RAPID GRANULAR FLOWS

Charles S. Campbell

Department of Mechanical Engineering, University of Southern California, Los Angeles, California 90089-1453

INTRODUCTION

A granular material is a collection of a large number of discrete solid particles. Generally, the interstices between the particles are filled with a fluid such as air or water, and thus, technically, a granular flow is a multiphase process. However, if the particles are closely packed or if they are much denser than the interstitial fluid, the particles alone—and not the fluid or the fluid-particle interactions—will play the greatest role in momentum transport within the material, in which case the interstitial fluid can be ignored in describing the flow behavior. Granular-material flows are generally taken to fall into this limiting category and thus may be considered dispersed single-phase rather than multiphase flows. This review addresses the fluidlike behavior of granular solids and, in particular, those flows for which the material is rapidly sheared. I outline what is currently known about rapid granular flows, discuss the various modeling techniques used to describe the motion of the bulk material, and point out many questions that remain to be answered.

Depending on the local stress conditions, a granular material may behave as either an elastic solid or a fluid. When a granular material is showing its elastic-solid behavior, it can support the large loads of building foundations or form into hills with finite slopes. However, much of the load is supported across frictional bonds between the particles, and, as such, the system’s strength is limited to the loads that those bonds can support. When enough of the bonds have been overcome, the system will fail and begin to flow. The initial failure will consist of many-particle blocks moving relative to one another along shear bands that roughly follow stress characteristics through the material. If the motion occurs slowly, particles will stay in contact and interact frictionally with their neighbors over long periods of time. The failure will continue in this
manner as long as the deformation remains fairly slow. This is the "quasi-static" regime of granular flow and has been classically studied using modified plasticity models based on a Coulomb friction criterion.

At the other extreme is the rapid-flow regime, which, as the name implies, corresponds to high-speed flows, far beyond the initial failure. [Other recent reviews of the field have been compiled by Savage (1984, 1988b).] Instead of moving in many-particle blocks, each particle moves freely and independently of even its nearest neighbors. Such a condition evolves naturally from the quasi-static behavior if the motion is rapid enough to transfer enough energy to the particles that neighbor the slip lines to break them free of their parent blocks; in this way the slip regions grow until every particle in the granular mass is moving independently. In the rapid-flow regime, the velocity of each particle may be decomposed into a sum of the mean velocity of the bulk material and an apparently random component to describe the motion of the particle relative to the mean. The analogy between the random motion of the granular particles and the thermal motion of molecules in the kinetic-theory picture of gases is so strong that the mean-square value of the random velocities is commonly referred to as the "granular temperature" or, simply, "temperature"—a term first used by Ogawa (1978). (In this paper, the temperature is denoted \( T = \langle u'^2 \rangle \), where \( u' \) is the instantaneous deviation from the mean velocity, and \( \langle \cdot \rangle \) represents an appropriate average. Thus \( T \) represents twice the energy per unit mass contained in the random motions of the particles.) As is spelled out in the remainder of this paper, both the thermodynamic and granular temperatures play similar roles in governing the behavior of their respective systems. In particular, both generate pressures and govern the internal transport rates of mass, momentum, and energy. Thus, while the term "temperature" sometimes leads to some semantical confusion for the uninitiated, the physical analogy between the two temperatures is so apt that its use has become standard throughout the field.

Since collisions between macroscopic solid particles are inelastic, granular and molecular systems have many fundamental differences. In particular, the energy associated with the granular temperature is always being dissipated by interparticle collisions; left to itself, the granular temperature would quickly dissipate away to nothing. In fact, this is common experience; shaking up a granular material—for example, a bottle of aspirin or a box of cereal—induces random temperaturelike motions between the particles, but once the shaking stops, the particles will almost instantaneously lose their granular temperature and cease all of their motion. To maintain the temperature, energy must be continually added into the random components of velocity to balance that which is lost to the
dissipative collisions. Unlike a molecular system, the granular temperature comes about as a by-product of the flow itself and cannot be specified independently of the fluid mechanics, as one could fix the temperature of a molecular system by putting it in contact with a large isothermal reservoir.

Granular temperature can be generated by the two mechanisms. The first is a by-product of interparticle collisions. When two particles collide, their resultant velocities will depend not only on their initial velocities, but also on the angle at which the two particles collide, the surface friction at the point of contact, and other factors that may affect the geometry of the collision impulse. Thus, even if the particles initially move with the mean velocities appropriate to their positions within the velocity field, their resultant velocities will contain apparently random velocity components. Note that the magnitude of the random velocities generated in this mode will be proportional to the relative velocities of the particles at the time of impact and hence must, in an averaged sense, be proportional to the mean velocity gradient within the material. The second mode of temperature generation is itself a by-product of the random particle velocities. Following its random path, a particle moving parallel to the local velocity gradient will pick up an apparently random velocity that is roughly equal to the difference in the mean velocity between its present location and the point of its last collision. Note that like the first mode of temperature generation, the magnitude of the random velocities so generated will also be proportional to the local velocity gradient. However, unlike the collisional temperature generation, this “streaming” mechanism can only generate one component of random velocity—the component that lies in the direction perpendicular to the mean velocity gradient. Consequently, the temperature so generated will be anisotropic, with its largest component in the direction of mean motion (although subsequent collisions with other particles will tend to randomize the direction of the random velocities). Obviously, collisional temperature generation will dominate at dense particle packings when particles cannot move far between collisions, and streaming temperature generation will dominate at loose particle packings when collisions are infrequent. Thus, one might anticipate anisotropic temperatures for loose packings, just as are observed in the computer-simulation results of Campbell (1989), Campbell & Gong (1986), and Walton & Braun (1986a,b).

This concept of a granular temperature is perhaps the single most important key to understanding the behavior of rapid granular flows, and understanding the granular temperature in turn requires an understanding of the internal energy flow illustrated in Figure 1. The source of all the energy is the work done on the system from the outside, either by body
forces (such as gravity) or through the motion of the system boundaries. Shear work (the product of the shear stress and the velocity gradient) performed on the system converts some of the kinetic energy of the mean motion into granular temperature (which is the kinetic energy associated with the random particle velocities); this reflects the fact that both of the temperature mechanisms described above are related to the velocity gradient. Finally, the inelastic collisions between particles will dissipate away the granular temperature into real thermodynamic heat. The magnitude of the granular temperature thus depends on a trade-off between the temperature generation by shear work and its dissipation by collisional inelasticity.

While the above describes the global energy path through the granular system, the local value of the granular temperature may be affected by its magnitude in neighboring areas. This is a conductionlike process and occurs for physical reasons exactly analogous to the thermodynamic conduction in a gas; that is, as a particle follows its random walk through the material, it carries its random kinetic energy with it (a streaming temperature transport), and when two particles collide they exchange some of their random kinetic energy (a collisional temperature transport). Both mechanisms result in a diffusion of granular temperature through the material, and like its thermodynamic counterpart, granular temperature

---

**Figure 1** The mechanical energy path within a rapidly flowing granular material.
will be conducted along its gradients in the direction of diminishing temperature. [The conduction of granular temperature is clearly evident in a case observed by Campbell & Brennen (1985b).]

