SUMMARY

"space"

CURVE in $\mathbb{R}^3$.

PARAMETRIZATION

of curve.

- For each curve, there are many parametrizations. (Infinitely many).

- For each vector function, there is only one corresponding curve.

- If $\vec{F}(t)$ is a vector function, $\vec{F}'(t)$ is

- If $\vec{F}(t)$ describes the position of a particle, $\vec{F}'(t)$ is its...
limits of Vector Functions.

Definition a) If \( \mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle \),
\[
\lim_{{t \to a}} \mathbf{r}(t) = \langle \lim_{{t \to a}} r_1(t), \lim_{{t \to a}} r_2(t), \lim_{{t \to a}} r_3(t) \rangle.
\]
provided each of these limits exist.

b) A vector function \( \mathbf{r}(t) \) is continuous at \( t = a \) if
\[
\lim_{{t \to a}} \mathbf{r}(t) = \mathbf{r}(a).
\]

Examples:

\begin{align*}
\text{continuous:} & & \text{discontinuous:} \\
\begin{array}{c}
\text{Curve is cont. but the parametrization is not.}
\end{array}
\end{align*}
Example.

\[ \mathbf{v}(t) = \begin{cases} 
  \langle 1, 1, t \rangle & t < 0, \\
  \langle 1, 1, t-1 \rangle & t \geq 0. 
\end{cases} \]
Derivatives of Vector-Valued Functions.

Definition: The derivative of a vector-value function \( \mathbf{r}(t) \) is

\[
\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},
\]

whenever this limit exists.

Note: as long as each component \( r_1(t), r_2(t), r_3(t) \) of \( \mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle \) is differentiable,

\[
\mathbf{r}'(t) = \langle r_1'(t), r_2'(t), r_3'(t) \rangle.
\]

e.g., \( \mathbf{r}(t) = \langle t^2, \sin t, e^{-t} \rangle \).

\[
\Rightarrow \mathbf{r}'(t) = \langle 2t, \cos t, -e^{-t} \rangle.
\]

The Picture:
Example. Find $\mathbf{r}'(t)$ if

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.$$ 

$$\Rightarrow \mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle.$$ 

**Sketch:**

At $t = 0$, 
$$\mathbf{r}'(0) = \langle 0, 1, 1 \rangle$$

At $t = \pi$, 
$$\mathbf{r}'(\pi) = \langle 0, -1, 1 \rangle$$
* Geometrically, \( \vec{r}'(t) \) is the tangent vector to the curve given by \( \vec{r}(t) \).

**Curriculum Question #3**

If \( \vec{r}(t) \) is the position of a bee at time \( t \), what is \( \vec{r}'(t) \)?

(A) velocity
(B) speed: \( |\vec{r}'(t)| \)
(C) acceleration: This would be \( \vec{r}''(t) \).
(D) other: \( \vec{r}''(t) \).
INTEGRALS OF VECTOR-VALUED FUNCTIONS.

Example:
Suppose that a bee flies with velocity,

\[ \vec{v}(t) = \langle e^{t^2}, \sin(\pi t), \cos(\pi t) \rangle \]

(c2) What is the bee's position at \( t = 0 \)?

A) \( \langle 0, 0, 0 \rangle \)

B) \( \langle 1, 0, 1 \rangle \) x \( \vec{v}(0) = \langle 1, 0, 1 \rangle \)

C) I don't know.

D) Other: ____________________________

If the bee's position is \( \langle 1, 0, 1 \rangle \) at \( t = 0 \),

where is it at \( t = \pi \) s?

E.g. Consider its x-coordinate.
In general, we see that we can integrate vector-valued functions coordinate by coordinate,

\[
\int_a^b \langle r_1(t), r_2(t), r_3(t) \rangle \, dt = \\
\langle \int_a^b r_1(t) \, dt, \int_a^b r_2(t) \, dt, \int_a^b r_3(t) \, dt \rangle.
\]

We can then write the fundamental theorem of calculus for vector-valued functions,

\[
\vec{r}(b) - \vec{r}(a) = \int_a^b \vec{r}'(t) \, dt
\]

Example:

Compare the following vector function.

\[
\vec{r}(t) = \langle e^t, 2e^t, e^t - 1 \rangle
\]

Which of the following best represents the corresponding "space curve"?

[Diagram with options A and B]