also we will assume that a discount bond is traded with maturity $T > 0$ and unit face value. The instantaneous interest rate $r > 0$ is supposed to be constant and identical for all maturities. To calculate expectations of the stock price process we will need to assume in Section 4 that the exponential moment of L exists. This section contains our main result: under mild Lipschitz continuity and boundedness assumptions on the option payoff function, and a natural integrability assumption on the underlying Lévy process, the discretely calculated American option prices converge to their continuous time versions as the number of multinomial steps tends to infinity, in the

sense of convergence of finite dimensional distributions for Lebesgue-almost all times in the interval $[0, T]$. This will be deduced from general criteria provided by Mulinacci and Pratelli (1998) for convergence in the topology used by Meyer and Zheng (1984).

In Section 5 we give two examples, pricing an American put on stock price processes which are (the exponential of) a variance gamma process and a normal inverse Gaussian process. For these, corresponding European put prices were obtained by Madan et al. (1998), against which we benchmark our computations. Section 6 summarises the results and discusses some possible extensions and other issues such as convergence rates. Proofs for Section 3 are in Section 7 and those for Section 4 are in Section 8.

3 Approximation Scheme and its Convergence

The tree-based scheme we introduce in this section sets out an approximating process, $L_t(n)$, of L_t , as an interpolation of a discrete time process with a finite number of states. The tree is designed to be recombining so as to facilitate a straightforward computation via a generalisation of the familiar and intuitively appealing backward induction scheme. For each $n = 1, 2, \dots$, the number of time steps per unit time is denoted by $N(n)$, and accordingly each time period is of duration $\Delta t(n) = 1/N(n)$. The increments of $L_t(n)$ take values on a ladder, of step size $\Delta(n)$, say; this gives the recombining property. The range of the increments is determined by the number of possible steps up: $m_+(n)$, and down: $m_-(n)$.

We choose sequences $\Delta(n) \downarrow 0$ and $N(n) \uparrow \infty$, as $n \rightarrow \infty$, such that

$$\liminf_{n \rightarrow \infty} \sqrt{N(n)} \Delta(n) > 0. \quad (3.1)$$

The integers $m_{\pm}(n) \geq 1$, $n = 1, 2, \dots$, are required to satisfy

$$\lim_{n \rightarrow \infty} \Delta(n) m_{\pm}(n) = \infty \quad (\text{thus, } \lim_{n \rightarrow \infty} m_{\pm}(n) = \infty). \quad (3.2)$$

Let $\mathcal{M}(n)$ denote the integers $\{-m_-(n), \dots, -2, -1, 1, 2, \dots, m_+(n)\}$ (note that 0 is absent), and let

$$I_k(n) = ((k - 1/2) \Delta(n), (k + 1/2) \Delta(n)], \quad k \in \mathcal{M}(n),$$

be non-overlapping intervals whose union is

$$\begin{aligned} \mathcal{I}(n) &= \bigcup_{k \in \mathcal{M}(n)} I_k(n) \\ &= (-(m_-(n) + 1/2) \Delta(n), (m_+(n) + 1/2) \Delta(n)) \setminus (-\Delta(n)/2, \Delta(n)/2]. \end{aligned} \quad (3.3)$$

We wish to define a random variable $X(n)$ on $\mathcal{I}(n)$, taking values $x_k(n)$, say, with $x_k(n) \in I_k(n)$ and $k \in \mathcal{M}(n)$. In order for the resulting tree to be recombining, the $x_k(n)$ should live on an equi-spaced grid. The simplest way to accomplish this, and achieve the required convergence, is to take $x_k(n) = k \Delta(n)$, and we will assume this throughout. (Other choices of $I_k(n)$ and of $x_k(n) \in I_k(n)$ may be made, and there may be some computational advantages in doing so – we discuss this in Section 6.) Also let $x_0(n) = 0$. Recall that $\Pi(\cdot)$ is the Lévy measure of L . Now we can define $X(n)$; it is to take values $x_k(n)$ with probabilities

$$\mathbb{P}(X(n) = x_k(n)) = \frac{1}{N(n)} \Pi(I_k(n)), \quad \text{for } k \in \mathcal{M}(n), \quad n = 1, 2 \dots, \quad (3.4)$$

and the value 0 with probability

$$\mathbb{P}(X(n) = 0) = 1 - \sum_{k \in \mathcal{M}(n)} \mathbb{P}(X(n) = x_k(n)) := 1 - A(n). \quad (3.5)$$

We must check that this gives a well defined probability distribution. Now

$$\begin{aligned} A(n) &= \frac{1}{N(n)} \Pi(\mathcal{I}(n)) \\ &= \frac{1}{N(n)} \Pi((- (m_-(n) + 1/2) \Delta(n), -\Delta(n)/2]) \\ &\quad + \frac{1}{N(n)} \Pi([\Delta(n)/2, (m_+(n) + 1/2) \Delta(n)]) \\ &\leq \frac{1}{N(n)} (\bar{\Pi}_-(\Delta(n)/2) + \bar{\Pi}_+(\Delta(n)/2)) \\ &= O(\Delta^2(n)) (\bar{\Pi}_-(\Delta(n)/2) + \bar{\Pi}_+(\Delta(n)/2)) \quad (\text{by (3.1)}), \end{aligned}$$

where

$$\bar{\Pi}_+(x) = \Pi((x, \infty)) \quad \text{and} \quad \bar{\Pi}_-(x) = \Pi((-\infty, -x)), \quad \text{for } x > 0,$$

denote the tails of $\Pi(\cdot)$. As a Lévy measure, $\lim_{x \rightarrow 0^+} x^2 \bar{\Pi}_\pm(x) = 0$, so we have $\lim_{n \rightarrow \infty} A(n) = 0$. We can thus assume n is chosen large enough, $n \geq n_0$, say, so that $A(n) < 1$; then (3.4) and (3.5) define a proper probability distribution.

Next, for a fixed $T > 0$ and each $n \geq n_0$, we take $X_j(n)$, $1 \leq j \leq N(n)T$, to be independent and identically distributed (i.i.d.) copies of $X(n)$. In the next theorem, a discrete approximating process $L_t(n)$ is defined in terms of the $X_j(n)$ and shown to converge to L_t . Let ‘ \xrightarrow{D} ’ denote convergence in distribution of one-dimensional distributions and ‘ \Rightarrow ’ denote weak convergence of càdlàg stochastic processes on a finite interval. Follow the convention that $\sum_1^0 = 0$. At this stage, we make no assumptions at all on L_t , other than to treat separately cases where a diffusion is or is not present.

Theorem 3.1. *Suppose the Lévy process L_t has triplet $(\gamma, 0, \Pi)$. Assume (3.1) and (3.2), define $X(n)$ by (3.4)–(3.5), and let $(X_j(n))_{1 \leq j \leq N(n)T}$ be i.i.d. copies of $X(n)$, $n \geq n_0$. Then a non-stochastic sequence $(a(n))_{n \geq 1}$ exists such that, as $n \rightarrow \infty$,*

$$L_t(n) := \sum_{j=1}^{\lfloor N(n)t \rfloor} (X_j(n) - a(n)) \Rightarrow L_t, \quad \text{in } D[0, T]. \quad (3.6)$$

The sequence $(a(n))_{n \geq 1}$ can be chosen so that

$$a(n) = -\frac{\gamma}{N(n)} + \mathbb{E}(X(n) \mathbf{1}_{\{|X(n)| \leq 1\}}) + b(n), \quad (3.7)$$

where $(b(n))_{n \geq 1}$ is any non-stochastic sequence which is $o(1/N(n))$ as $n \rightarrow \infty$.

Remark 3.2 (Centering sequence). *If we suppose in addition that the Lévy process L has finite first order moment, or, equivalently (Sato, 1999, p. 159), $\int_{|u|>1} |u| \Pi(du) < \infty$, then we can define the centering sequence $a(n)$ alternatively by*

$$a(n) = -\frac{\gamma}{N(n)} + \mu(n) - \frac{1}{N(n)} \int_{|u|>1} u \Pi(du) + b(n), \quad (3.8)$$

where $\mu(n) = \mathbb{E}(X(n))$ and $(b(n))_{n \geq 1}$ is any $o(1/N(n))$ non-stochastic sequence. (Note that the multinomial $X(n)$ always has a finite expectation.) Since $\mathbb{E}L_1 = \gamma + \int_{|u|>1} u \Pi(du)$, we then have

$$a(n) = -\frac{1}{N(n)} \mathbb{E}L_1 + \mu(n) + b(n), \quad (3.9)$$

and we can rewrite (3.6) equivalently as the convergence of mean-centered processes:

$$M_t(n) := \sum_{j=1}^{\lfloor N(n)t \rfloor} (X_j(n) - \mu(n)) \Rightarrow L_t - t \mathbb{E}L_1, \quad \text{in } D[0, T]. \quad (3.10)$$

Remark 3.3 (Truncation Point). *The choice of truncation point in (3.7) is arbitrary, and “1” can be replaced by any other positive real number by readjusting the constants. Moreover, since Π has only a countable number of discontinuities, we can choose this value to be a continuity point of Π . Thus we may assume without loss of generality that 1 is also a continuity point of Π .*

Remark 3.4 (Adding in a Diffusion). *By grafting on to ours the usual binomial scheme of Cox et al. (1979), we can add a diffusion into the Lévy process. Given a constant $\sigma > 0$, let the rv $Y(n)$ take values $\pm\sigma/\sqrt{N(n)}$ with*

probabilities $1/2$ each. Let $(Y_j(n))_{1 \leq j \leq N(n)T}$ be independent copies of $Y(n)$ which are also independent of the $X_j(n)$. Then

$$\sum_{j=1}^{\lfloor N(n)t \rfloor} Y_j(n) \Rightarrow \sigma B_t, \quad \text{in } D[0, T], \quad (3.11)$$

where B is a standard Brownian motion, and if we let $Z_j(n) = X_j(n) - a(n) + Y_j(n)$, for $1 \leq j \leq N(n)T$, with $a(n)$ as in (3.7), then Theorem 3.1 and (3.11) give

$$\sum_{j=1}^{\lfloor N(n)t \rfloor} Z_j(n) \Rightarrow \tilde{L}_t, \quad \text{in } D[0, T], \quad (3.12)$$

where \tilde{L} is a Lévy process with triplet (γ, σ, Π) .

