The interest sensitivity of wealth in the life cycle model

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Received 20 December 1993; accepted 21 March 1994

Abstract

This paper examines Modigliani's (American Economic Review, 1986, 76, 297–313) conjecture that in the simple life cycle model with Leontief consumption preferences, an increase in the interest rate from zero will cause aggregate wealth to decline. I find that the conjecture does not hold in general, but depends on the age of retirement.

JEL classification: E21

1. Introduction

In his 1986 Nobel Prize address, Modigliani asserted that in the simplest version of the life cycle model, with constant wages, zero population growth, and Leontief preferences, increasing the interest rate from 0% causes aggregate wealth to decline. I show that while this is the case, as conjectured, when retirement age is close to life expectancy, it does not hold in general. If retirement occurs sufficiently early in an individual's lifetime, the conjecture is overturned.

It is well known that in this model the interest elasticity of consumption is positive.1 In order to satisfy the budget constraint, the consumer must equate the present values of lifetime consumption and income. With consumption and income assumed constant during the periods in which they occur, and assuming a period of retirement in which no income is earned, the duration of consumption will be greater than that of income. So following an increase in the interest rate, the present value of future income falls, but the present value of future consumption falls by a greater percentage. The consumption path must therefore rise to re-equate the budget constraint.

The interest rate's effect on wealth in this life cycle economy is somewhat more complicated. If we refer to the previously described effect as the 'consumption channel' of

1 The elasticity of consumption is defined as ΔC/Δr|(r + Δr)/(C + ΔC)|.
interest rates, we may refer to another as the ‘compounding channel’. Higher interest rates obviously cause saved assets to throw off more income. So while higher interest rates cause a lower savings–income ratio, they cause those assets that are saved to grow faster. I find that either effect can dominate the other depending on the parameter values chosen.

2. A model of wealth

Born with zero wealth and no bequest motive, the consumer’s problem is to maximize discounted lifetime utility,

\[ U = \int_{0}^{L} u(C_t) e^{-\theta t} dt, \]

subject to a budget constraint

\[ \int_{0}^{T} Y_t e^{-r t} dt \geq \int_{0}^{L} C_t e^{-r t} dt, \]

where \( C_t \) and \( Y_t \) are consumption and non-property income at time \( t \), \( \theta \) is the consumer’s discount rate, \( r \) is the interest rate, \( T \) is the age of retirement, and \( L \) is the life span. In a perfect capital market, consumption is thus independent of current income, and depends only on the present value of lifetime income.

If we assume that non-property income is constant at \( \bar{Y} \) until retirement and that utility is CRRA with elasticity of intertemporal substitution \( \sigma \), consumption at age \( t \) is given by

\[ C_t = \frac{\bar{Y}}{r} \left(1 - e^{-rT}\right) \left(\frac{r - \sigma(r - \theta)}{1 - e^{-(r-\sigma(r-\theta))L}}\right)(e^{\sigma(r-\theta)Y}). \]

In the Leontief case, individuals wish to consume the same amount at every age. Therefore an increase in the return on savings will not induce a postponement of consumption, but rather raise the amount consumed at every instant. Defining \( \bar{C} = \lim_{\sigma \to 0} C_t \), we find the simplified expression for the constant level of consumption:

\[ \bar{C} = \begin{cases} 
1 - e^{-rT} \bar{Y}, & \text{if } r \neq 0, \text{ or} \\
\frac{T}{L} \bar{Y}, & \text{if } r = 0.
\end{cases} \]

Note that the discount rate parameter drops out. We now find the derivative of this consumption level with respect to the interest rate,

\[ \frac{d\bar{C}}{d r} = \frac{L(1 - e^{rT}) - T(1 - e^{rL})}{e^{r(T + L)}(1 - e^{-rL})^2} \bar{Y}, \]
which, as expected, can be shown to be positive over all $r \neq 0$, and equal to $(L - T)T/2L$ as $r$ approaches zero.\footnote{The expression is positive if $L - e^{-r}L - T + e^{\epsilon T}T > 0$. Let $Q = L - e^{-r}L - T + e^{\epsilon T}T$. So $dQ/dr = TL(e^{\epsilon T} - e^{-r})$. Since $dQ/dr > 0$ when $r > 0$, $dQ/dr < 0$ when $r < 0$, and $\lim_{r \to 0} Q = 0$, $Q$ is positive as long as $r \neq 0$.}

Eq. (6) gives the law of motion for assets $A_t$:
\[
d\frac{A_t}{dt} = Y_t - \bar{C} + rA_t. \tag{6}
\]

Remembering that $Y_t = \bar{Y}$ for $t \leq T$ and $Y_t = 0$ for $t > T$, we find that an individual with no assets at time zero ($A_0 = 0$), facing a non-zero interest rate, will hold assets as follows:
\[
A_t = \begin{cases} 
(\bar{Y} - \bar{C})(-1 + e^{\epsilon T}), & \text{if } t \leq T, \\
A_T e^{\epsilon(t - T)} + \bar{C}(1 + e^{\epsilon(t - T)}), & \text{if } t > T.
\end{cases} \tag{7}
\]