One further point should be understood about the granular temperature. In the kinetic-theory picture of gases, the random velocities of the molecules are so large that the imposition of a mean velocity field provides only a first-order correction to the thermal velocities. In contrast, for granular materials, the random velocities are proportional to the mean velocity gradient and consequently will be much smaller than the mean velocities of the particles. Thus, when viewing a granular flow, the particle fluctuations are not as violent as one might expect.

**An Illustrative Example**

Figure 2 is a plot taken from a computer-simulation study by Campbell & Brennen (1985b). Computer simulations of granular flow are descendents of molecular-dynamics calculations in which each particle within a mechanical system of particles is followed exactly as it moves and interacts with its neighbors. The simulations depend only on models for the individual particle properties and on the ways in which the particles interact with each other and with the boundaries of the system, and hence they may be considered idealized experiments performed on idealized materials. These simulations may be divided into two types. The rigid-particle simulation used by Campbell (1982, 1988, 1989), Campbell & Brennen (1985a,b), and Campbell & Gong (1986, 1987) assumes that the particles have infinite elastic moduli, so that any particle collisions are instantaneous. The soft-particle models used by Haft (1987), Haft & Werner (1986), Walton (1984, 1986), and Wakita & Aoki (1985, 1986) assume that the particles have finite elastic moduli, so that any particle collisions are instantaneous. The soft-particle models used by Haft (1987), Haft & Werner (1986), Walton (1984,

---

**Figure 2**  Velocity, solid fraction, and temperature profiles from a computer simulation of the flow of two-dimensional disks down a plane inclined at 30° with respect to the horizontal. Figure taken from Campbell & Brennen (1985b).
1986b), Walton et al. (1988), Walton & Braun (1986a,b), Werner (1988), and Werner & Haff (1985, 1986, 1988) assume finite elastic properties and thus must examine the collisions in more detail. In addition, Monte Carlo simulation methods have also been adapted to granular flows by Hopkins (1985, 1987), Hopkins & Shen (1988), Shen & Hopkins (1988), and Shen et al. (1986, 1987, 1988). These do not follow particle trajectories exactly but instead generate distribution functions from which the system properties can be inferred; unfortunately, the method assumes that the particles are independently distributed both spatially and statistically within the distribution function—an assumption that breaks down at large density. Computer simulation has played a valuable role in the development of granular flow theory because, literally, everything is known about the simulated systems. In contrast, experimental studies are hampered by a general lack of instrumentation and experimental strategies for performing work in a dense, abrasive medium that does not take well to intrusive probing. Thus, most detailed measurements of density and temperature profiles and stress tensors come from computer simulation rather than direct experiments. The computer-simulation field has been reviewed by Campbell (1986b) and Walton (1986a).

This particular study (Figure 2) is a simulation of gravity-driven granular flow down a plane inclined at an angle of $30^\circ$ with respect to the horizontal. Here, $y$ is the coordinate normal to the channel bottom, and $H$ is twice the mass mean depth and is therefore an approximation to the height of the free surface. The flow is bounded on the bottom ($y = 0$) by a solid wall but is unbounded on the top. The material in this study, as in many others, is composed of disks rather than spheres, and the flow is confined to a two-dimensional motion within the plane of the disks. The three panels show the velocity, density, and granular temperature on the horizontal axis, all plotted against the height above the solid wall on the vertical axis. Collectively they illustrate the complex interaction between the fluid mechanics and the granular temperature. The velocity profile shows a large degree of slip at the bottom wall, and near the wall it shows a region with a large velocity gradient. From the discussion in the previous section, one would expect a large degree of temperature generation where the velocity gradient (and thus the shear work) is large—a speculation that is confirmed by examining the temperature profile in the third panel. The most interesting part of this plot is the density profile, shown in the second panel. What is plotted here is the “solid fraction,” a dimensionless density equal to the fraction of a unit volume (or unit area in this two-dimensional case) that is occupied by solid material. The plot shows that the density is low near the bottom of the channel, reaches a maximum toward the center, and then tails off again near the free surface. The low
density near the channel bottom has been observed experimentally in chute flows by Bailard (1978) and Ridgway & Rupp (1970), in slurry flows by Shook et al. (1968), and is the logical explanation for the reduction in heat transfer at high velocities observed by Spelt et al. (1982), Patton (1985), and Patton et al. (1986). The flow can maintain a region of low density near the bottom wall, despite the large overburden of material, because of the large granular temperature, clearly illustrating the analogy between the granular and thermodynamic temperatures.

Note that the low density near the wall contributes in its own way to the relatively large local magnitude of the granular temperature. As illustrated in Figure 1, the magnitude of the granular temperature is determined by the balance between the temperature generation via the shear work and the temperature dissipation via the inelasticity of particle collisions. The collision rate is a function of the density and the granular temperature (which governs the rate at which particles are driven together), and when the particle density is small the collision rate is correspondingly reduced unless there is a corresponding increase in the granular temperature. Furthermore, in order to dissipate a given amount of energy at a reduced collision rate requires that more energy be dissipated per collision, implying large impact velocities and therefore large granular temperatures. Hence, the large granular temperature near the wall can be attributed to a combination of increased temperature generation due to the large shear rate and the effects of the lowered density on temperature dissipation.

**CONSTITUTIVE BEHAVIOR**

The pioneering work on the constitutive behavior of rapid granular flows was Bagnold's (1954) experimental study of wax spheres, suspended in a glycerin-water-alcohol mixture and sheared in a coaxial cylinder rheometer. He found that at reasonably large concentrations and shear rates, the generated stresses depended on the square of the imposed shear rate and obeyed a relation of the form

$$\tau_{ij} = \rho_p R^2 f_i(v) \gamma^2,$$

(1)

where $\rho_p$ is the particle density, $R$ is the particle radius, $\gamma$ is the shear rate, and $f_i$ is a tensor-valued function of the solid fraction $v$. This type of relation has been confirmed experimentally in shear-cell experiments of both wet and dry mixtures by Craig et al. (1986, 1987a,b), Hanes (1983), Hanes & Inman (1985), Savage & McKeown (1983), and Savage & Sayed (1984); in computer simulations by Campbell (1982, 1989), Campbell &
Brennen (1985a,b), Campbell & Gong (1986), Hopkins (1985), Hopkins & Shen (1986, 1988), and Walton & Braun (1986a,b); and in just about every theoretical study.

Such universal agreement is really not surprising. As long as the interstitial fluid is ignored, the only available dimensional quantities are the particle properties, $p_p$ and $R$, which (along with a time scale supplied by the velocity gradient $\gamma$) requires, owing to dimensional necessity, that the stresses be of the form in Equation (1). It is interesting to note also that as the granular temperature has units of squared velocity, it must be proportional to $R^2\gamma^2$, again owing to dimensional necessity. Equation (1) states that the effective viscosity of a granular material varies linearly proportional to the shear rate and is therefore proportional to the square root of the granular temperature in much the same way as kinetic theory predicts that the viscosity of a gas varies as the square root of the thermodynamic temperature. Again, this should not be surprising; as long as there are no long-range forces between particles, the transport of any property must be accomplished by the random motion of the particles, and the transportation rate must be proportional to the magnitude of the random velocity (i.e. to the square root of the granular temperature). In this context, it is interesting to note that Campbell & Wang (1986) showed that the apparent thermal conductivity of a dry granular material also varies linearly proportional to the shear rate and consequently is also proportional to the square root of the granular temperature—as again would be anticipated from the kinetic theory of gases.

