Last day - 2 Big Questions.

1. What is the slope of \( z = f(x,y) \) at the point \( (x_0, y_0) \) in the direction \( \hat{u} = \langle a, b \rangle \).

\[
D_\hat{u} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.
\]

\[
= \nabla f(x_0, y_0) \cdot \hat{u}
\]

Example:

C1. Let \( f(x,y) = x^2y^3 \).

Find the slope of the graph \( z = f(x,y) \) at the point \( (x_0, y_0) = (1,1) \) in the direction of \( \langle 1,2 \rangle \).

(A) \( \langle 2,3 \rangle \)

(B) \( \langle 1,0 \rangle \)

(C) \( \varnothing \)

(D) \( \varnothing / \sqrt{5} \)

(E) Other:

\[ D_\hat{u} f(1,1) = \nabla f(1,1) \cdot \hat{u} \]

\[
\hat{u} = \langle 1,2 \rangle \cdot \langle 1,2 \rangle = \frac{1}{\sqrt{5}}
\]

\[
\nabla f(1,1) = \langle 2,3 \rangle.
\]

Solution:

\[ \langle 2,3 \rangle \cdot \frac{1}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \]

\[ \hat{u} = \frac{\langle 1,2 \rangle}{\sqrt{5}} \]
**BIG QUESTION #2:** At any given point \((x_0, y_0)\), in which direction is the graph \(z = f(x, y)\) the steepest?

we investigate this question using a couple of different approaches...

**EXHIBIT A:** (Looking at the graph \(z = f(x, y)\) (the surface), we noticed the direction maximum increase at \((x_0, y_0)\) was orthogonal to the level curve at that point.

**EXHIBIT B:** Here's an example where we have \(z = f(x, y)\) and the contour map of a function only.

At various points \((x_0, y_0)\), sketch a vector that points in the direction of maximum increase.
EXHIBIT C: Use the formula \( \mathbf{a} = \nabla f \cdot \mathbf{u} \) from last day.

For each point \((x_0, y_0)\), we are seeking for the direction \(\mathbf{u}\) such that \(\nabla f(x_0, y_0)\) is the largest.

\[
\nabla f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}
\]

This is maximized when
\( \mathbf{u} \parallel \nabla f(x_0, y_0) \).

\[
\nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta
\]

\[
= |\nabla f| \cos \theta \cdot \text{largest when } \theta = 0.
\]

SIGNIFICANCE OF THE GRADIENT VECTOR:

1. \(\nabla f(x,y)\) points in the direction of maximum increase of \(f\) at the point \((x,y)\).

2. \(|\nabla f(x,y)|\) is the maximum value of the directional derivative at \(x,y\).

3. \(\nabla f(x,y)\) is orthogonal to the level curve of \(f\) passing through \((x,y)\).
1. The height $z$ of a mountain above the point $(x, y)$ is given by

$$z = f(x, y) = x(2 - \sin y).$$

A goat stands on the mountain above the point $(1, \frac{\pi}{2})$.

(a) In what direction should the goat move to go up the mountain as fast as possible?

$$\nabla f(x, y) = \left< 2 - \sin y, -x \cos y \right>$$

$$\Rightarrow \nabla f(1, \frac{\pi}{2}) = \left< 2 - \sin(\frac{\pi}{2}), -\cos(\frac{\pi}{2}) \right>$$

$$= \left< 1, 0 \right>$$

(b) If the goat moves in the $(1, 1)$-direction with speed $1 \text{ m/s}$, how fast does its height change?

(c) In what direction(s) should the goat move to go uphill at a rate only half as steep as the steepest incline? (Give your answer(s) in the form of unit vector(s)).
Extend to Functions of 3 Variables.

For a function of 3 variables, \( F(x,y,z) \),

* the gradient vector is:
  \[ \nabla F = \left< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right> \]

* the level things are:
  A. curves
  B. surfaces: \( F(x,y,z) = k \)
  C. blobs
  D. other

The same facts from the previous page carry over to the 3D case.

1. \( \nabla F(x,y,z) \) points in the direction of maximum increase.

2. \( |\nabla F(x,y,z)| \) is the maximum value of the directional derivative.

3. \( \nabla F(x,y,z) \) is orthogonal to the level surface at \( (x,y,z) \).
Example.

a) Let

\[ F(x, y, z) = xyz^2 \]

Find the equation for the tangent plane to the level surface of \( F \) at \( (x, y, z) = (3, 2, 1) \).