How can we combine functions together to make new functions?

1. Sums of functions:
   - Let \( f \) and \( g \) be any two functions.
   - For any \( x \) in the domain of both \( f \) and \( g \),
     \[
     (f+g)(x) = f(x) + g(x)
     \]

Example:
Let \( f(x) = \frac{1}{x-2} \) and \( g(x) = \sqrt{x} \).
Then
\[
(f+g)(x) = f(x) + g(x) = \frac{1}{x-2} + \sqrt{x}
\]

what is the domain of \( f+g \)?
\[
\{ x \in \mathbb{R} : x \neq 2 \text{ and } x \geq 0 \} = [0, 2) \cup (2, \infty)
\]
\[(f+g)(x) = f(x) + g(x)\]

1. In general, the domain of \(f+g\) is the set of all numbers that are in both the domain of \(f\) and the domain of \(g\).
   
   \[D(f+g) = D(f) \cap D(g)\]

   Similarly, the difference \(f-g\) is also a function given by:
   
   \[(f-g)(x) = f(x) - g(x)\]

   Also, for \(a, b \in \mathbb{R}\):
   
   \[(af + bg)(x) = af(x) + bg(x)\]

2. **Products and Quotients of Functions.**

   - The product and quotient of \(f\) and \(g\) are defined by the rules:
     
     \[(fg)(x) = f(x)g(x)\quad \text{and}\quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\]

   - The domain of \(fg\) is the set of all real numbers \(x\) such that \(x\) is both in the domain of \(f\) and the domain of \(g\):
     
     \[D(fg) = D(f) \cap D(g)\]

   True or False: The domain of \(\frac{f}{g}\) is also the set of all real numbers in the domains of both \(f\) and \(g\). 
   
   False! We also require, for \(g(x) \neq 0\) \(\forall x\) in the domain of \(\frac{f}{g}\) is the set of all \(x\).
Such that \( x \) is in both the domains of \( f \) and \( g \) \( \text{AND} \ g(x) \neq 0 \).

**Example.**
what is the domain of \( \frac{f}{g} \), where \( f(x) = \sqrt{x} \) and \( g(x) = x^2 - 9 \)?

\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x^2 - 9}.
\]

The domain is the set of all \( x \) such that \( x > 0 \) \( \text{AND} \) \( x \in \mathbb{R} \) \( \text{AND} \) \( x \neq \pm 3 \). \( \text{domain of } f \) \( \text{domain of } g \) \( g(x) \neq 0 \).

i.e. \( x > 0 \) and \( x \neq 3 \)
i.e. \( [0, 3) \cup (3, \infty) \)

**Compositions of Functions.**

Let \( f \) and \( g \) be functions.

\[
\begin{array}{c}
\text{input} \\
\uparrow \\
\bigtriangleup
\end{array} \xrightarrow{g} \bigtriangleup \xrightarrow{g(x)} \bigtriangleup \xrightarrow{f} \bigtriangleup \xrightarrow{f(g(x))} \text{output}
\]

This is a function we call \( f \circ g \) "the composition of \( f \) and \( g \)."

**Real-life Examples.**
Let $x$ = mercury level in the algae.

$u = \text{the fish} = g(x)$

$y = \text{the human} = f(u) = f(g(x))$

We have

$y = f(g(x)) = f \circ g(x)$

Can you come up with a real-life example?

- **Russian nesting dolls**.

Examples: using our favourite functions from this class.

1. $f(x) = \text{ }$
   $g(x) = \text{ }$

2. $f(x) = \text{ }$
   $g(x) = \text{ }$