Equation (1) may lead to the mistaken conclusion that a rapid granular flow behaves like a power-law fluid for which the viscosity is linearly proportional to the shear rate. Unfortunately, as the previous paragraph suggests, this is only true in cases where the square root of the temperature is linearly proportional to the velocity gradient. (For most molecular fluids the temperature is independently fixed, and thus flow-induced temperature variations do not account for a power-law behavior.) Remember that granular temperature is produced by shear work, which is in turn proportional to the velocity gradient. Thus it might be argued that Equation (1) is valid under any conditions in which the local magnitude of the granular temperature is governed solely by the local production rate. However, for the chute flow used in Figure 2, there are large gradients of temperature within the flow, in which case the local value of the temperature—and consequently the local viscosity—may be affected by the temperature in surrounding regions through the conduction mechanism discussed earlier. [Ahn et al.’s (1989b) analysis of chute flows shows that the conductivity is very significant in determining the final behavior of the
system.] Campbell & Gong (1987) have shown that, especially with the proper choice of boundary conditions, the velocity gradient, density, and granular temperature are nearly uniform in simple shear flows (such as Bagnold's experiments), and consequently that there will be minimal conduction effects. This explains why Equation (1) is valid in this context, but it also indicates that simple shear flows are anomalously simple cases of granular flow. Thus, this is a case where Couette rheometer experiments may indicate constitutive laws that cannot be generalized to other than simple shear flows.

Note that because the function $f_I(\nu)$ is tensor valued, Equation (1) indicates that a simple shear flow may generate stresses within the material that are not simply reaction forces to the imposed velocity gradient. In particular, the simple shear motion generates granular temperature, which in turn generates normal stresses in exactly the same way as the thermodynamic temperature generates thermodynamic pressure in a gas. These normal stresses were first observed experimentally by Bagnold (1954), and he referred to them as "dispersive stresses," as they tend to cause dilatation of the material (or "disperse" the particles).

Due to their random thermal motions, every particle in a rapid granular flow will be in relative motion with even its nearest neighbors. As a result, interparticle contacts are momentary, and it is usual to idealize particle interactions as instantaneous collisions. As in common fluids, what are perceived as continuum stresses are a by-product of the microscale mechanisms of momentum transfer within the material. For granular materials and hard-sphere models of gases, momentum is transferred in two modes that are exactly analogous to the two modes in which the granular temperature is generated and conducted. The "streaming," or "kinetic," mode describes the transport of momentum as a particle moves through the material carrying the momentum of its motion with it. By arguments such as those presented in Chapman & Cowling (1970), the resulting contribution to the stress tensor is

$$\tau_s = -\rho_v \langle u'u' \rangle,$$

where $u'$ is the instantaneous deviation from the mean velocity. Note that (2) is essentially the same as a Reynolds-stress tensor and is obviously closely related to the granular temperature. [In fact, the granular temperature $T$ is just the trace of $\tau_s$ divided by $(-\rho_v v)$.] The "collisional" mode, as the name implies, reflects the transfer of momentum between particles when they collide. The effect of a collision is a transport of momentum equal to the collision impulse $J$ over the distance $2R$ between the particle centers. If $k$ denotes the unit vector in the direction that
connects the centers of the colliding particles at impact, the effective
momentum transport by a single collision across a surface separating the
particles is \(2R\mathcal{F} \cdot n\), where \(n\) is the normal vector to the surface. The
transport rate is determined by the collision frequency \(\mathcal{F}\), which is related
to the relative velocity between colliding particles and, hence, is related to
the square root of the granular temperature. (Note that \(\mathcal{F}\) will also be a
strong function of the solid fraction \(v\) and will become infinite when the
concentrations are large enough so that particles are always in contact.)
The average of all such collisions, multiplied by the collision frequency, will
yield the collisionally induced traction force on that surface, \(2R\mathcal{F} \langle Jk \rangle \cdot n\).
This expression should be the dot product between a stress tensor and an
outward-pointing normal unit vector, and thus that portion of the stress
tensor that is due to interparticle collisions, \(\tau_c\), is given by

\[
\tau_c = 2R\mathcal{F} \langle Jk \rangle .
\]  
Obviously the streaming mode will dominate at low densities, where col-
lisions are infrequent and particles move long distances between collisions,
while the collisional mode will dominate at high densities, where collisions
are frequent and particles cannot move far before colliding. (It was com-
mon to ignore the streaming contribution in many of the early theoretical
studies—an assumption that is somewhat justified because most granular
flows occur at very high densities, where the collisional contributions are
dominant.) The complete stress tensor is found by summing the con-
tributions of both the streaming and collisional modes. Note that the
transport rates for both modes are controlled by the granular temperature,
and that the amount of momentum that can be exchanged by either
mechanism is that carried by the inertia of the random particle motions.
Consequently, Bagnold (1954) used the term “grain-inertia” regime to
describe the class of flows that obey Equation (1).

Some measurements of the shear-stress function \(f_{xy}(v)\) in simple shear
flows are shown in Figure 3a. Its companion, Figure 3b, shows the cor-
responding predictions of many of the theories that are discussed in the
next section. The subscripts “xy” reflect that these come from the shear
stresses induced in a material by a \(y\)-direction gradient of \(x\)-direction

---

Figure 3  (a) Measurements of the solid fraction function \(f_{xy}(v)\) for simple shear flows of spherical particles. These data are taken from various computer simulations (assuming a coefficient of restitution \(\varepsilon = 0.8\)) and experimental studies. The rigid-particle simulations are from Campbell (1989), the soft-particle simulations are from Walton (1986b) and Walton & Braun (1986b), and the Monte Carlo simulations are provided by M. A. Hopkins (personal communication, 1986). This figure is taken from Campbell (1986b).  (b) The corresponding theoretical predictions.
velocity. Most of these results were generated by computer simulations of uniformly sized spheres by Campbell (1989), Campbell & Gong (1986), Walton (1986b), Walton & Braun (1986b), and M. A. Hopkins (personal communication, 1986); they are plotted along with data from shear-cell experiments on glass spheres by Savage & Sayed (1984), and with other glass-bead data that are unpublished results from the heat-transfer experiments of Campbell & Wang (1986). The single component \( f_{xy} \) was chosen as an illustrative example, since it is reported extensively in the literature, but all other components of \( f_{ij} \) show the same qualitative behavior. In the computer simulations, the material properties are represented by several surface-friction coefficients \( \mu \), but with a single coefficient of restitution \( \varepsilon = 0.8 \). (The coefficient of restitution specifies the ratio of the approach to recoil velocities in the center of mass and thus affects the rate of energy dissipation in the collision.) The curves show the characteristic "U" shape, with asymptotes to \( \infty \) both at large \( v \) and as \( v \to 0 \). The large-\( v \) asymptote occurs because there is a maximum concentration (about \( v = 0.6 \) for uniformly sized spheres) at which an assemblage of uniformly sized particles can support a shear flow no matter what the applied stress. The reason behind the \( v \to 0 \) asymptote is more elusive but can be understood in terms of the energy path shown in Figure 1. Recall that temperature is produced by shear work \( (= \tau_{xy} \dot{\gamma}) \) and dissipated by the inelasticity of the collisions. But as \( v \to 0 \), the collision rate and hence the energy dissipation both go to zero. The collision rate is proportional to the probability of finding two particles in contact, the probability of finding a particle at any given location is proportional to \( v \), and at low density (where the presence of a particle will have little effect on the location of its neighbors) the probability of a second particle in contact with the first is also proportional to \( v \). Hence, both the collision rate and the energy-dissipation rate are proportional to \( v^2 \) for small values of \( v \). The temperature generation, on the other hand, is dominated by the shear work performed by the streaming contributions to the stress tensor \( \tau_{xy} \dot{\gamma} \), and as can be seen in Equation (2), \( \tau_{xy} \) is only linearly proportional to the solid fraction \( v \). Hence, as \( v \to 0 \), the temperature-dissipation rate goes to zero faster than the temperature-production rate, which results in infinite granular temperatures that are accompanied by infinite stresses, since the streaming contribution to the stress tensor is strongly related to the granular temperature.