4 Convergence of American Option Prices

Having constructed $L_t(n)$, our discrete time approximation to the continuous time exponential Lévy process (2.1) is then specified by

$$S_t(n) = S_0(n) e^{L_t(n)}, \quad \text{for } 0 \leq t \leq T, \quad n \in \mathbb{N}_0 := \{n \in \mathbb{N} : n \geq n_0\}, \quad (4.1)$$

where $L_t(n)$ is defined in (3.6), and $S_0(n) > 0$, the starting value of the discrete stock price process, is assumed independent of $(L_t(n))_{0 \leq t \leq T}$, for each $n \in \mathbb{N}_0$, and to satisfy $S_0(n) \xrightarrow{D} S_0$, as $n \rightarrow \infty$.

To calculate risk-neutral expectations we must assume also that the exponential moment of L , $\mathbb{E}e^{L_t}$, exists. Thus by Theorem 25.17 of Sato (1999), $\Psi(1)$ is assumed to be finite, where

$$\Psi(\theta) = \log \mathbb{E}e^{\theta L_1} = \gamma \theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{\theta x} - 1 - \theta x \mathbf{1}_{\{|x| \leq 1\}}) \Pi(dx), \quad \theta > 0. \quad (4.2)$$

Assume also that $\mathbb{E}S_0 < \infty$. Then, recalling (2.1), we can write

$$\mathbb{E}(S_t) = \mathbb{E}(S_0) \mathbb{E}(e^{L_t}) = \mathbb{E}(S_0) e^{t\Psi(1)}, \quad \text{finite.} \quad (4.3)$$

For no-arbitrage pricing, the discounted stock price process, $(e^{-rt}S_t)_{t \geq 0}$, in an equilibrium, with either a complete or an incomplete market, must constitute a martingale; see Harrison and Pliska (1981), and Duffie (1988). Based on Girsanov's theorem (Jacod and Shiryaev, 1987, p. 159), we can find a measure transformation giving a martingale probability measure, \mathbb{Q} , say, equivalent to \mathbb{P} , for which the discounted stock price process is a martingale. In a complete

market, the equivalent martingale measure (EMM) \mathbb{Q} is uniquely defined, but in our general Lévy setup it is not.

In general the process L will not be a Lévy process under an equivalent measure \mathbb{Q} . For our purposes we assume that L is also a Lévy process when the measure has been changed from the “physical” measure \mathbb{P} to the EMM \mathbb{Q} . In other words, we assume the setup is such that the discounted stock price is already a martingale under \mathbb{Q} . Consequently, throughout the paper, (γ, σ, Π) is henceforth understood as the Lévy triplet of L under the measure \mathbb{Q} and, further, \mathbb{E} is expectation with respect to \mathbb{Q} . Any EMM \mathbb{Q} is a *risk-neutral measure* because, under it, the growth rate $\Psi(1)$ of the stock price equals the risk-free rate $r > 0$.

The value of an option is the expected discounted payoff under the EMM, see Harrison and Pliska (1981), and Duffie (1988). An American option can be exercised at any time in the interval $[0, T]$. We denote by $\pi_t(n)$ the price process of the not-exercised option. To specify this, let $(\mathcal{F}_t^n)_{0 \leq t \leq T}$ be the filtration generated by $\{S_0(n), X_1(n), \dots, X_{\lfloor N(n)t \rfloor}(n)\}$, or, equivalently, $\{S_0(n), \dots, S_t(n)\}$. Modify the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ defined in Section 2 slightly by adjoining the σ -algebra $\sigma(S_0)$ to it, thus $(\mathcal{F}_t)_{0 \leq t \leq T}$ now denotes the filtration generated by $(S_t)_{0 \leq t \leq T}$. Let $\mathcal{S}_{t,T}(n)$ be the set of stopping times (with respect to the filtration $(\mathcal{F}_t^n)_{0 \leq t \leq T}$) taking values in $[t, T] \cap \{\frac{1}{N(n)}\mathbb{N}\} = \{\frac{k}{N(n)} : k \in \mathbb{N}, N(n)t \leq k \leq N(n)T\}$, and let $\mathcal{S}_{t,T}$ be the set of stopping times (with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$) taking values in $[t, T]$. Then, for an American option with payoff function $g(\cdot) : [0, \infty) \mapsto [0, \infty)$, assumed measurable, the discounted price process of the not-exercised option is given by the Snell envelope² of the discounted payoff process; thus

$$\pi_t(n) = \text{ess sup}_{\tau \in \mathcal{S}_{t,T}(n)} \mathbb{E} \left(e^{-r(\tau-t)} g(S_\tau(n)) \mid \mathcal{F}_t^n \right), \quad \text{for } 0 \leq t \leq T. \quad (4.4)$$

Our aim is to prove the convergence of the prices $\pi_t(n)$ obtained under our discrete scheme to their continuous time counterpart

$$\pi_t = \text{ess sup}_{\tau \in \mathcal{S}_{t,T}} \mathbb{E} \left(e^{-r(\tau-t)} g(S_\tau) \mid \mathcal{F}_t \right), \quad \text{for } 0 \leq t \leq T. \quad (4.5)$$

For this we use the following technical lemmas which correspond to two conditions of Mulinacci and Pratelli (1998). They involve kinds of continuity and boundedness conditions on the payoff function $g(\cdot)$, and uniform integrability conditions on the discrete stock price process.

²See, e.g., Bingham and Kiesel (2004, p. 89), and their references, for the Snell envelope.

Lemma 4.1. Assume $g(\cdot) \in Lip(\beta)$ for some $\beta > 0$, that is,

$$g(x) - g(y) = O(|x - y|^\beta), \text{ as } |x - y| \rightarrow 0, \quad (4.6)$$

and also that the following global boundedness condition holds: for constants $C_1 > 0$, $C_2 > 0$, and some $m \geq 1$,

$$|g(x)| \leq C_1 + C_2 x^m, \quad \text{for } x \geq 0. \quad (4.7)$$

Suppose also that, for this value of m ,

$$\int_{\mathbb{R}} u^2 e^{m|u|} \Pi(du) < \infty, \quad (4.8)$$

and that $\{S_0^m(n)(\log S_0(n))^2\}_{n \geq n_0}$ is uniformly integrable; thus, for some $C_0 < \infty$, we have that $\mathbb{E}S_0^m(n)(\log S_0(n))^2 \leq C_0$, $n = 1, 2, \dots$, and

$$\lim_{a \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E}(S_0^m(n)(\log S_0(n))^2 \mathbf{1}_{\{S_0^m(n)(\log S_0(n))^2 > a\}}) = 0. \quad (4.9)$$

Then, for any $\tau \in \mathcal{S}_{0,T}(n)$,

$$\lim_{t \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{0 \leq s \leq t} \mathbb{E} |g(S_{\tau+s}(n)) - g(S_\tau(n))| = 0. \quad (4.10)$$

(We follow the convention that the interval of definition of τ is extended so that we can write $\tau + s$ rather than $(\tau + s) \wedge T$, etc., throughout.)

Lemma 4.2. Suppose that (4.8) and (4.9) hold for a given value of $m \geq 1$. Then for any $t \in (0, T]$ and $\tau \in \mathcal{S}_{t,T}(n)$, $\sup_{n \geq n_0} \mathbb{E}S_\tau^m(n)$ is finite, and

$$\lim_{a \rightarrow \infty} \sup_{\tau \in \mathcal{S}_{t,T}(n)} \sup_{n \geq n_0} \mathbb{E} (S_\tau^m(n) \mathbf{1}_{\{S_\tau^m(n) > a\}}) = 0. \quad (4.11)$$

The following theorem, our main result, establishes the convergence of the discretely obtained American option price process (the Snell envelope of the discounted discrete payoff process), in the Meyer-Zheng topology, to the continuous time value. Convergence in the Meyer-Zheng topology (see Meyer and Zheng 1984) is denoted by \xrightarrow{MZ} . Meyer-Zheng convergence is weaker than convergence in distribution but it implies convergence of finite dimensional distributions on a set of full Lebesgue measure, see Meyer and Zheng (1984), Theorem 5.

