Math 224e: Lecture #13

Plan:
- Functions of 2 variables
- Functions of 3 variables
- Limits of functions of 2 & 3 variables

Functions of 3 Variables:

Domain: a region in $\mathbb{R}^3$
Range: a region in $\mathbb{R}$

- To create a graph for such a function, we'd require 3 dimensions for the inputs and 1 dimension for the outputs: a total of 4 dimensions.

- Example: Temperature of each point in 3D space.
Example

Suppose that a small heater sits at a point (we call this origin), and the temperature at point \( (x,y,z) \) is

\[
T(x,y,z) = \frac{1}{x^2+y^2+z^2}
\]

C2) What are the level sets \( T(x,y,z) = k \)?

- (A) paraboloids
- (B) spheres
- (C) balls (solid)
- (D) other.

In general, for nice “smooth” functions \( f \) of 3 variables, the level sets

\[
f(x,y,z) = k \quad \text{(for constant } k)\]

are surfaces.

We call them “LEVEL SURFACES”
Example: \( f(x, y) = \frac{xy^2}{x^2 + y^4} \)

Goal: Find \( \lim_{(x, y) \rightarrow (0, 0)} \frac{xy^2}{x^2 + y^4} \) (if it exists)

**IDEAS**

How could \( (x, y) \) approach \((0, 0)\)?

Along \( y = 0 \), we take \( x \rightarrow 0 \)....

\[
\lim_{(x, y) \rightarrow (0, 0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \rightarrow 0} \frac{x(0)^2}{x^2 + (0)^4} = \lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = 0.
\]

Along \( x = 0 \),

\[
\lim_{(x, y) \rightarrow (0, 0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \rightarrow 0} \frac{(0)y^2}{0^2 + y^4} = \lim_{y \rightarrow 0} 0 = 0.
\]
we can approach along any other straight line, \( y = mx \), for some constant \( m \).

Then

\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \to 0} \frac{x(mx)^2}{x^2 + (mx)^4} = \lim_{x \to 0} \frac{m^2x^3}{x^2 + m^4x^4} = \lim_{x \to 0} \frac{m^2x}{1 + m^4x^2} = \frac{0}{0} = 0.
\]

Hence, if we approach \((0,0)\) along any straight line, we find a limit of \(0\).

**However...**

Consider approaching \((0,0)\) along the parabola \( x = y^2 \).

Then

\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \to 0} \frac{y^2 y^2}{y^4 + y^4} = \lim_{y \to 0} \frac{y^4}{2y^4} = \lim_{y \to 0} \frac{1}{2} = \frac{1}{2}.
\]
Therefore, \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \) DOES NOT EXIST.

What is going on here??

From above:

\[ z = \frac{1}{2} \]

Mountain range / ridge
* In order for a limit \( \lim_{(x,y) \to (a,b)} f(x,y) \) to exist, we must get the same answer along EVERY path.

* You can prove that the limit doesn't exist by finding different limits along two different paths.

* How can we prove that a limit DOES exist?

**Example:**

Prove that \( \lim_{(x,y) \to (0,0)} y^2 \sin \left( \frac{1}{x} \right) = 0 \)

**Proof:**

\[
\begin{align*}
-\frac{y^2}{2} & \leq y^2 \sin \left( \frac{1}{x} \right) \leq \frac{y^2}{2} , \\
\Rightarrow \quad \lim_{x \to 0} y^2 \sin \left( \frac{1}{x} \right) & = 0.
\end{align*}
\]

\[
\therefore \lim_{(x,y) \to (0,0)} y^2 \sin \left( \frac{1}{x} \right) = 0.
\]