There is some question as to whether the \( v \to 0 \) asymptote has any real-world significance. Note that it appears in Figure 3 only in the computer simulation data and is not apparent in the experimental results. This may just point out the difficulties of doing experiments at low densities in rotating systems, where centrifugal forces may cause density stratification. But another possible explanation is the drag that the interstitial air applies...
to the particles. Recall that the low-density asymptote comes about because collisions provide the only dissipation mechanism, but in real experimental systems, aerodynamic forces on the particles will damp the granular temperature independently of the collision rate and may be sufficient to eliminate the low-density asymptote. There are, however, some observations in the chute-flow experiments of Patton (1985) and Patton et al. (1987) that indicate that the mechanisms behind the low-density asymptote may still be active, even in the presence of interstitial air. For large, heavy particles (which are nearly immune to aerodynamic forces), they noted high-speed, low-density chute flows in which the depth of the flow continually increased—indicating that the system was becoming progressively more dilute—as the flow accelerated down the chute. (If there were no density changes, the depth would decrease as the flow accelerates.) This indicates that the granular temperature increases with the flow velocity, causing the material to dilate. This continual depth increase, which does not appear to approach a steady state, may be an indication that the flow is experiencing the instability inherent in the low-density asymptote of the $f_{xy}$ curve. That is, once the flow has dilated to a solid fraction below the minimum in the $f_{xy}$ curve, any further acceleration causes an increase in granular temperature along with a corresponding decrease in density, which, by reducing the dissipation rate of the granular temperature, results in still larger granular temperatures and consequently larger pressures and shallower depths. This process will continue and can only be stabilized when the combined effect of air drag and the few remaining collisions is large enough to dissipate the granular temperature at the rate it is being generated.

A related occurrence is the appearance of normal-stress differences in granular flows at low densities that has been documented by both two- and three-dimensional computer simulations of Campbell (1989), Campbell & Gong (1986), Walton & Braun (1986a,b), and Hopkins (1987). The results show that for a simple shear flow in the $x$-$y$ plane with the velocity gradient in the $y$-direction, the $\tau_{xx}$ normal stress is always larger than $\tau_{yy}$, and that the ratio $\tau_{xx}/\tau_{yy}$ can be of order 10 as $v \to 0$ but drops to approaching a value only slightly larger than 1 as $v$ approaches the shearable limit. Some measurements of this by computer simulation are shown in Figure 4. Furthermore, the effect is large for small coefficients of restitution (large collisional energy loss) and almost insignificant for coefficients of restitution near 1. This effect is attributable to the two modes by which granular temperature can be generated. Recall that the streaming mode, which is dominant at low densities, will only generate temperature in the direction perpendicular to the velocity gradient (in this case, the $x$-direction), whereas the collisional mode, dominant at high densities, will
generate granular temperatures that are more or less isotropic. Consequently, the streaming mode only contributes to the $\tau_{xx}$ component of the streaming stress tensor, and this explains why, at lower concentrations, the ratio $\tau_{xx}/\tau_{yy}$ is large. The dependence on the coefficient of restitution has a similar explanation. The smaller the coefficient of restitution, the larger the degree of energy dissipation and the smaller the granular temperature that will be generated by a collision. This alters the balance between the collisional and streaming modes of temperature generation in favor of the streaming mode and results in larger values of $T_{xx}/T_{yy}$. Campbell (1989) has also reported normal-stress differences between $\tau_{xx}$ and $\tau_{yy}$ and the out-of-the-shear-plane normal stress $\tau_{zz}$.

**KINETIC-THEORY MODELING**

The physical similarity between rapid granular flows and the kinetic-theory view of gases has led to a great deal of work on creating similar models for granular materials. The idea is to derive a set of continuum equations entirely from microscopic models of individual particle interactions. All of the models assume that particles interact by instantaneous collisions, which implies that only binary, or two-particle, interactions need be considered. The particle properties are modeled simply, usually using a con-

---

*Figure 4* Normal-stress differences for spheres as a function of the solid fraction $\nu$, taken from computer simulations (assuming a coefficient of restitution $\varepsilon = 0.8$). The rigid-particle simulations are from Campbell (1989), the soft-particle simulations are from Walton (1986b) and Walton & Braun (1986b), and the Monte Carlo simulations are provided by M. A. Hopkins (personal communication, 1986). Also shown are the predictions of the smooth-sphere theory of Richman (1989b).
stant coefficient of restitution, to represent the energy dissipated by the impact normal to the point of contact between the particles, and for the most part surface friction or any other particle interactions tangential to the point of contact are ignored. Furthermore, molecular chaos (Boltzmann's *stosszahlansatz*) is generally assumed, implying that the random motion of even closely neighboring particles are independently distributed (a condition that should break down at high concentrations). While these models are still in the early stages of development, they have been applied to problems as diverse as flow down inclined chutes (Savage 1983a,b, Ahn et al. 1989b, Richman & Marciniec 1988), landslides (Hutter et al. 1986a,b, Szidarovsky et al. 1987), the motion of pack ice (Shen et al. 1986, 1987), grain-size reduction (Richman & Akkoc 1987, Richman & Chou 1989), and the rings of Saturn (Borderies et al. 1985, Goldreich & Tremaine 1978).

Even though the various theories derive from significantly different calculational principles, they all result in roughly the same type of continuum equations, which are similar in form to the Navier-Stokes equations. There is, first of all, a mass-conservation equation:

$$\frac{dp}{dt} + \rho \nabla \cdot \mathbf{u} = 0,$$

where $\rho$ is the local density, $\mathbf{u}$ is the local velocity vector, and $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is the Lagrangian derivative. There is also a conservation-of-momentum equation:

$$\frac{\rho d\mathbf{u}}{dt} = -\nabla \cdot \tau + \rho \mathbf{g},$$

where $\tau$ ($\tau = \tau_s + \tau_v$) is the total-stress tensor, and $\mathbf{g}$ is the body-force vector. (If the calculation includes particle surface-friction rotation, a similar equation should be included for the conservation of angular momentum.) Finally, there is an energy-conservation equation for the kinetic energy contained in the granular temperature:

$$\frac{1}{2} \rho \frac{dT}{dt} = -\nabla \cdot \mathbf{q} + \tau : \nabla \mathbf{u} - \Gamma,$$

where, as before, $T = \langle \mathbf{u}^2 \rangle$ is twice the fluctuating kinetic energy per unit mass (hence the $\frac{1}{2}$ as a multiplier); $\mathbf{q}$ is the "granular heat" flux vector, which represents the conduction of granular temperature within the material; $\tau : \nabla \mathbf{u}$ is the granular-temperature generation by shear work; and $\Gamma$ is the dissipation of the granular temperature into thermodynamic heat. (Again, if the theory considers particle surface friction and rotation, then
either an additional equation is required for the spin energy—through a granular spin temperature—or the spin energy must be included in the granular temperature $T$.) The major difference between all the theories is the constitutive dependence of the quantities $\tau$, $q$, and $\Gamma$ on the flow properties $V$, $\rho$, and $T$ and on the properties and sizes of the solid particles.