Theorem 4.3. Assume (3.1) – (3.5), (4.1) and (4.3), suppose that g satisfies conditions (4.6) and (4.7), and that (4.8) and (4.9) hold for the value of m in (4.7). Then $(S_t(n), g(S_t(n))) \Rightarrow (S_t, g(S_t))$ in $D[0, T]$ as $n \rightarrow \infty$. Further, $(g(S_t(n)), \pi_t(n)) \xrightarrow{MZ} (g(S_t), \pi_t)$, and, for t in a set of full Lebesgue measure in $[0, T]$, the finite dimensional distributions of $(g(S_t(n)), \pi_t(n))$ converge to those of $(g(S_t), \pi_t)$, as $n \rightarrow \infty$.

Corollary 4.4. *Under the conditions of Theorem 4.3, the American option prices obtained under the discrete time approximation scheme converge in distribution to their continuous time counterparts, i.e. $\pi_t(n) \xrightarrow{D} \pi_t$, as $n \rightarrow \infty$, for almost all $t \in [0, T]$.*

Remark 4.5. *A drawback of the Meyer-Zheng topology is that we can claim convergence of the finite dimensional distributions of $\pi_t(n)$ to π_t only for t in a subset of full Lebesgue measure in $[0, T]$. Present methods do not yield convergence for all t in $[0, T]$, though it plausibly holds under our conditions.*

We remark that condition (4.9) is essentially harmless since in most cases we can take $S_0(n) = S_0$, a known constant.

5 Examples

The approximation scheme is illustrated in this section by computing American put option prices for the variance gamma (VG) and normal inverse Gaussian (NIG) models. The parameters we use for the VG model are the same as those estimated from S&P 500 index options data by Madan et al. (1998) for that model. The parameters we use for the NIG were obtained by matching its first four moments (centered, except for the first) to those of the variance gamma.³

The put prices were computed for various exercise prices, and times to maturity $T = 0.25$ and $T = 1$. The defining parameters for the tree were chosen as $N = 800$, $\Delta t = 0.00125$, $\Delta = 0.004785$, and $m_{\pm} = 200$ for time to maturity $T = 0.25$, and as $N = 200$, $\Delta t = 0.0050$, $\Delta = 0.019142$, and $m_{\pm} = 200$ for time to maturity $T = 1$. (Thus there were $n = 200$ time steps in each tree.) In each case we set $\Delta = \Delta(n) = \sqrt{v\Delta t(n)}$, where v is the variance rate of the driving Lévy process.

The accuracy of the approximation for these choices of tree parameters was assessed by comparing the European option prices obtained under the multinomial scheme to those obtained using the closed form solution based on Fourier transform methods as in Madan et al. (1998). At most a discrepancy of 2.5% was achieved in this comparison with these parameter choices. The approximation scheme was set up such that the measure \mathbb{Q}_n of the approximation was an EMM for the stock price process $S_t(n)$ in discrete time, by choosing the appropriate value for the deterministic $b(n)$ in Theorem 3.1 (see (8.8) below).

³Këllezi and Webber (2004) took a similar approach. We note that matching moments in general is not an optimal way to calibrate model parameters; in a practical implementation, calibration should be carefully done, e.g., by solving the corresponding inverse problem. For our examples, however, the parameters serve only illustrative purposes.

Table 5.1 shows the European (E) and American (A) put prices obtained under a VG model. The relative error in the European option prices obtained under the multinomial scheme (E-MN) as compared to those obtained from the closed form solution (E-FFT), for both $T = 0.25$ and $T = 1$, is less than 1% at the lowest exercise price, $K = \$90$, falling below 0.2% for higher exercise prices. This suggests that 200 steps is adequate for reasonable accuracy. The American multinomial prices obtained by our method are denoted by A-MN. We see that the early exercise premium increases with the strike price from around \$0.03 to \$2.44 for $T = 0.25$, and from \$0.26 to \$5.03 for $T = 1$ respectively. For the highest exercise prices, the American option value equals the intrinsic value of the option, implying that the American option is exercised immediately.

Table 5.1: American and European Put Prices under a Variance Gamma Model
The table shows American (A) and European (E) put option prices (in \$) under a VG model for various exercise prices. Prices obtained under the multinomial model are indicated by MN. The European prices were obtained using Fourier transforms in Madan et al. (1998). The initial stock price is \$100, the risk free rate is $r = 0.10$, and the times to maturity T of the options are 0.25 years and 1 year. The Lévy density can be expressed as $\exp(\alpha x/\beta^2) \exp(-(|x|/\beta)\sqrt{2/\nu + \alpha^2/\beta^2}/(\nu|x|))$, where $\nu = 0.20$, $\alpha = -0.14$, and $\beta = 0.12$. (These parameters are chosen so that $\Psi(1) = r$, cf. (4.2) and (4.3); γ was set equal to 0 for this purpose. The parameters are on a per annum basis. The truncation at 1 in (4.2) can be ignored for this model.)

Exercise Price	$T = 0.25$			$T = 1$		
	E-FFT	E-MN	A-MN	E-FFT	E-MN	A-MN
90	0.2304	0.2298	0.2651	0.5347	0.5319	0.7924
95	0.6218	0.6212	0.7270	1.0300	1.0274	1.5925
100	1.5708	1.5720	1.8836	1.8538	1.8532	3.0151
105	3.6925	3.6974	5	3.1277	3.1321	5.4374
110	7.5572	7.5594	10	4.9617	4.9697	10

Table 5.2 shows the NIG results. The relative pricing error, comparing the European prices obtained under the multinomial scheme to the closed form solutions, is bounded by 2.5%, occurring for $T = 1$ and exercise price $K = \$90$, and falling below 1% when the exercise price exceeds \$100. The VG model produced significantly smaller pricing errors in comparable situations, highlighting

the fact that the accuracy of the multinomial approximation may depend on the specific Lévy process driving the model. (Of course greater accuracy could be obtained with a smaller step size.) Nevertheless, the put option prices and the early exercise premia of both models, VG and NIG, are very similar. It is worth noting that, in a similar setting, Këllezi and Webber (2004), investigating call option prices under the VG and NIG models, experienced problems in increasing the number of steps higher than 40. Thus for example they could not identify clearly those exercise prices which led to immediate exercise.

Table 5.2: American and European Put Prices under a Normal Inverse Gaussian Model
The table shows American (A) and European (E) put option prices (in \$) under an NIG model for various exercise prices. Prices obtained under the multinomial model are indicated by MN. The European prices were obtained using Fourier transforms in Madan et al. (1998). The initial stock price is \$100, the risk free rate is $r = 0.10$, and the times to maturity T of the options are 0.25 years and 1 year. The Lévy density can be expressed as $\delta\pi|x|e^{\beta|x|}K_1(\alpha|x|)/\alpha$, with K_1 as the modified Bessel function of the second kind with index 1. We set $\alpha = 28.421$, $\beta = -15.086$, and $\delta = 0.317$. (The parameters are chosen so that $\Psi(1) = r$, cf. (4.2) and (4.3); γ is set equal to 0.0585 for this purpose. The parameters are on a per annum basis. The truncation at 1 in (4.2) can be ignored for this model.)

Exercise Price	$T = 0.25$			$T = 1$		
	E-FFT	E-MN	A-MN	E-FFT	E-MN	A-MN
90	0.2203	0.2184	0.2501	0.5287	0.5155	0.7615
95	0.5994	0.5961	0.6924	1.0203	1.0007	1.5383
100	1.5670	1.5628	1.8665	1.8437	1.8190	2.9440
105	3.7659	3.7647	5	3.1235	3.0955	5.4099
110	7.5508	7.5511	10	4.9693	4.9425	10

The variance gamma model and the normal inverse Gaussian model are both infinite activity models, i.e., the Lévy measure is not finite. Amin (1993) studies the finite activity jump-diffusion model. We attempted to include also a finite activity model in this paper, in order to compare with the infinite activity case. A comparable compound Poisson model with diffusion was specified by matching its lower order centered moments to those of the variance gamma process used in Table 5.1. However, the jump component in this specification turned out to be

so small that the model essentially reduced to the standard diffusion setup. As we noted earlier (see footnote 3), moment matching is not necessarily an optimal procedure. Nevertheless, the results of this exercise appear to support informal observations by Madan and others that infinite activity models are needed to properly model departures from the Black-Scholes/geometric Brownian motion situation.

6 Summary and Discussion

As outlined in Section 1, there are various ways to approach the computation of American option prices in a given model; we made brief mention, for example, of PDE, PIDE and simulation methods, in addition to tree-based methods.⁴ Our aim in this paper was to present a transparent tree-based scheme which works in principle whenever the stock price follows an exponential Lévy process model for which the canonical measure is explicitly prescribed or easily calculated. We showed that the proposed method does indeed “work in principle” in the sense that the American option price calculated from the tree will approach the “true” continuous time price as the number of steps becomes large, provided the defining parameters of the tree are related in the way we specified, and provided the payoff function and the process satisfy some quite mild technical restrictions.

An advantage of tree (or lattice) methods is that the model as well as the valuation principles are easily set up and understood without deep knowledge of the underlying financial, mathematical and probabilistic foundations. Further, the method is easily programmed in a straightforward manner in standard computer packages. But one may argue that the price to pay for the transparency of the tree method is a lack of numerical efficiency in computations, and a lack of easily applicable mathematical machinery for establishing the rate of convergence of the discrete time approximation to the continuous time limit.

