Compositions of Functions (continued)

Examples:

a) \( f(x) = x^2 + \sqrt{x-1} \)
   \( g(x) = x + 2 \)

\[ f \circ g (x) = f(g(x)) = f(x+2) = (x+2)^2 + \sqrt{(x+2)-1} = (x+2)^2 + \sqrt{x+1} \]

\[ g \circ f (x) = g(f(x)) = g(x^2 + \sqrt{x-1}) = x^2 + \sqrt{x-1} + 2 \]

Domain of \( f \circ g \) = \( \{ x \in \mathbb{R} : x \geq 1 \} \)
Domain of \( g \circ f \) = \( \{ x \in \mathbb{R} : x \geq 1 \} \)

"Such that"
b) Consider \( h(x) = (\sqrt{x+1})^3 - \frac{1}{\sqrt{x+1}} \). 

Express \( h(x) \) as the composition of two functions: \( h = f \circ g \)

\[ f(x) = x^3 - \frac{1}{x} \]
\[ g(x) = \sqrt{x+1} \]

Then \( f(g(x)) = f(\sqrt{x+1}) = (\sqrt{x+1})^3 - \frac{1}{\sqrt{x+1}} \)

Another way of doing this...

\[ f(x) = (\sqrt{x})^3 - \frac{1}{\sqrt{x}} \]
\[ g(x) = x+1 \]

In this case, the domain of \( f \circ g \) was the set of all real numbers \( x \) such that:
- \( x \geq -1 \) and \( x \neq -1 \).

i.e. \( x \geq -1 \)

In general, the domain of \( f \circ g(x) = f(g(x)) \)

is the set of all real numbers \( x \) such that:
- \( x \) is in the domain of \( g \) and
- \( g(x) \) is in the domain of \( f \).
**Inverse Functions**

\[ \text{Caution: } f^{-1} \neq \frac{1}{f} \]

The "inverse" of a function \( f \) satisfies:

\[ f^{-1}(f(x)) = f(f^{-1}(x)) = x \quad \text{i.e. } \quad f^{-1}(y) = x \text{ whenever } f(x) = y \]

**Example.**

**Example 1** (let \( f(x) = x^3 + 1 \)).

Find a formula for \( f^{-1}(x) \) and sketch its graph.

Let:

\[ y = f(x) = x^3 + 1 \]

\[ y - 1 = x^3 \]

\[ \Rightarrow 3\sqrt{y-1} = x \]

\[ \Rightarrow x = f^{-1}(y) = 3\sqrt{y-1} \]

\[ \therefore f^{-1}(x) = 3\sqrt{x-1} \text{ for any } x \in \mathbb{R} \]

**Check:**

\[ f^{-1}(f(x)) = 3\sqrt{x^3+1-1} = 3\sqrt{x^3} = x \]

\[ f(f^{-1}(x)) = (3\sqrt{x-1})^3 + 1 = (3\sqrt{x-1})^3 + 1 \]

\[ = x - 1 + 1 = x \]

[Ref. 3]