The first calculation to relate the constitutive behavior of a rapid granular material to the microscopic properties was performed in a heuristic manner by Bagnold (1954) and yielded the stress/strain-rate behavior described by Equation (1). Similar and slightly more sophisticated calculations have been performed by McTigue (1978) and Pasquarell et al. (1988). The calculations that yield a complete set of all the transport coefficients vary from the heuristic (although very intuitive and physically motivated) calculations of Haff (1983) through a large number of calculations by Ackermann & Shen (1982), Babić & Shen (1989), Hopkins & Shen (1986), Raymond & Shen (1986), Shen & Ackermann (1982, 1984, 1986), Kanatani (1979, 1980), and Ogawa et al. (1980), all of which are based on simplifying assumptions about such properties as collisional frequencies and the particle interactions, to a final category derived from the formalisms of the kinetic theory of nonuniform dense gases such as are described in Chapman & Cowling (1970). The first calculation of this last type (Savage & Jeffrey 1981) derived the collisional portion of the stress tensor for uniform-sized smooth spheres based on the assumption that the random particle velocities followed a Maxwellian distribution. [Measurements of the velocity distribution by the computer simulations of Campbell & Brennen (1985a) and Walton & Braun (1986a,b) and in the chute-flow experiments of Drake (1988) show that the velocity distribution is not far from Maxwellian.] But the theory was incomplete in the sense that it did not include an energy equation such as (6) and thus could not predict the magnitude of the granular temperature. Savage & Jeffrey did realize that the imposition of a velocity gradient whose square was of the same order as the granular temperature would induce anisotropy in the distribution of angles at which collisions would occur between particles. Their anisotropy predictions were later confirmed by Campbell & Brennen (1985a). The next step, by Jenkins & Savage (1983), was to solve the energy balance for the granular temperature but to do so required an approximation to the anisotropy of the collision distribution; although later, the same calculation using the more exact collision distribution of Savage & Jeffrey (1981) was performed in the first part of the paper by Lun et al. (1984). More importantly, the second half of that paper was the first attempt using any method to compute the streaming contribution to the stress tensor. Up until this point, all calculations had assumed Maxwellian velocity distributions and had given only the collisional por-
tion of the stress tensor. Strictly, the velocity distribution should be Maxwellian only if the system is in equilibrium (i.e. no velocity and temperature gradients and no energy dissipation). A Maxwellian velocity distribution would predict that all components off the diagonal in the streaming stress tensor are zero, but the nonzero streaming stresses, which are strongly apparent at low solid fractions, indicate that there are strong, non-Maxwellian effects active in the flow despite the apparent similarity between a Maxwellian and the measured distributions. Lun et al. (1984) computed the first correction to the Maxwellian velocity distribution and obtained surprisingly accurate predictions of the streaming stresses; but, to do so, however, required abandoning the anisotropic collision distribution. Similar calculations were almost simultaneously performed by Jenkins & Richman (1985a). The techniques have recently been reviewed by Richman (1986) and Jenkins (1987a,b), and the predictions of many of the theories are shown in Figure 3b. As originally written, much of the difference between these theories lies in their choice of the form of the functional dependence of the density on the stresses. In all cases, the density dependence is determined independently of the rest of the solution, usually from curve fits to molecular-dynamics models of gases. To eliminate any variance in the results due to the choice of this function, all of the curves plotted in Figure 3b were recomputed using the density dependence given in Babić & Shen (1989). Many of the early theories considered only the collisional components of the stress tensor; these are easy to distinguish in Figure 3b because the curves go to zero, rather than to infinity, as \( v \to 0 \).

Problems have arisen in the generalization of these results, largely due to the great complexity of the calculations. For example, the Lun et al. (1984) theory does not predict any normal-stress differences. To include these requires a more complicated analysis, as has been performed for smooth disks by Jenkins & Richman (1988) in the dense (collision-dominated) and dilute (collision-free) limits and for smooth spheres in the dilute limit by Richman (1989a). Richman (1989b) has also developed a heuristic theory for the normal-stress differences; some of his results are plotted in Figure 4 and compare extremely well with the computer-simulation data for smooth spheres. [The exact theory has recently been extended numerically to cover the entire range of solid concentrations (M. W. Richman, personal communication, 1989), which, at least for \( \varepsilon = 0.8 \), agrees within a few percent with the more heuristic theory (Richman 1989b).] The inclusion of rotation and surface friction in the calculations (even while avoiding normal-stress differences) has also proven to be very difficult. Lun & Savage (1987) based their frictional particle work on the earlier Jenkins & Savage (1983) theory, which considers only collisional stresses, as they found friction to be too complicated to include in the Lun et al.
(1984) calculations. Jenkins & Richman (1985b) performed a calculation similar to that of Lun et al. for rough disks but found that the theory would only be valid for unrealistically small values of the particle surface friction. Nakagawa (1988a) has performed a calculation valid for any surface friction but was restricted to the dilute limit. These complications are mirrored in the computational complexity that Lun & Savage (1986) uncovered when they tried to include a velocity-dependent coefficient of restitution to their earlier calculation (even though they used an exponentially decaying coefficient of restitution, which melded easily into their collision integrals.)

BOUNDARY CONDITIONS

Conventional fluid mechanics has the luxury of a no-slip condition in which both the velocity and the thermodynamic temperature at a solid boundary assume the properties externally imposed on the boundary. This allows the properties of the material at the boundary to be specified independently of whatever is happening in the rest of the flow field. Granular materials are not so blessed, and the flow behavior at a solid or free surface is an integral part of the solution for the entire flow field. For example, as in Figure 2, there will generally be velocity slip at the boundary and this opens a whole host of problems because the slip velocity is determined by the manner in which the flow interacts with the boundary and thus acts to match the physical nature of the boundary with the flow far from the boundary. Similarly, the granular temperature at the wall cannot be specified independently. Because there is a slip velocity, the boundary will be performing shear work and generating granular temperature at a rate equal to the product of the wall shear stress and the slip velocity. In this way the wall may be considered to be a local source of temperature. But, at the same time, collisions with the wall are dissipating energy. Thus, under different conditions, the wall may act as a source or sink of granular temperature.