In the present paper we have made no effort to investigate the rate of convergence of the method or to improve the computational efficiency of our implementation of it. Our implementation, however, turned out to be adequate for the American put calculations we report in Section 5. (The computations were programmed in Matlab 6.5 on a Pentium 4 computer with 2.53GHz speed.)

⁴The PDE, PIDE and tree approaches are to some extent inter-related. The derivative price, expressed as a solution of a PDE or PIDE, is usually approximated on a grid, or lattice, when not explicitly obtainable. With the tree based methods, the lattice aspect is evident, and the approaches can be related when studying the convergence rate as in Lamberton (1998).

Now, the scheme can also be adapted to incorporate stock price processes that pay continuous or discrete dividends, as well as models with stochastic interest rates, and it is sufficiently flexible to use for a wide range of exotic option problems including, e.g., path-dependent options such as barrier options and lookback options. Early experience we have gained with the latter reveals that research into the rate of convergence issue is indicated, and the computational efficiency needs to be improved substantially before any more extensive numerical investigations of exotic options could be done.

Research into convergence and convergence rates for American options in tree (or lattice) models is quite recent. The binomial model for pricing (especially) American options in the Black-Scholes setup was advocated by Cox, Ross and Rubinstein in their seminal paper in 1979. But convergence and convergence rate results for this model were not established until about 2 decades later, see, e.g., Amin and Khanna (1994) and Lamberton (1998), and the general theory for tree models based on semi-martingales, which we applied herein, was given by Mulinacci and Pratelli only in 1998. There appear to be substantial technical problems to overcome in establishing convergence rate results for methods like this, which we do not address here, other than to mention that it is likely that this task might be carried out analogously to the analysis of Lamberton (1998) relating the binomial setup to partial differential calculus; and we expect that convergence rates for the multinomial method might be established by invoking results of Matache et al. (2005), who studied the pricing of American options in a general Lévy process setting via partial-integro differential inequality methods.

We conclude with a few comments regarding numerical efficiency. Our approximation scheme is designed to cope with infinite activity in the Lévy process, so it ignores the value of the Lévy measure in a neighbourhood of 0. A key point in our methodology is that the omitted neighbourhood must shrink in the limit at the right speed, not too fast, as determined by (3.1). A possible alternative approach, whereby an extra Brownian motion is introduced to take the place of the “small” jumps in the Lévy process,⁵ was used by Asmussen and Rosinski (2001) to improve convergence speed in a certain practical implementation. In preliminary investigations involving the examples we gave in Section 5, we found that this made little difference in our setup, probably because the singularities in the VG and NIG models are relatively mild.

The discrete time approximation, as we defined it in order to ensure a recombining scheme, is rather restrictive, and the choice of states, and the prob-

⁵This alternative was suggested by Ole E. Barndorff-Nielsen in discussions at the Oberwolfach workshop “Statistics in Finance”, January 2004.

abilities we specified for them, are probably not optimal. However, these can be modified in various ways by adding/subtracting small order terms which are unimportant asymptotically but may be influential in the finite situation. For example, a modification of the probabilities which matches the moments (or other characteristics) of an increment of the multinomial random variable $X(n)$ with the corresponding characteristics of the increment of the Lévy process would be expected to generate a faster converging approximation scheme; see, e.g., Killezi and Webber (2004), who implemented such a refinement in their Bermudan scheme. Again we leave such investigations to future research.

7 Proofs for Section 3

Proof of Theorem 3.1. First consider the case $t = 1$. We wish to show that

$$\sum_{j=1}^{N(n)} (X_j(n) - a(n)) \xrightarrow{D} L_1, \quad (7.1)$$

where L_1 is an infinitely divisible random variable with triplet $(\gamma, 0, \Pi)$. The convergence in (7.1) is the convergence in distribution of the row sums of a centered triangular array, $(X_j(n))_{1 \leq j \leq N(n), n \geq n_0}$, and we can apply general theory as in e.g., Petrov (1975) or Jacod and Shiryaev (1987). Note that

$$\mathbb{P}(|X_j(n)| > \varepsilon) = \mathbb{P}(|X(n)| > \varepsilon) \leq \mathbb{P}(X(n) \neq 0) = A(n),$$

for $\varepsilon > 0$, $1 \leq j \leq n$. Since $\lim_{n \rightarrow \infty} A(n) = 0$ we get

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq N(n)} \mathbb{P}(|X_j(n)| > \varepsilon) = 0, \quad \text{for } \varepsilon > 0,$$

or, equivalently, $(X_j(n))_{1 \leq j \leq N(n)}$ is uniformly asymptotically negligible. Thus, by, e.g., Theorem 7, p. 81, of Petrov (1975), necessary and sufficient conditions for (7.1) are:

$$\sum_{j=1}^{N(n)} \mathbb{P}(X_j(n) \leq -y) \rightarrow \bar{\Pi}_-(y), \quad \text{and} \quad \sum_{j=1}^{N(n)} \mathbb{P}(X_j(n) > y) \rightarrow \bar{\Pi}_+(y), \quad (7.2)$$

$$\lim_{y \searrow 0} \limsup_{n \rightarrow \infty} \sum_{j=1}^{N(n)} [\mathbb{E}(X_j^2(n) \mathbf{1}_{\{|X_j(n)| \leq y\}}) - \mathbb{E}^2(X_j(n) \mathbf{1}_{\{|X_j(n)| \leq y\}})] = 0, \quad (7.3)$$

and

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{N(n)} [\mathbb{E}(X_j(n) \mathbf{1}_{\{|X_j(n)| \leq y\}}) - a(n)] = \gamma + \int_{1 < |u| \leq y} u \Pi(du), \quad (7.4)$$

for all continuity points $y > 0$ of the limits.

To verify (7.2) for our setup (we deal with the righthand convergence in (7.2) first), fix $y > 0$, a continuity point of $\Pi(\cdot)$, then choose n so large that $\Delta(n)/2 < y \leq \Delta(n)(m_+(n) + 1/2)$. This is possible since $\Delta(n) \downarrow 0$ and $\Delta(n)m_+(n) \rightarrow \infty$ by (3.2), and it means that $y \in I_k(n)$ for some k , by (3.3). Thus we can define

$$k(n, y) = \min \{k \geq 1 : x_k(n) > y\},$$

then $x_{k(n, y)-1}(n) \leq y < x_{k(n, y)}(n)$. Since the $X_i(n)$ are i.i.d. as $X(n)$, it suffices to compute

$$\begin{aligned} N(n) \mathbb{P}(X(n) > y) &= N(n) \sum_{k \in \mathcal{M}(n): k \geq k(n, y)} \left(\frac{1}{N(n)} \Pi(I_k(n)) \right) \quad (\text{by (3.4)}) \\ &= \bar{\Pi}_+((k(n, y) - 1/2)\Delta(n)) - \bar{\Pi}_+(\Delta(n)(m_+(n) + 1/2)). \end{aligned} \quad (7.5)$$

The subtracted term in (7.5) has limit 0 as $n \rightarrow \infty$ since $\lim_{n \rightarrow \infty} \Delta(n)m_+(n) = \infty$ and of course $\lim_{x \rightarrow \infty} \bar{\Pi}_+(x) = 0 = \lim_{x \rightarrow \infty} \bar{\Pi}_-(x)$. Now

$$(k(n, y) - 3/2)\Delta(n) \leq y < (k(n, y) + 1/2)\Delta(n),$$

and thus $\lim_{n \rightarrow \infty} k(n, y)\Delta(n) = y$. Since y is a continuity point of $\Pi(\cdot)$, (7.5) gives

$$\lim_{n \rightarrow \infty} N(n) \mathbb{P}(X(n) > y) = \bar{\Pi}_+(y),$$

as required for the righthand convergence in (7.2). Similar reasoning gives

$$\lim_{n \rightarrow \infty} N(n) \mathbb{P}(X(n) < -y) = \bar{\Pi}_-(y),$$

when $-y < 0$ is a continuity point of $\Pi(\cdot)$, as required for the lefthand convergence in (7.2).

To verify (7.3), take $y > 0$ a continuity point of $\Pi(\cdot)$ and write

$$\begin{aligned} \sum_{j=1}^{N(n)} \mathbb{E}(X_j^2(n) \mathbf{1}_{\{|X_j(n)| \leq y\}}) &= N(n) \mathbb{E}(X^2(n) \mathbf{1}_{\{|X(n)| \leq y\}}) \\ &= N(n) \sum_{k \in \mathcal{M}(n): |x_k(n)| \leq y} x_k^2(n) \left(\frac{1}{N(n)} \Pi(I_k(n)) \right) \\ &= \sum_{k \in \mathcal{M}(n): |x_k(n)| \leq y} x_k^2(n) \int_{I_k(n)} \Pi(du). \end{aligned} \quad (7.6)$$

Now $x_k(n) \in I_k(n)$, which is an interval of width $\Delta(n)$, so if $u \in I_k(n)$, then $|x_k(n) - u| \leq \Delta(n)$, thus $|x_k(n)| \leq |u| + \Delta(n)$. Also, since no $I_k(n)$ has a point

closer to 0 than $\Delta(n)/2$, $|u| \geq \Delta(n)/2$. Hence $|x_k(n)| \leq 3|u|$. Then (7.6) is bounded by 9 times

$$\sum_{k \in \mathcal{M}(n): |x_k(n)| \leq y} \int_{I_k(n)} u^2 \Pi(du) \leq \int_{\{|u| \leq y + \Delta(n)\}} u^2 \Pi(du).$$

The last term converges to $\int_{|u| \leq y} u^2 \Pi(du)$, as $n \rightarrow \infty$. This integral is finite and tends to 0 as $y \rightarrow 0$. Hence (7.3) follows.