If the interest rate is zero, the asset function simplifies somewhat:
\[
A_t = \begin{cases} 
(\bar{Y} - \bar{C})t, & \text{if } t \leq T, \\
A_T - \bar{C}(t - T), & \text{if } t > T.
\end{cases} \tag{8}
\]

In an economy with a stationary population and no wage growth (either within lifetimes or across generations), aggregate wealth is proportional to the integral of individual assets over the average life span. So
\[
W = k \int_0^L A_t dt, \tag{9}
\]

where $k$ is population size. Substituting in Eqs. (4) and (7), we can now write per capita aggregate wealth as
\[
W = \frac{L(1 - e^{-rT}) - T(1 - e^{-T})}{r(1 - e^{-rT})} \bar{Y}. \tag{10}
\]

Tables 1 and 2 present the percent increases in consumption and wealth that result from a one percentage point increase in the interest rate, assuming a life span of 50 years. The interest rate shown on the left is the rate prior to the 1% increase.

Table 1 verifies the positive relationship between the interest rate and consumption. Over the range of $r$ and $T$ shown, increasing the interest rate by a point can raise consumption by as much as 50% at every moment. The magnitude of some of these numbers can be explained by the fact that when the interest rate is negative, consumption as a fraction of pre-retirement wages can be very low. Thus a small increase in consumption resulting from a rate increase may be substantial relative to the old level of consumption. For example, when $r$ is $-10\%$ ($-0.10$) and $T = 30$, an individual's consumption is just 13% of the pre-retirement wage. If $r$
Table 1
Percent increases in consumption resulting from a one percentage point increase in the interest rate

<table>
<thead>
<tr>
<th>Interest rate</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20%</td>
<td>52.20</td>
<td>46.70</td>
<td>40.70</td>
<td>34.40</td>
<td>28.20</td>
<td>22.00</td>
<td>16.10</td>
<td>10.50</td>
<td>5.12</td>
</tr>
<tr>
<td>-15%</td>
<td>49.70</td>
<td>44.70</td>
<td>39.30</td>
<td>33.50</td>
<td>27.60</td>
<td>21.70</td>
<td>16.00</td>
<td>10.40</td>
<td>5.09</td>
</tr>
<tr>
<td>-10%</td>
<td>45.10</td>
<td>40.70</td>
<td>35.90</td>
<td>30.90</td>
<td>25.70</td>
<td>20.40</td>
<td>15.20</td>
<td>9.99</td>
<td>4.92</td>
</tr>
<tr>
<td>-5%</td>
<td>36.40</td>
<td>32.70</td>
<td>28.80</td>
<td>24.80</td>
<td>20.80</td>
<td>16.60</td>
<td>12.50</td>
<td>8.29</td>
<td>4.13</td>
</tr>
<tr>
<td>0%</td>
<td>24.00</td>
<td>20.90</td>
<td>18.00</td>
<td>15.20</td>
<td>12.40</td>
<td>9.78</td>
<td>7.22</td>
<td>4.73</td>
<td>2.33</td>
</tr>
<tr>
<td>5%</td>
<td>13.20</td>
<td>10.80</td>
<td>8.65</td>
<td>6.79</td>
<td>5.18</td>
<td>3.79</td>
<td>2.60</td>
<td>1.59</td>
<td>0.73</td>
</tr>
<tr>
<td>10%</td>
<td>7.23</td>
<td>5.26</td>
<td>3.72</td>
<td>2.56</td>
<td>1.71</td>
<td>1.09</td>
<td>0.65</td>
<td>0.35</td>
<td>0.14</td>
</tr>
<tr>
<td>15%</td>
<td>4.34</td>
<td>2.71</td>
<td>1.62</td>
<td>0.93</td>
<td>0.51</td>
<td>0.27</td>
<td>0.13</td>
<td>0.06</td>
<td>0.02</td>
</tr>
<tr>
<td>20%</td>
<td>2.84</td>
<td>1.49</td>
<td>0.73</td>
<td>0.34</td>
<td>0.15</td>
<td>0.06</td>
<td>0.03</td>
<td>0.01</td>
<td>0.00</td>
</tr>
</tbody>
</table>

*Note:* This table assumes a life span of 50 years.

Rises just one percentage point, consumption increases to 15.6% of the pre-retirement wage, a 20% increase.

The pattern also reflects that as retirement age $T$ gets closer to zero, an individual consumes less and less out of current income, and thus becomes increasingly vulnerable to interest rate fluctuations. As individuals work longer and finance their consumption more out of current income, interest rates become less important. In the extreme case where the entire life is spent working, consumption is immune to interest rate effects.

In Table 2 we see the effect that raising the interest rate has on aggregate wealth. We can explain the variation within the table by separating this effect into the consumption channel and the compounding channel.