The effect of the boundaries has also been noted experimentally. Savage & Sayed (1984) and Hanes & Inman (1985) both performed nearly identical shear-cell tests on nearly identical materials. The only major difference between the two studies was that Savage & Sayed's walls were roughed with sandpaper—which typically had a roughness size much smaller than that of the test material—while Hanes & Inman roughed their walls by gluing on particles of the test material. Whatever the case, Hanes & Inman's measured stresses were up to three times larger than Savage & Sayed's for otherwise equivalent conditions. The effect of boundary conditions on shear-cell studies of metal powders has been studied by Craig
et al. (1987a); these indicate that the drive surfaces are more important than the test material in determining the results of the test. Concerns about boundary conditions may also explain discrepancies in some chute-flow experiments and simulations. For example, the velocity profiles shown in Figure 2 are very similar to those observed by Bailard (1978), Drake & Shreve (1986), and Augenstein & Hogg (1978), yet they lack the inflection point observed by Savage (1979) and Ishida & Shirai (1979). Also, the corresponding solid-fraction profile is very similar to that observed by Ridgway & Rupp (1970), but the low-density zone near the chute bottom is nearly absent from the data of Ahn et al. (1989a). The differences may be attributed not only to the character of the boundary, but also to the fact that these studies were performed on different materials that may interact with similar boundaries in very different ways.

Recently, several theoretical studies have incorporated boundary conditions into their problem solution. The first of these (Hui et al. 1984) modeled the wall interaction as an equivalent-roughness coefficient but neglected the shear work performed by the boundary due to the velocity slip at the surface. Jenkins & Richman (1986) developed a set of boundary conditions for two-dimensional smooth circular disk flows in the neighborhood of a boundary composed of semicircular bumps glued to a flat wall. They assumed that the particle's velocity distribution was Maxwellian, and that the spacing of the glued particles was such that the free particles would always hit a glued particle and could not collide with the flat portion of the boundary. However, a complication became apparent in examining the corresponding solutions to Couette-flow problems. The results showed that a steady flow was possible for a given separation distance between the driving surfaces for exactly one average value of the solid fraction. In contradiction, shear-cell experiments and computer simulations of these flows indicate that steady flows are possible for any average concentration. This work was later extended to non-Maxwellian velocity distributions for disks by Richman & Chou (1988) and to spheres by Richman (1988). These latter two works suggested a heuristic solution to the uniqueness problem by not strictly applying the continuity of normal stress at the boundary. But the resolution of the uniqueness problem has recently been independently found by Gutt & Haff (1988) and Hanes et al. (1988). They noted that the presence of a wall causes a local distortion in the arrangement of particles in its immediate vicinity. Hence, they made the concentration of the flow in the immediate neighborhood of the boundary a free parameter to be determined as a matter of course in the application of the boundary conditions. Allowing the density at the wall to vary introduced enough freedom in the solution to allow steady flows at any average concentration. [It turns out that this solution to the problem
is essentially equivalent to the heuristic arguments used by Richman & Chou (1988) and Richman (1988).] Richman & Marcinkec (1988) have also analyzed the flow down an inclined chute with a bumpy wall.

Other problems arise if particle surface friction is included in the analysis. In Figure 2, the large temperature generation near the wall is a direct result of the large frictional coupling between the particle surface and the wall. These simulations assumed that on departure after a collision with the boundary, there would be no slip between the surface of the particle and the boundary; that is, the particle would adopt a rotational velocity $\omega$ such that its surface velocity $R\omega$ exactly equaled its slip velocity at the wall. For the first boundary condition, the slip in the mean velocity indicates that on collision with the wall, a particle will pick up a large rotation rate. This large rotation is not itself random or temperaturelike in the sense that all the particles that collide with the wall will rotate at nearly the same rate. However, on subsequent collisions, the particle will transfer this rotational energy to other particles through the surface-frictional coupling. Some of the transferred rotational energy will go into rotational modes of motion, and some will go into translational modes. But the generated translational velocities will depend on the geometry of the collision and thus be more or less randomly distributed so that the coherent rotational motion will be very quickly transformed into granular temperature. Hence, the friction between the solid surface and the particle makes the boundary a large source of granular temperature. This problem has been studied in more detail by Campbell & Gong (1987) and Campbell (1988). Both observed that the high rotation rate at the boundary induced asymmetries in the stress tensor. They concluded that such asymmetries in steady flow imply that the granular material is behaving like a Cosserat continuum, and that the internal torques implied by the stress asymmetries had to be balanced by gradients in a couple-stress tensor. (The couple-stress tensor yields the moment exerted across a surface internal to the material, just as the stress tensor describes the force exerted across a surface.) Obviously, in steady flow, a wall collision induces a large torque, and thus large couple stresses to the material which must be balanced by countertorques applied by the material. This process has been observed by Campbell (1988), who has measured the couple stress generated internal

---

1Campbell & Brennen (1985b) also briefly examined an alternative boundary condition where, in an approximation of a classic no-slip condition, the velocity of the particle center following a collision with the wall equals the wall’s velocity with no change in the rotation rate. They showed that the large velocity gradient, granular temperature, and low-density zones in the neighborhood of the wall, which were the prominent features of Figure 2, largely disappeared for the second boundary type. Similar observations were made for the same two boundary conditions in the Couette-flow simulations of Campbell & Brennen (1985a).
RAPID GRANULAR FLOWS

to the material. These effects are strongly tied to the particle surface friction and thus await the development of constitutive models that can incorporate finite friction before they can be dealt with theoretically.

PRESSING CONCERNS

There are many aspects of rapid granular flows that have barely been addressed or still have yet to be addressed. Here I mention a few of them and speculate a bit on their possible importance.

**Material Properties**

Both the theoretical models and the computer-simulation studies of rapid granular flows start with models for particle properties and interactions and extrapolate from them the behavior of the entire granular mass. With few exceptions, the work to date has assumed that the energy loss in a collision is largely governed by a constant coefficient of restitution $\varepsilon$, the ratio of the relative approach to recoil velocities normal to the point of impact on the particle. By definition, $0 \leq \varepsilon \leq 1$, where $\varepsilon = 1$ implies a perfectly elastic collision with no energy dissipation, and $\varepsilon = 0$ implies a completely inelastic collision, for which all of the kinetic energy contained in the initial normal relative velocity is lost during the collision. However, measurements by Goldsmith (1952, 1960), Lun & Savage (1986), Raman (1918), and Sondergaard et al. (1989) show that the coefficient of restitution is a strong function of the relative impact velocity. Physically, the energy dissipation comes from plastic deformation of the particle’s surface. Thus, one expects completely elastic collisions ($\varepsilon = 1$) unless the impact is strong enough to cause yielding within the solid material, and indeed all of the measurements show that the coefficient of restitution approaches unity at zero impact velocity. Apparently, though, significant plastic dissipation occurs for any velocity, as all the measured coefficients of restitution immediately drop from unity. Velocity-dependent coefficients of restitution have been incorporated into Walton & Braun’s (1986a) computer simulations and into Lun & Savage’s (1986) theoretical analysis. But there is some evidence that incorporating simple velocity dependence is little more than a first-order improvement. For example, Raman (1918) notes that he could only obtain repeatable results in his experiments by polishing his particles between tests. This indicates that the plastic deformation resulting from an earlier test is sufficient to affect the results of later tests. [This could also explain the significant scatter in the results of Lun & Savage (1986).] Thus, a truly realistic microscopic model would have to include the deformation history of the particle’s surface. Such a correction is probably intractable in any theoretical model or even in the computer
simulations, so the question remains as to what constitutes an adequate microscopic model upon which accurate macroscopic models can be constructed.