To verify (7.4), define $a(n)$ as in (3.7). Then for $y > 0$ a continuity point of $\Pi(\cdot)$,

$$\begin{aligned} & \sum_{j=1}^{N(n)} [\mathbb{E}(X_j(n) \mathbf{1}_{\{|X_j(n)| \leq y\}}) - a(n)] \\ &= \gamma + N(n) \mathbb{E}(X(n) \mathbf{1}_{\{|X(n)| \leq y\}}) - N(n) b(n) \\ &= \gamma + \sum_{1 < |x_k(n)| \leq y} x_k(n) \int_{I_k(n)} \Pi(du) + o(1), \end{aligned} \quad (7.7)$$

where the summation is for k taking values in $\mathcal{M}(n)$. Now (7.4) will hold if the second expression on the righthand side of (7.7) converges to $\int_{1 < |u| \leq y} u \Pi(du)$ for $n \rightarrow \infty$. For fixed y , we use (3.2) to choose n large enough that $m_{\pm}(n) \Delta(n) > y$. If $u \in I_k(n)$ we have $|x_k(n) - u| \leq \Delta(n)$ and

$$\bigcup_{1 < |x_k(n)| \leq y} I_k(n) \subseteq \{1 - \Delta(n) < |u| \leq y + \Delta(n)\}.$$

So, for $n \rightarrow \infty$,

$$\sum_{1 < |x_k(n)| \leq y} \int_{I_k(n)} |x_k(n) - u| \Pi(du) \leq \Delta(n) \int_{1 - \Delta(n) < |u| \leq y + \Delta(n)} \Pi(du) \rightarrow 0.$$

Further, y is a continuity point of Π and we assumed in Section 3 that 1 is also. Thus, since

$$\begin{aligned} \left| \int_{1 < |u| \leq y} u \Pi(du) - \sum_{1 < |x_k(n)| \leq y} \int_{I_k(n)} u \Pi(du) \right| &\leq \int_{1 - \Delta(n) \leq |u| \leq 1 + \Delta(n)} (1 + \Delta(n)) \Pi(du) \\ &\quad + \int_{y - \Delta(n) \leq |u| \leq y + \Delta(n)} (y + \Delta(n)) \Pi(du), \end{aligned}$$

and the righthand side tends to 0 as $n \rightarrow \infty$, we have proved (7.4). Accordingly the convergence in (7.1) is established.

To extend (7.1) to (3.6), use the functional limit theorem of Jacod and Shiryaev (1987), Theorem 2.29, Ch. VIII, p. 426. Note that this theorem is

not quite in the form we need (we would like the centering sequence $a(n)$ to be explicit), but it is easily modified to give (3.6). Alternatively, use Theorem 4.1 of Durrett and Resnick (1978). \square

8 Proofs for Section 4

Throughout this section we assume the setup in Sections 3 and 4, including (3.1) – (3.5), (4.1) and (4.3). Note that, as a multinomial rv, $X_j(n)$ has finite moments of all orders, in fact a finite moment generating function, for each $j \geq 1$, $n \geq 1$, and the same is true of $L_t(n)$ for each $t \in [0, T]$ and $n \geq 1$ (see (3.6)).

Before proceeding we need the estimates in the following lemma. Recall that $\mu(n) = \mathbb{E}X(n)$.

Lemma 8.1. *Assume (3.1) – (3.5), (4.1) and (4.3), suppose (4.8) holds for some $m > 0$, and choose $0 < \theta \leq m$. Then, as $n \rightarrow \infty$,*

$$\mu(n) = o(1/\sqrt{N(n)}), \quad (8.1)$$

and

$$\phi_n(\theta) := \mathbb{E}(e^{\theta(X(n) - \mu(n))}) = 1 + O(1/N(n)). \quad (8.2)$$

Proof of Lemma 8.1. Assume (3.1) – (3.5), (4.1), (4.3), and (4.8) for some $m > 0$. We can write

$$\mu(n) = \mathbb{E}(X(n)) = \frac{1}{N(n)} \sum_{k \in \mathcal{M}(n)} x_k(n) \Pi(I_k(n)).$$

Take $n \in \mathbb{N}_0$, $k \in \mathcal{M}(n)$, and $u \in I_k(n)$. We showed in the proof of Theorem 3.1 that, then, $|u| \geq \Delta(n)/2$ and $|x_k(n)| \leq 3|u|$. Thus, using (3.1), for some $C_3 > 0$ and for any $\eta > 0$, once n is large enough,

$$\begin{aligned} C_3 \sqrt{N(n)} |\mu(n)| &\leq \Delta(n) \left| \sum_{k \in \mathcal{M}(n)} \int_{I_k(n)} x_k(n) \Pi(du) \right| \\ &\leq 3\Delta(n) \int_{|u| \geq \Delta(n)/2} |u| \Pi(du) \\ &= 3\Delta(n) \left(\int_{\Delta(n)/2 \leq |u| < \eta} |u| \Pi(du) + \int_{|u| \geq \eta} |u| \Pi(du) \right) \\ &\leq 3 \left(2 \int_{\Delta(n)/2 \leq |u| < \eta} u^2 \Pi(du) + \Delta(n) \int_{|u| \geq \eta} |u| \Pi(du) \right). \end{aligned}$$

Since $\mathbb{E}|L_1| < \infty$ by (4.8), the integral in the second expression in the brackets is finite and the expression goes to 0 as $n \rightarrow \infty$ because $\Delta(n) \rightarrow 0$. The first expression in the brackets can be made arbitrarily small by choosing η close to 0, so we have proved (8.1).

Now we estimate $\phi_n(\theta)$ for $0 < \theta \leq m$. Since $\mathbb{E}(X(n) - \mu(n)) = 0$ we have, using the inequality $|e^z - 1 - z| \leq z^2 e^{|z|}$,

$$\begin{aligned} |\phi_n(\theta) - 1| &= |\mathbb{E}(e^{\theta(X(n) - \mu(n))} - 1 - \theta(X(n) - \mu(n)))| \\ &\leq \theta^2 \mathbb{E} \left((X(n) - \mu(n))^2 e^{\theta|X(n) - \mu(n)|} \right). \end{aligned}$$

Next, since $\mu(n) = o(1/\sqrt{N(n)})$, by (8.1), and $\Delta(n) \geq C_3/\sqrt{N(n)}$ for $C_3 > 0$, for large n , by (3.1), and since no $x_k(n)$ is closer to the origin than $\pm\Delta(n)/2$, we have that $|x_k(n)| \geq \Delta(n)/2 \geq |\mu(n)|/2$, thus $|x_k(n) - \mu(n)| \leq 3|x_k(n)|$, for large n . Also, $|x_k(n)| \leq 3|u|$ for $u \in I_k(n)$. Additionally, for $u \in I_k(n)$,

$$|x_k(n) - \mu(n)| \leq |x_k(n) - u| + |u - \mu(n)| \leq \Delta(n) + |u| + |\mu(n)| = |u| + o(1),$$

where “ $o(1)$ ” does not depend on u . Using these inequalities, and (8.1) again, gives

$$\begin{aligned} &\mathbb{E} \left((X(n) - \mu(n))^2 e^{\theta|X(n) - \mu(n)|} \right) \\ &= \frac{1}{N(n)} \sum_{k \in \mathcal{M}(n)} (x_k(n) - \mu(n))^2 e^{\theta|x_k(n) - \mu(n)|} \Pi(I_k(n)) \\ &\quad + \mu^2(n) e^{\theta|\mu(n)|} (1 - A(n)) \\ &\leq \left(\frac{81}{N(n)} \right) \sum_{k \in \mathcal{M}(n)} \int_{I_k(n)} u^2 e^{\theta|u| + o(1)} \Pi(du) + o \left(\frac{1}{N(n)} \right) \end{aligned} \quad (8.3)$$

Keeping in mind that $\int_{\mathbb{R}} u^2 e^{\theta|u|} \Pi(du) < \infty$ for $0 < \theta \leq m$, the second last term here is $O(1/N(n))$. Thus (8.2) holds. \square

Proof of Lemma 4.1. Assume (3.1) – (3.5), (4.1) and (4.3), and in addition suppose (4.6) – (4.9) hold. Take $\tau \in \mathcal{S}_{0,T}(n)$. By (4.6) there are constants $C_4 > 0$, $\eta \in (0, 1)$, such that

$$|g(x) - g(y)| \leq C_4|x - y|^\beta, \text{ whenever } |x - y| \leq \eta. \quad (8.4)$$

