11.7 Maximum and Minimum Values.

At any point \((a, b)\) where \(f(x, y)\) achieves a local maximum or minimum value, 
\[ \frac{\partial f}{\partial x}(a, b) = 0 \text{ and } \frac{\partial f}{\partial y}(a, b) = 0, \]
provided that \(\frac{\partial f}{\partial x}(a, b)\) and \(\frac{\partial f}{\partial y}(a, b)\) both exist.

DEF. A **critical point** for a function \(f\) is a point \((a, b)\) that satisfies either both 
\[ \frac{\partial f}{\partial x}(a, b) = 0 \text{ and } \frac{\partial f}{\partial y}(a, b) = 0, \]

or at least one of \(\frac{\partial f}{\partial x}(a, b)\) or \(\frac{\partial f}{\partial y}(a, b)\) does not exist.

**Local Max/Min** \(\Rightarrow\) \((a, b)\) is a critical point.

\(\text{Local Max/Min at } (a, b)\)
(a) \( \text{Example (from an old midterm 2)} \)

\[
\text{let } f(x,y) = y(4-x^2-y^2) \\
= 4y - yx^2 - y^3
\]

(a) Find all critical points.

\[
\nabla f = \langle -2xy, 4 - x^2 - 3y^2 \rangle = \langle f_x, f_y \rangle
\]

at a critical point \((x,y)\), \(\nabla f(x,y) = 0,0\)

\[
\therefore \begin{cases} \quad -2xy = 0 \quad (1) \\
4-x^2-3y^2 = 0 \quad (2) \end{cases}
\]

From (1), \(x = 0\) or \(y = 0\).

If \(x = 0\),

\[
(2) \Rightarrow 4 - 3y^2 = 0 \\
\Rightarrow 4 = 3y^2 \\
\Rightarrow y = \pm \frac{2}{\sqrt{3}}
\]

\[\therefore \text{The critical points are } \left(0, \frac{2}{\sqrt{3}}\right), \left(0, -\frac{2}{\sqrt{3}}\right), \left(2, 0\right) \text{ and } \left(-2, 0\right).\]
Each critical point could be a local maximum, local minimum, or neither.

Examples:

local max    local min    saddle point

How can we mathematically distinguish these different cases?

SECOND DERIVATIVES TEST:

Suppose that \((a, b)\) is a critical point for a smooth function \(f\).

\[
D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2
\]

(a) If \(D(a, b) > 0\) and \(f_{xx}(a, b) > 0\), then \(f\) has a local MINIMUM at \((a, b)\).
b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f$ has a local **maximum** at $(a,b)$.

c) If $D(a,b) < 0$, then $f$ has a saddle point at $(a,b)$.

**Summarize with a chart:**

<table>
<thead>
<tr>
<th>$D$</th>
<th>$f_{xx}$</th>
<th>Behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+$</td>
<td>$+$</td>
<td>local min</td>
</tr>
<tr>
<td>$+$</td>
<td>$-$</td>
<td>local max</td>
</tr>
<tr>
<td>$-$</td>
<td>n/a</td>
<td>saddle</td>
</tr>
</tbody>
</table>

**Notes:**

1. $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$

2. If $D = 0$, the second derivative test is indeterminate.
Example (b)

Use the second derivative test to classify the critical points of the function.

\[ f(x, y) = y(4 - x^2 - y^2) \]

\[
\begin{align*}
f_{xx}(x, y) &= -2y \\
f_{xy}(x, y) &= -2x \\
f_{yy}(x, y) &= -2y
\end{align*}
\]

\[
\begin{align*}
f_{yy} &= \frac{\partial}{\partial y} f_x \\
&= \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left( y(4 - x^2 - y^2) \right) \right) \\
&= \frac{\partial}{\partial y} \left( -2xy \right) \\
&= -2x
\end{align*}
\]

\[
\begin{align*}
D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\
&= (-2y)(-2y) - (-2x)^2 \\
&= 4y^2 - 4x^2 = 4(3y^2 - x^2)
\end{align*}
\]

At \((0, \pm \frac{2}{\sqrt{3}})\),

\[
D(0, \pm \frac{2}{\sqrt{3}}) = 4 \left( 3 \left( \pm \frac{2}{\sqrt{3}} \right)^2 - 0^2 \right) > 0
\]

\[
f_{xx}(0, \frac{2}{\sqrt{3}}) = -2 \left( \frac{2}{\sqrt{3}} \right) < 0
\]

\(\Rightarrow\) \(f\) has a local max at \((0, \frac{2}{\sqrt{3}})\)

\[
f_{xx}(0, -\frac{2}{\sqrt{3}}) = -2 \left( -\frac{2}{\sqrt{3}} \right) > 0
\]

\(\Rightarrow\) \(f\) has a local min at \((0, -\frac{2}{\sqrt{3}})\).

At \((\pm 2, 0)\),

\[
D(\pm 2, 0) = 4 \left( 3(0)^2 - (\pm 2)^2 \right) < 0
\]

\(\Rightarrow\) \(f\) has saddle points at both \((2, 0)\) and \((-2, 0)\).
ABSOLUTE/Global Extrema

Single-variable case

\[ z = f(x(t)) = g(t) \]

"Closed interval method"

Theorem:

A continuous function on a closed bounded set \( D \) in \( \mathbb{R}^2 \) must have an absolute maximum and absolute minimum in \( D \).

How can we find the absolute maximum and minimum values of \( f \) on \( D \)?

1. Find all critical points in \( D \) and test the function values there.
2. Find extreme values on the boundary of \( D \).
3. Compare all function values at critical points and boundary points.
Examples.

\( f(x, y) = x^2 + y^2 - x + 1. \)

Find the absolute maximum and minimum values of \( f \) on the closed disk \( x^2 + y^2 \leq 1. \)
Examples (cont'd)

2. Find the absolute maximum and minimum values of
\[ f(x,y) = x^2 - 2xy + 2y \]
on the set
\[ D = \{(x,y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}. \]