Similar problems occur in the modeling of particle surface friction. A standard surface-friction coefficient $\mu$ represents a yield condition; there will be no slip between particle surfaces unless the ratio of tangential to normal forces at the point of impact exceeds $\mu$, and if $\mu$ is exceeded, the tangential force transmitted across the point of contact will be just $\mu$ times the normal force. Sondergaard et al. (1989) measured the surface friction actually experienced during a particle collision with a wall. The tests were reasonably repeatable up to the point that slip occurred between the particle and wall surface. However, there was considerable scatter in the results when the particle was slipping, indicating that there is no unique value of the particle surface-friction coefficient. This may be expected if one assumes that the microscopic origin of surface friction lies in the interaction of microroughnesses on the two contacting surfaces. On impact the normal stresses will seldom be large enough to cause significant deformation of the particles, so that the contact area will be small and only a small fraction of the microroughnesses will play a part in the collision. Hence, the collision result is very sensitive to local variations in the surface friction. Furthermore, it is evident that collisions shear off the microroughness, causing changes in even the average surface friction as the flow progresses.

An actual surface-friction coefficient is very difficult to incorporate in theoretical models because the decision of slip/no-slip between the particle surfaces introduces a discontinuity in the collision integrals. In its stead, most theoreticians employ a tangential coefficient of restitution $\beta$, representing the ratio of final to initial relative tangential velocities between the particle surfaces. By definition, $-1 \leq \beta \leq 1$, where $\beta = 1$ represents smooth particles (i.e. no frictional interaction), $\beta = 0$ represents no-slip between the particle surfaces, and $\beta = -1$ represents a reversal of the relative velocity [i.e. not only is there frictional interaction, but also tangential energy is stored and released through elastic deformation of the particle surfaces; the effects of energy storage due to tangential deformation are evident in the experiments of Sondergaard et al. (1989)]. Several theories (Jenkins & Richman 1985b, Lun & Savage 1987) have assumed that $\beta$ is a constant, but this is not a good approximation, since in a truly frictional material, $\beta$ will depend strongly on collision geometry (see Goldsmith 1960). The geometrical dependence has been incorporated into the theory of Nakagawa (1988a,b), although he assumes that the particle surfaces always slip relative to one another, thus avoiding the slip/no-slip discontinuity in the collision integrals.
Microstructure

The shear-cell experiments of Savage & Sayed (1984), along with the computer simulations of Campbell (1989), Campbell & Gong (1986), and Walton & Braun (1986a,b), showed that the bulk friction coefficient $\tau_{xy}/\tau_{yy}$ of a shearing granular material decreases with particle concentration. Some examples of these measurements are shown in Figure 5. This is a rather surprising observation because standard soil-mechanics tests to measure the yield strength of a static sample indicate that the yield friction coefficient increases with concentration. These two observations collectively indicate that the reduction in $\tau_{xy}/\tau_{yy}$ is an attribute of fully developed granular flows that is not present in the undisturbed sample. The computer simulations of Campbell (1989) and Campbell & Gong (1986) indicate that the observed reduction can be attributed to two causes. The friction coefficient will be large at low densities, where the streaming stresses are active, and will decrease with increasing density as the collisional stresses become more important; exactly the same behavior is predicted by theories such as those of Lun et al. (1984) and Richman (1989b). At higher densities, where the collisional stresses are dominant, a further reduction in friction coefficient can be attributed to the development of an internal microstructure within the material. (Notice that the

![Figure 5](image-url) The friction coefficient for spheres as a function of the solid fraction $v$. The rigid-particle simulations are from Campbell (1989), the soft-particle simulations are from Walton (1986b) and Walton & Braun (1986b), and the Monte Carlo simulations are provided by M. A. Hopkins (personal communication, 1986). All of these assume that $\varepsilon = 0.8$. Also shown are the predictions of the smooth-sphere theory of Richman (1989b) and polystyrene bead data from shear-cell experiments by Savage & Sayed (1984).
theoretical and the Monte Carlo simulation results—both of which, by their construction, do not allow for microstructure formation—do not show this additional drop-off at large densities.) The microstructure affects the collisional stresses indirectly by inducing strong anisotropies in the collision-angle distribution (i.e. the probability that a collision will occur at a given unit vector $\mathbf{k}$ connecting the particles' centers). As the collisional stress tensor is formed by the average of the dyadic product $\langle \mathbf{Jk} \rangle$, favored values of $\mathbf{k}$ can strongly affect both the absolute and relative magnitudes of the stress-tensor components.

The development of the microstructure and its effect on the distribution of collision angles have been documented by Campbell & Brennen (1985a) for simple shear flows of two-dimensional disks. They showed that in order to maintain a shear flow at large density, the particles align themselves into layers oriented in the direction of mean flow; this organization allows almost unrestricted motion between the layers in the direction along the layers and thus permits a shear flow at concentrations that, without the layer formation, would probably exhibit solid behavior. Such a microstructure induces a preference for a particle to collide with particles within its own layer along lines parallel to the layer and with particles in the immediately neighboring layers at angles roughly perpendicular to the direction of flow. This biases the preferred collision angles, and a heuristic calculation by Campbell (1986a) incorporating just such a microstructure shows the expected drop-off in the collisional friction coefficient. No observations of equivalent microstructure development have been reported for assemblies of rigid spheres, although the action of such a microstructure is apparent in the experimental and simulated results shown in Figure 5. However, molecular-dynamics studies of Leonard-Jones molecules performed by Heyes (1986) indicate that shearing forces the molecules to align themselves into linear "strings" of molecules pointing roughly in the direction of flow (corresponding to the $x$-direction in the current simulations). The strings are themselves organized in a triangular packing in the plane perpendicular to the flow direction. Thus, a shear motion can be maintained at high density within such a packing by relative motion between the strings in much the same way as a two-dimensional shear motion was maintained by relative motion between the layers. As such an arrangement is the least restrictive that still permits a shear motion, it seems reasonable to expect that a similar microstructure forms at high density in granular shear flows. The resulting restrictions on the collision geometry would be pretty much the same as in the two-dimensional case. That is, collisions between particles within the same string occur about the poles of a particle, while collisions between a particle and those in neighboring strings would be more or less evenly distributed about the
equator. One might guess, however, that owing to the extra degree of translational freedom, the three-dimensional microstructure is much less restrictive than its two-dimensional counterpart.

Notice that the arguments in the above paragraph are purely kinematic—i.e. they take the form of “in order for a shear flow to exist, the particles must assume this arrangement” but describe nothing of the process of microstructure formation. Unfortunately, there are no good theoretical models to describe the development of a microstructure, and there is not even a good intuitive understanding of the forces that motivate the development. Thus, it will not be possible to incorporate microstructure development into the theoretical models until a method exists to describe how the microstructure comes about. [In his heuristic study, Campbell (1986a) simply created a microstructure that was similar to that of Campbell & Brennen (1985a).] In fact, existing theories, such as that of Richman (1989b) plotted in Figure 5, predict that the bulk friction coefficient actually increases slightly with concentration as the shearable limit is approached. Hence, this is an area for which a great deal of very interesting future work is required that may have relevance well outside the field of granular flow.