If (8.4) holds for a given value of $\beta > 0$ it holds for any smaller value of β , so we may without loss of generality assume $\beta < 1 \wedge m = 1$. Fix $t > 0$ and take $0 \leq s \leq t$. Use (8.4) and (4.7) to write

$$\begin{aligned} |g(S_{\tau+s}(n)) - g(S_\tau(n))| &\leq C_4|S_{\tau+s}(n) - S_\tau(n)|^\beta \mathbf{1}_{\{|S_{\tau+s}(n) - S_\tau(n)| \leq \eta\}} \\ &\quad + (2C_1 + C_2(S_{\tau+s}^m(n) + S_\tau^m(n))) \mathbf{1}_{\{|S_{\tau+s}(n) - S_\tau(n)| > \eta\}}. \end{aligned} \quad (8.5)$$

We estimate the expectation of each of the terms in this decomposition. Begin with

$$\mathbb{E}|S_{\tau+s}(n) - S_\tau(n)|^\beta = \mathbb{E}S_\tau^\beta(n)\mathbb{E}|e^{L_{\tau+s}(n)-L_\tau(n)} - 1|^\beta. \quad (8.6)$$

We need a uniform bound for $\mathbb{E}S_\tau^\beta(n)$. Let $M_t(n)$ denote the mean-centered sum, as in (3.10). Then, using (3.6) and (3.9), write

$$\begin{aligned} L_t(n) &= \sum_{j=1}^{\lfloor N(n)t \rfloor} (X_j(n) - \mu(n)) + \lfloor N(n)t \rfloor (\mathbb{E}L_1/N(n) - b(n)) \\ &= M_t(n) + c_n(t), \text{ say,} \end{aligned} \quad (8.7)$$

where (recall that $b(n) = o(1/N(n))$)

$$|c_n(t)| \leq t|\mathbb{E}L_1 + o(1)| \leq 2t\mathbb{E}|L_1|, \text{ for large } n.$$

Let

$$\hat{S}_t(n) := S_0(n)e^{M_t(n)}/(\phi_n(1))^{\lfloor N(n)t \rfloor}, \text{ for } t > 0, n \geq n_0, \quad (8.8)$$

where $\phi_n(\theta)$ is defined in (8.2). It's straightforward to prove that $\{\hat{S}_t(n), \mathcal{F}_t^n\}_{t \geq 0}$ is a martingale for each $n = 1, 2, \dots$. From (8.2) we have $\phi_n(\theta) \leq 1 + C_5/N(n)$, for $\theta > 0$, for some $C_5 > 0$, if n is large enough. Thus, since $\tau \leq T$ a.s., we have, for $u > 0$,

$$\begin{aligned} (\phi_n(1))^{\lfloor N(n)(\tau+u) \rfloor} &\leq (1 + C_5/N(n))^{N(n)(\tau+u)} \\ &\leq (1 + C_5/N(n))^{N(n)(T+u)} \leq e^{C_5(T+u)}, \text{ a.s.} \end{aligned}$$

Similarly, $(\phi_n(1))^{\lfloor N(n)(\tau+u) \rfloor} \geq e^{-C_5(T+u)/2}$ for all large n , thus the expression $(\phi_n(1))^{\lfloor N(n)(\tau+u) \rfloor}$ is bounded away from zero and infinity for large enough n for any τ and $u \geq 0$. Also, $\lim_{n \rightarrow \infty} \phi_n^{N(n)}(1) = \mathbb{E}e^{L_1} = e^{\Psi(1)} = e^r$, thus

$$\lim_{n \rightarrow \infty} (\phi_n(1))^{\lfloor N(n)t \rfloor} = e^{rt}. \quad (8.9)$$

Now, since $\beta \leq 1$, $x \mapsto x^\beta$ is concave on $[0, \infty)$, and $\hat{S}_t(n)$ is a non-negative martingale. Also, for $u \geq 0$, $e^{c_n(\tau+u)} \leq e^{2\mathbb{E}|L_1|T}$ a.s., for large n . (Recall our convention that the interval of definition of τ is extended so that $\tau + u$ can be replaced by $(\tau + u) \wedge T$.) Thus we have by Jensen's inequality and optional sampling, for any $u \geq 0$,

$$\begin{aligned} \mathbb{E}S_{\tau+u}^\beta(n) &= \mathbb{E}\left(e^{\beta c_n(\tau+u)}(\phi_n(1))^{\beta \lfloor N(n)(\tau+u) \rfloor} \hat{S}_{\tau+u}^\beta(n)\right) \\ &\leq e^{(2\mathbb{E}|L_1|+C_5)(T+u)} \left(\mathbb{E}\hat{S}_{\tau+u}(n)\right)^\beta \\ &= e^{(2\mathbb{E}|L_1|+C_5)(T+u)} (\mathbb{E}(S_0(n)))^\beta \\ &\leq C_6^\beta e^{(2\mathbb{E}|L_1|+C_5)(T+u)}, \text{ say, c.f. (4.9).} \end{aligned}$$

Consequently, taking $u = 0$, we get the required uniform bound:

$$\sup_{n \geq n_0} \mathbb{E} S_\tau^\beta(n) \leq C_6^\beta e^{(2\mathbb{E}|L_1| + C_5)T}. \quad (8.10)$$

The second expectation on the righthand side of (8.6) is the expectation of

$$\begin{aligned} |e^{L_s(n)} - 1|^\beta &= (e^{L_s(n)} - 1)^\beta \mathbf{1}_{\{L_s(n) > 0\}} + (1 - e^{L_s(n)})^\beta \mathbf{1}_{\{L_s(n) \leq 0\}} \\ &\leq (1 - e^{-L_s(n)})^\beta e^{\beta L_s(n)} \mathbf{1}_{\{L_s(n) > 0\}} + |L_s(n)|^\beta \\ &\leq L_s^\beta(n) e^{\beta L_s(n)} \mathbf{1}_{\{L_s(n) > 0\}} + |L_s(n)|^\beta. \end{aligned} \quad (8.11)$$

Now, recall we chose $\beta < m$. Take $\nu > 0$ with $\beta + \nu < m$, and choose $a = a(\nu) > 1$ so large that $x^\beta e^{\beta x} \leq e^{(\beta + \nu)x} \leq e^{mx}$ whenever $e^{\beta x} > a$. Then the last expression in (8.11) is bounded by

$$\begin{aligned} &a |L_s(n)|^\beta \mathbf{1}_{\{1 < e^{\beta L_s(n)} \leq a\}} + e^{(\beta + \nu)L_s(n)} \mathbf{1}_{\{e^{\beta L_s(n)} > a\}} + |L_s(n)|^\beta \\ &\leq (a + 1) |L_s(n)|^\beta + e^{mL_s(n)} \mathbf{1}_{\{e^{mL_s(n)} > a\}}. \end{aligned} \quad (8.12)$$

Use (8.7) together with the elementary inequality $(a + b)^\beta \leq 2^\beta (a^\beta + b^\beta)$, $a, b > 0$, to get

$$|L_s(n)|^\beta \leq 2^\beta (|M_s(n)|^\beta + s^\beta (\mathbb{E}|L_1| + o(1))) \leq 2^\beta (|M_s(n)|^\beta + 2s^\beta (\mathbb{E}|L_1|)),$$

for large n . Since $\beta \leq 1$ and $x \mapsto x^\beta$ is concave, Jensen's inequality gives

$$\begin{aligned} \sup_{0 \leq s \leq t} (\mathbb{E}|M_s(n)|^\beta)^{2/\beta} &\leq \sup_{0 \leq s \leq t} (\mathbb{E}|M_s(n)|)^2 \leq \sup_{0 \leq s \leq t} \mathbb{E} M_s^2(n) \\ &= \mathbb{E} \left(\sum_1^{\lfloor N(n)t \rfloor} (X_j(n) - \mu(n))^2 \right) \\ &\leq N(n)t \mathbb{E}(X(n) - \mu(n))^2 \leq C_7 t, \end{aligned} \quad (8.13)$$

by (8.3) with $\theta = 0$.