The consumption channel, which dominates at higher retirement ages and interest rates, is based on the earlier conclusion that raising the interest rate increases consumption and thus lowers savings at every age. When the interest rate is high and the retirement age is high, as

Table 2
Percent increases in aggregate wealth resulting from a one percentage point increase in the interest rate

<table>
<thead>
<tr>
<th>Interest rate</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20%</td>
<td>5.22</td>
<td>5.19</td>
<td>5.14</td>
<td>5.04</td>
<td>4.86</td>
<td>4.53</td>
<td>3.96</td>
<td>3.01</td>
<td>1.54</td>
</tr>
<tr>
<td>-15%</td>
<td>6.81</td>
<td>6.68</td>
<td>6.47</td>
<td>6.17</td>
<td>5.72</td>
<td>5.06</td>
<td>4.13</td>
<td>2.84</td>
<td>1.12</td>
</tr>
<tr>
<td>-10%</td>
<td>8.81</td>
<td>8.31</td>
<td>7.70</td>
<td>6.94</td>
<td>6.01</td>
<td>4.86</td>
<td>3.49</td>
<td>1.87</td>
<td>-0.03</td>
</tr>
<tr>
<td>-5%</td>
<td>9.49</td>
<td>8.30</td>
<td>7.02</td>
<td>5.64</td>
<td>4.16</td>
<td>2.57</td>
<td>0.89</td>
<td>-0.90</td>
<td>-2.79</td>
</tr>
<tr>
<td>0%</td>
<td>6.45</td>
<td>4.64</td>
<td>2.88</td>
<td>1.16</td>
<td>-0.52</td>
<td>-2.15</td>
<td>-3.75</td>
<td>-5.30</td>
<td>-6.82</td>
</tr>
<tr>
<td>5%</td>
<td>2.12</td>
<td>0.16</td>
<td>-1.59</td>
<td>-3.16</td>
<td>-4.56</td>
<td>-5.81</td>
<td>-6.92</td>
<td>-7.90</td>
<td>-8.77</td>
</tr>
<tr>
<td>10%</td>
<td>-0.30</td>
<td>-2.17</td>
<td>-3.60</td>
<td>-4.78</td>
<td>-5.71</td>
<td>-6.44</td>
<td>-7.00</td>
<td>-7.43</td>
<td>-7.77</td>
</tr>
<tr>
<td>15%</td>
<td>-1.23</td>
<td>-2.83</td>
<td>-3.97</td>
<td>-4.75</td>
<td>-5.27</td>
<td>-5.61</td>
<td>-5.83</td>
<td>-5.97</td>
<td>-6.05</td>
</tr>
<tr>
<td>20%</td>
<td>-1.55</td>
<td>-2.92</td>
<td>-3.75</td>
<td>-4.22</td>
<td>-4.48</td>
<td>-4.61</td>
<td>-4.68</td>
<td>-4.72</td>
<td>-4.74</td>
</tr>
</tbody>
</table>

*Note:* This table assumes a life span of 50 years.
seen in Table 1, a small change in interest rates will have only a small percent effect on consumption. This is due to the fact that the fixed level of consumption at these parameter values is very close to its global maximum, $\bar{Y}$. Since savings is therefore just as close to its global minimum, 0, even a small rate-induced decrease in savings can change the savings to income ratio dramatically, affecting aggregate wealth significantly.

The compounding channel of interest rates dominates in the opposite case. When retirement is early and interest rates are low, the greater part of income must be saved in order to maintain a fixed consumption path through retirement. This high savings to income ratio will not be changed dramatically by a small change in the interest rate. As the retirement age $T$ gets closer to zero, assets remain idle for longer periods of time and are subject to substantial appreciation or depreciation, thus affecting aggregate wealth noticeably.

Modigliani’s conjecture about the negative effect of the interest rate on wealth, however, was restricted to the case at $r = 0$ in which the interest rate is then increased. To verify this claim, the derivative of wealth with respect to the real interest rate must be calculated:

$$\frac{dW}{dr} = \bar{Y} \left[ \frac{T - L}{r^2} \right] + \frac{\bar{Y}L}{r} \left[ \frac{Te^{-rT} - Te^{-rL} - L(1 - e^{-rT})(1 - e^{-rL})}{(1 - e^{-rT})^2} \right].$$

We can calculate the limit of $\frac{dW}{dr}$ as $r \to 0$:

$$\lim_{r \to 0} \frac{dW}{dr} = \frac{(L - 2T)(L - T)T}{12} \bar{Y}.$$

Aside from the extremes $T = 0$ or $T = L$ where saving is non-existent, the sign of the derivative of wealth with respect to the interest rate, and thus the sign of the interest elasticity of wealth, depend solely on the ratio $T/L$, the fraction of life worked. If this fraction is greater than one-half, then the effect of the increase in consumption outweighs the higher interest earned on savings, and the derivative is negative. A fraction smaller than a half implies a positive derivative, because the compounding effect dominates. If the fraction of life worked is exactly one-half, these effects cancel each other out and the derivative is zero. Modigliani’s statement that wealth will decrease following an increase in the interest rate from 0% is therefore not valid in all cases.

Acknowledgements

The opinions expressed are those of the author and do not necessarily reflect the views of the Board of Governors of the Federal Reserve System. I wish to thank Athanasios Orphanides and David Wilcox for helpful comments.

Reference