We estimate the expectation of the second term on the righthand side of (8.12) as follows. By (8.7), we can replace $e^{L_s(n)}$ with $e^{M_s(n)}$ (up to a non-random multiple, which is unimportant). Now, $M_s(n) = M_k(n)$ for some $0 \leq$

$k \leq N(n) s \leq N(n) t$. Then, for $a > 1$,

$$\begin{aligned}
\mathbb{E} \left(e^{m M_k(n)} \mathbf{1}_{\{e^{m M_k(n)} > a\}} \right) &= \int_{(a, \infty)} y \, d\mathbb{P}(e^{m M_k(n)} \leq y) \\
&\leq \frac{1}{(\log a)^2} \int_{(0, \infty)} y (\log y)^2 \, d\mathbb{P}(e^{m M_k(n)} \leq y) \\
&= \frac{1}{(\log a)^2} \mathbb{E} \left(e^{m M_k(n)} (m M_k(n))^2 \right) \\
&= \frac{m^2}{(\log a)^2} \int_{\mathbb{R}} y^2 e^{m y} \, d\mathbb{P}(M_k(n) \leq y) \\
&= \frac{m^2}{(\log a)^2} \frac{\partial^2}{\partial \theta^2} \int_{\mathbb{R}} e^{\theta y} \, d\mathbb{P}(M_k(n) \leq y) \Big|_{\theta=m} \\
&= \frac{m^2}{(\log a)^2} \frac{\partial^2}{\partial \theta^2} \phi_n^k(\theta) \Big|_{\theta=m} \\
&\leq 2m^2 C_7^2 (t^2 + t) e^{C_5 t} / (\log a)^2. \tag{8.14}
\end{aligned}$$

To verify the last step, observe that

$$\frac{\partial^2}{\partial \theta^2} \phi_n^k(\theta) = k(k-1) \phi_n^{k-2}(\theta) (\phi_n'(\theta))^2 + k \phi_n^{k-1}(\theta) \phi_n''(\theta).$$

Now

$$\begin{aligned}
|\phi_n'(\theta)|_{\theta=m} &= |\mathbb{E}(X_{1n} - \mu_n) e^{m(X_{1n} - \mu_n)}| \\
&= \left| \mathbb{E}(X_{1n} - \mu_n) \left(e^{m(X_{1n} - \mu_n)} - 1 \right) \right| \\
&\leq \mathbb{E} \left((X_{1n} - \mu_n)^2 e^{m|X_{1n} - \mu_n|} \right) \\
&\leq C_7 / N(n).
\end{aligned}$$

The last estimate follows from (8.3) with $\theta = m$. Similarly

$$\phi_n''(\theta)|_{\theta=m} = \mathbb{E} \left((X_{1n} - \mu_n)^2 e^{m(X_{1n} - \mu_n)} \right) \leq C_7 / N(n).$$

Also, since $k \leq N(n) t$,

$$\phi_n^k(\theta) = (1 + \phi_n(\theta) - 1)^k \leq (1 + C_5 / N(n))^k \leq (1 + C_5 / N(n))^{N(n) t} \leq e^{C_5 t}. \tag{8.15}$$

So we get, for large n ,

$$\frac{\partial^2}{\partial \theta^2} \phi_n^k(\theta) \leq C_7^2 (t^2 + t) \phi_n^{k-2}(\theta) \leq 2C_7^2 (t^2 + t) e^{C_5 t},$$

giving the result (8.14).

Putting together (8.11)– (8.14) we get, for all $t > 0$ and all large n ,

$$\sup_{0 \leq s \leq t} \mathbb{E}|e^{L_s(n)} - 1|^\beta \leq (a+1)2^\beta \left(C_7^{\beta/2} t^{\beta/2} + 2t^\beta \mathbb{E}|L_1| \right) + 2C_7^2(t^2+t)e^{C_5 t}/(\log a)^2. \quad (8.16)$$

But by (8.6) and (8.10),

$$\sup_{0 \leq s \leq t} \mathbb{E}|S_{\tau+s}(n) - S_\tau(n)|^\beta \leq C_6^\beta e^{(2\mathbb{E}|L_1|+C_5)T} \sup_{0 \leq s \leq t} \mathbb{E}|e^{L_s(n)} - 1|^\beta. \quad (8.17)$$

Hold a fixed, take the limsup as $n \rightarrow \infty$, then let $t \rightarrow 0$ in (8.16), to get 0 for this term.

Going back to (8.5), we now deal with

$$\begin{aligned} & \mathbb{E} \left(2C_1 \mathbf{1}_{\{|S_{\tau+s}(n) - S_\tau(n)| > \eta\}} \right) \\ & \leq 2C_1 \mathbb{E} \left(\mathbf{1}_{\{|e^{L_{\tau+s}(n)} - L_\tau(n)} - 1| > \eta/a\}} + \mathbf{1}_{\{S_\tau(n) > a\}} \right) \\ & \leq \frac{2C_1 a}{\eta} \mathbb{E}|e^{L_s(n)} - 1| + \left(\frac{2C_1}{a} \right) \mathbb{E}S_\tau(n) \\ & \leq \frac{2C_1 a}{\eta} \mathbb{E}|e^{L_s(n)} - 1| + \left(\frac{2C_1}{a} \right) C_6 e^{(2\mathbb{E}|L_1|+C_5)T}. \end{aligned} \quad (8.18)$$

Here $a > 0$. In the last two steps we used Markov's inequality twice and the bound in (8.10) with $\beta = 1$. The first term on the righthand side of (8.18) is a special case of (8.16) with $\beta = 1$. Holding a and η fixed, taking the sup over $0 \leq s \leq t$, then the limsup as $n \rightarrow \infty$, then letting $t \rightarrow 0$, gives 0 for this term, just as for (8.17). The second term on the righthand side of (8.18) does not depend on n or t and tends to 0 as $a \rightarrow \infty$.

The next term to deal with in (8.5) is

$$\begin{aligned} & \mathbb{E} \left(C_2 S_{\tau+s}^m(n) \mathbf{1}_{\{|S_{\tau+s}(n) - S_\tau(n)| > \eta\}} \right) \\ & \leq C_2 \mathbb{E} \left(S_{\tau+s}^m(n) \mathbf{1}_{\{S_{\tau+s}^m(n) \leq a\}} \left(\mathbf{1}_{\{|e^{L_{\tau+s}(n)} - L_\tau(n)} - 1| > \eta/a\}} + \mathbf{1}_{\{S_\tau(n) > a\}} \right) \right) \\ & \quad + C_2 \mathbb{E} \left(S_{\tau+s}^m(n) \mathbf{1}_{\{S_{\tau+s}^m(n) > a\}} \right) \\ & \leq aC_2 \left(\mathbb{P} \left(|e^{L_s(n)} - 1| > \eta/a \right) + \mathbb{P} \left(S_\tau(n) > a \right) \right) \\ & \quad + C_2 \mathbb{E} \left(S_{\tau+s}^m(n) \mathbf{1}_{\{S_{\tau+s}^m(n) > a\}} \right). \end{aligned} \quad (8.19)$$

Here we keep $a > 1$. Use Markov's inequality to bound the first term by $C_2(a^2/\eta) \sup_{0 \leq s \leq t} \mathbb{E}|e^{L_s(n)} - 1|$, which, by (8.16), tends to 0 as $n \rightarrow \infty$ then $t \rightarrow 0$, for fixed a and η . Also use Markov's inequality to bound the second term in (8.19) by $a^{1-m} C_2 \mathbb{E}(S_\tau^m(n) \mathbf{1}_{\{S_\tau^m(n) > a^m\}})$. This is of the same form as the third term (with $s = 0$, and recall $m \geq 1$), so we treat both at once.

Proceed as follows. First keep $m > 1$. Replace $S_t(n)$ by the martingale $\hat{S}_t(n)$, as is possible since the ratio $S_t(n)/\hat{S}_t(n)$ is nonrandom and bounded between finite nonzero limits as $n \rightarrow \infty$, for all $0 \leq t \leq T$. Next write, for $u \geq 0$ and $b > 1$,

$$\begin{aligned} & \mathbb{E} \left(\hat{S}_{\tau+u}^m(n) \mathbf{1}_{\{\hat{S}_{\tau+u}^m(n) > b\}} \right) \\ & \leq \left(\frac{m}{(\log b)^2} \right) \mathbb{E} \left(\hat{S}_{\tau+u}^m(n) (\log \hat{S}_{\tau+u}^m(n))^2 \mathbf{1}_{\{\hat{S}_{\tau+u}^m(n) > b\}} \right) \\ & \leq \left(\frac{m}{(\log b)^2} \right) \mathbb{E} \left(\sup_{0 \leq t \leq T+u} \left(\hat{S}_t(n) (\log \hat{S}_t(n))^{2/m} \mathbf{1}_{\{\hat{S}_t(n) \geq 1\}} \right)^m \right). \end{aligned} \quad (8.20)$$

Now let $\ell(x) := x(\log x)^{2/m} \mathbf{1}_{\{x \geq 1\}}$ for $x \geq 0$. If $m \leq 2$, $\ell(\cdot)$ is convex on $[0, \infty)$; define $\tilde{\ell}(x) = \ell(x)$ and let $x_m := 1$ in this case. If $m > 2$, $\ell''(x) > 0$ for all $x \geq x_m := e^{1-2/m} > 1$. In this case define $\tilde{\ell}(x) = \ell(x) \vee \ell(x_m)$ for $x \geq 0$. In either case $\tilde{\ell}(x)$ is nonnegative and convex on $[0, \infty)$, and so the process $\tilde{\ell}(\hat{S}_t(n))$ is a nonnegative submartingale. Also $\ell(x) \leq \tilde{\ell}(x)$. So for $m > 1$, by Doob's inequality (Ethier and Kurtz 1986, p. 63), the expectation in (8.20) is bounded by

$$\mathbb{E} \left(\sup_{0 \leq t \leq T+u} \left(\tilde{\ell}(\hat{S}_t(n)) \right)^m \right) \leq \left(\frac{m}{m-1} \right)^m \mathbb{E} \left(\tilde{\ell}^m(\hat{S}_{T+u}(n)) \right).$$

The last expression is bounded by

$$\begin{aligned} & \left(\frac{m}{m-1} \right)^m \left(\tilde{\ell}^m(x_m) + \mathbb{E} \left(\tilde{\ell}^m(\hat{S}_{T+u}(n)) \mathbf{1}_{\{\hat{S}_{T+u}(n) \geq x_m\}} \right) \right) \\ & \leq \left(\frac{m}{m-1} \right)^m \left(\tilde{\ell}^m(x_m) + \mathbb{E} \left(\hat{S}_{T+u}^m(n) (\log \hat{S}_{T+u}(n))^2 \right) \right). \end{aligned} \quad (8.21)$$

The last expectation is asymptotically (as $n \rightarrow \infty$) a constant multiple of

$$\mathbb{E} \left(S_0^m(n) e^{mM_{T+u}(n)} \left(\log((\phi_n(1))^{-\lfloor N(n)(T+u) \rfloor} S_0(n)) + M_{T+u}(n) \right)^2 \right),$$

which is uniformly bounded in n by similar arguments as before, and invoking the fact that $\{S_0^m(n)(\log S_0(n))^2\}$ is uniformly integrable. Letting $b \rightarrow \infty$ in (8.20) then gives the result for $m > 1$.

Next, suppose $m = 1$, and modify the above argument, using “log” instead of “(log)²” in (8.20), and the nonnegative convex function $x \mapsto x \log x \mathbf{1}_{\{x \geq 1\}}$, to get an upper bound of the form (c.f. Chow and Teicher (1988), Theorem 8, Ch. 7.4, p. 247) $e/(e-1)$ times

$$1 + \mathbb{E} \left(\hat{S}_{T+u}(n) \log \hat{S}_{T+u}(n) \log \left(\hat{S}_{T+u}(n) \log \hat{S}_{T+u}(n) \mathbf{1}_{\{\hat{S}_{T+u}(n) \log \hat{S}_{T+u}(n) \geq 1\}} \right) \right)$$

(instead of (8.21)). This is bounded by a multiple of $\mathbb{E}(\hat{S}_{T+u}(n)(\log \hat{S}_{T+u}(n))^2)$ and hence can be dealt with as before.

Finally, in (8.5), we deal with $\mathbb{E}S_\tau^m(n)\mathbf{1}_{\{|S_{\tau+s}(n)-S_\tau(n)|>\eta\}}$ in the same way as we handled (8.19). \square

Proof of Lemma 4.2. That $\mathbb{E}S_\tau^m(n)$ is finite and uniformly bounded in n follows from a variant of the argument in (8.20), while (4.11) follows from (8.20) with $u = 0$, and the subsequent bounds, after replacing $S_\tau(n)$ with $\hat{S}_\tau(n)$, as in the proof of Lemma 4.1. We omit further details. \square

Proof of Theorem 4.3. We assumed in (4.1) that $S_0(n) \xrightarrow{D} S_0$, which random variables are independent of $L_t(n)$ and L_t , so $S_t(n) = S_0(n)e^{L_t(n)} \Rightarrow S_0e^{L_t} = S_t$ in $D[0, T]$ follows immediately from (3.6). This further implies the convergence of $g(S_t(n))$ to $g(S_t)$ as $n \rightarrow \infty$ in $D[0, T]$ since $g(\cdot)$ is continuous by (4.6).

The functional convergence of $\pi_t(n)$ in the MZ topology then follows from Theorem 3.5 and Example 4.3 of Mulinacci and Pratelli (1998). The latter requires a setting where $S_t = S_0 \mathcal{E}(X_t)$ and $S_t(n) = S_0(n) \mathcal{E}(X_t(n))$, where X_t and $X_t(n)$ have independent increments, and \mathcal{E} denotes the stochastic exponential. To see that this representation obtains in our model, viz, with $S_t = S_0 e^{L_t}$ and $S_t(n) = S_0(n) e^{L_t(n)}$, where L_t is a Lévy process, use Itô's formula, see, e.g. Protter (2003, p. 78), to write

$$\frac{dS_t}{S_{t-}} = dL_t + \frac{1}{2} d[L, L]_t^c + (e^{\Delta L_t} - 1 - \Delta L_t).$$

Equating this to dX_t defines the required Lévy process X_t in the continuous time setting. Similarly we define the required random walk $X(n)$ in discrete time.

In such a setting, convergence in the Meyer-Zheng topology holds, i.e., Theorem 3.5 of Mulinacci and Pratelli (1998) applies, if $g(S_t(n)) \Rightarrow g(S_t)$ in $D[0, T]$, as we established above, if the tightness criterion (4.10) is satisfied, and if the discounted payoff process $e^{-rt}g(S_\tau(n))$ is uniformly integrable (these are conditions (3.2) and (3.1), resp., of Mulinacci and Pratelli (1998), in our setup). Lemma 4.1 shows that tightness holds for $g(S_\tau(n))$ and Lemma 4.2 establishes the uniform integrability of $S_\tau^m(n)$ and hence of $g(S_\tau(n))$, and these conditions transfer immediately to the discounted processes. This proves the convergence of the discounted option price process in the Meyer-Zheng topology. Finite dimensional convergence of the $\pi_t(n)$ to π_t almost everywhere in $[0, T]$ follows directly from Theorem 5 of Meyer and Zheng (1984). \square

Proof of Corollary 4.4. This is a direct consequence of Theorem 4.3. \square

References

- [1] Albanese C., and A. Kuznetsov (2003) Discretization schemes for subordinated processes. Forthcoming: *Mathematical Finance*.???
- [2] Amin, K. (1993) Jump Diffusion Option Valuation in Discrete Time. *Journal of Finance* **48**, 1833–1863.
- [3] Amin K., and A. Khanna (1994) Convergence of American option values from discrete to continuous-time financial markets. *Mathematical Finance* **4**, 289–304.
- [4] Asmussen, S., and J. Rosinski (2001) Approximations of small jumps of Levy processes with a view towards simulation. *Journal of Applied Probability* **38**, 482–493.
- [5] Barndorff-Nielsen, O.E. (1998) Processes of normal inverse Gaussian type. *Finance and Stochastics* **2**, 41–68.
- [6] Bertoin, J. (1996) *Lévy Processes*. Cambridge University Press, Cambridge.
- [7] Bingham, N.H., and Kiesel, R. (2004) *Risk-Neutral Valuation*, 2nd Ed., Springer, Berlin.
- [8] Black, F., and M. Scholes (1973) The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637–654.
- [9] Chow, Y.S., and H. Teicher (1988) *Probability Theory: Independence, Interchangeability, Martingales*. Springer, Heidelberg.
- [10] Cont, R., and E. Voltchkova (2005) Integro-differential equations for option prices in exponential Lévy models. *Finance & Stochastics* **9**, 299–325.
- [11] Cox, J.C., S.A. Ross, and M. Rubinstein (1979) Option pricing: A simplified approach. *Journal of Financial Economics* **3**, 125–144.
- [12] Duffie, D. (1988) *Security Markets: Stochastic Models*. Academic Press, NY.
- [13] Durrett, R., and S. Resnick (1978) Functional limit theorems for dependent random variables. *Annals of Probability* **6**, 829–846.
- [14] Ethier and Kurtz (1986) *Markov Processes*, Wiley, NY.
- [15] Fu, M.C., Laprise, S.B., Madan, D.B., Su, Y., and Wu, R. (2001) Pricing American options: a comparison of Monte Carlo simulation approaches. *Journal of Computational Finance* **2**, 62–73.
- [16] Glasserman, P. (2003) *Monte Carlo Methods in Financial Engineering*. Springer, New York.

- [17] Harrison, M., and S. Pliska (1981) Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications* **11**, 215–260.
- [18] Hirs A., and D. Madan (2003) Pricing American options under variance-gamma. Forthcoming: *Journal of Computational Finance*.
- [19] Jacod, J. and A.N. Shiryaev (1987) *Limit Theorems for Stochastic Processes*. Springer, Heidelberg.
- [20] K ellezi, E., and N. Webber (2004) Valuing Bermudan options when asset returns are L evy processes. *Quantitative Finance* **4**, 87–100.
- [21] Lamberton, D. (1998) Error estimates for the binomial approximation of American put options. *Annals of Applied Probability* **8**, 206–233.
- [22] Madan, D.B., P.P. Carr, and E.C. Chang (1998) The Variance Gamma process and option pricing. *European Finance Review* **2**, 79–105.
- [23] Matache, A.-M., P.A. Nitsche, and C. Schwab (2005) Wavelet Galerkin pricing of American options on L evy driven assets. *Quantitative Finance* **5**, 403–424.
- [24] Merton, R.C. (1976) Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* **3**, 125–144.
- [25] Meyer, P.A., and W.A. Zheng (1984) Tightness criteria for laws of semimartingales. *Ann. Inst. Henri Poincar e* **20**, 353–372.
- [26] Mulinacci, S. (1996) An Approximation of American option prices in a jump-diffusion model. *Stochastic Processes and their Applications* **62**, 1–17.
- [27] Mulinacci, S., and M. Pratelli (1998) Functional convergence of Snell envelopes: Application to American options approximations. *Finance & Stochastics* **2**, 311–327.
- [28] Neveu, J. (1975) *Discrete Parameter Martingales*. North-Holland, Amsterdam.
- [29] Petrov, V.V. (1975) *Sums of Independent Random Variables*. Springer, Heidelberg.
- [30] Protter, P. (1990) *Stochastic Integration and Differential Equations: A New Approach*. Springer, New York.
- [31] Sato, K. (1999) *L evy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.