Figure 5 may give some insight into the complexity of this problem. Notice that all of the computer simulations exhibit the large \( \nu \) drop in \( \tau_{xy}/\tau_{yy} \) at about \( \nu = 0.6 \), which seems to correspond to the development of the hexagonal prismatic microstructure observed by Heyes (1986). But the experimental results show the drop at a much smaller solid fraction, at about \( \nu = 0.52 \), which may indicate the development of a microstructure in which the spheres are organized into layers perpendicular to the velocity gradient. Either microstructure will allow a shear flow to be maintained at large density, and both have a similar effect on \( \tau_{xy}/\tau_{yy} \). Obviously, then, one cannot make an a priori assumption about which form the flow will adopt, and the apparent difference in the microstructure choice for the two cases must have something to do with the different flow situations. The computer simulations assumed a gravity-free environment and, by assuming periodically repeated control volumes, approximated uniform shearing in an infinite medium. The shear-cell tests were performed in a finite volume in a rotating system with a gravitational acceleration vector pointing in the direction of the imposed velocity gradient (which is probably perpendicular to the layers that are formed.) But there is no way of knowing whether the microstructure choice represents some long-range order induced by the test-volume geometry, whether it is some complex action of gravity or centrifugal forces, or, for that matter, whether the discrepancy between the experimental and simulation results in Figure 5 is actually a reflection of microstructural differences.
Nonuniform Particle Size and Segregation

So far, the majority of the experimental and theoretical studies have assumed that the granular material was composed of uniformly sized disks or spheres. But real materials, formed by crushing or mining operations, have a wide distribution of particle sizes, which may significantly change the properties of the flow. For example, Zeininger et al. (1983) found that the flow rate from hoppers could be dramatically improved for binary mixtures (two particle sizes), that the flow rate was a maximum when the concentrations of each size were about equal, and that the effect was more dramatic as the particle-size ratio increased.

Many of the kinetic-theory studies (Shen 1984a, Farrell et al. 1986, Jenkins & Mancini 1987, Shen & Hopkins 1988) have been generalized for binary mixtures, and a few have been generalized for distributions of particle sizes (Shen 1984b, Shen & Hopkins 1988). Of these, however, only Jenkins & Mancini’s analysis for two-dimensional binary disk mixtures considers both the collisional and streaming components of the stress tensor. The general procedure for binary mixtures is to treat the collection of particles of each size as a separate medium that interacts with itself and with the particles of the other size, a process that leads to individual phase equations for each component and a combined phase equation for the mixture.

But such a procedure will not yield all of the interesting properties of particle mixtures. One problem that has plagued industry and effectively eliminates the possibility of laboratory testing of the above theories is the strong tendency of granular flows to segregate according to size under the action of any body force such as gravity or the centrifugal forces encountered in a shear cell. [For example, very rapid separation rates are evident in chute-flow experiments on binary mixtures performed by Savage & Lun (1988).] Savage (1987) has recently reviewed this subject. Simply put, the random motion of the particle mass is more likely to open a gap large enough for a small particle to fall into than one that can accommodate a large particle. Hence, the small particles migrate to the bottom of the flow, and the large particles migrate to the top. Note that this process is not an analog to the molecular diffusion of species; in fact, it is quite the opposite, because a homogeneous mixture moves toward complete separation rather than the other way around. Furthermore, this effect could not be anticipated using the theories outlined in the last paragraph, as the surface and body forces that appear in the individual phase momentum equations would not drive this type of process. Instead, it is a by-product of a complex interaction between the particle sizes that is not well under-
stood, yet one that should appear somewhere in the interaction terms of the individual phase equations.

**Nonspherical Particles**

Granular materials that result from crushing or mining operations will generally be highly angular, and pellets of extruded plastics will often have a cylindrical shape. Yet all of the analyses and most of the computer simulations have been performed for perfect spheres or disks. The few exceptions are Walton’s (1984) and Hopkins & Shen’s (1988) computer simulations of polygonal particles, although none of their particles had very large aspect ratios. Sphericity has been assumed in most studies for many reasons. In theoretical work and computer simulations it is easy to detect a collision of round particles, as particles are in contact whenever their centers are two radii apart. For out-of-round particles, the contact decision becomes much more complicated, as the orientation of the particle—which changes as the particle rotates—must be taken into account. Also, from an experimental point of view, aspherical particles are undesirable because the roughnesses tend to break off as the experiment proceeds, causing a slow degradation of the particle properties.

Spherical particles are probably good approximations for sand particles, which, although angular, are still roughly spherical and become more so as collisions break off any protuberances. However, extruded pellets and some agricultural grains such as rice have extremely large aspect ratios, and it is unclear what effect this might have. Large-aspect-ratio particles may have preferred orientations of their principal axes within a given flow field. Will such particles tend to align themselves randomly, in the direction of the local velocity gradient, perpendicular to it, or will they be unstable and flop around like a disk in a viscous shear flow? The answers may significantly affect the flow properties, as random orientation of such particles would tend to disrupt the freedom of particle motion and, in particular, may disrupt the formation of the microstructure.

**Interstitial Fluid Effects**

Bagnold’s (1954) original experiments that led to Equation (1) were performed on densely loaded suspensions of wax spheres in a glycerin-water-alcohol mixture. He found that the validity of Equation (1) depended on a dimensionless quantity, which has subsequently been called the Bagnold number:

\[
\text{Ba} = \frac{4 \rho_p R^2 \gamma (v)^{1/2} \gamma}{\eta},
\]  

(7)
where \( \rho_p \) is the density of the particle (or, in this case, the fluid, since the densities must be the same in a suspension), \( \eta \) is the dynamic viscosity of the interstitial fluid, and \( \lambda(v) = (v_0/v - 1)^{-1/3} \) is the "linear concentration" in which \( v_0 \) is the maximum static volume concentration. [The function \( \lambda(v) \) is roughly the ratio of the particle diameter to the average free space between particles and was Bagnold's approximation to the \( f_\lambda(v) \) function in Equation (1).] If \( Ba < 40 \), which he dubbed the "macro-viscous" regime, he found that the suspension behaved like a Newtonian viscous fluid with a viscosity proportional to \( \lambda^{3/2} \eta \). If \( Ba > 440 \), i.e. the "grain-inertia" regime, the viscosity varied proportional to \( \rho R^2 \lambda^2 \gamma \), much the same as in Equation (1). Thus the Bagnold number is the ratio of the grain-inertia and the macro-viscous viscosities. These experiments were repeated by Savage & McKeown (1983). They found, however, that the inertial stresses were more or less independent of particle size and did not show the \( R^2 \) dependence as determined by Bagnold (and dimensional analysis). Savage & McKeown concluded from this observation that liquid turbulence—which acts over length scales on the order of the shear gap width rather than the particle size—was playing a larger role in the results than Bagnold had allowed. Bagnold performed his experiments with only one particle size, and the \( R^2 \) dependence was inferred from his analyses. In Bagnold's defense, however, it should be noted that while both sets of experiments were performed in coaxial cylinder rheometers, Savage & McKeown rotated the inner cylinder of the rheometer, whereas Bagnold rotated the outer cylinder. This would make Savage & McKeown's flows subject to Taylor instabilities that will both make the flow more prone to turbulence and also induce flows that act over a length scale proportional to the gap spacing. Thus, it is still possible then that both are correct, and that Savage & McKeown's measurements are dominated by the length scale of the instability, while other experiments could be dominated by particle size.

But all of this raises the question, what mechanism generated the forces that Bagnold measured in the grain-inertia regime? As these experiments were performed on suspensions, the density of the fluid equals the density of the solid, and thus one cannot assume that the forces carried by particle inertia dominate over those carried by fluid inertia. Compounding this confusion, Equation (1) is strongly reminiscent of Prandtl's mixing-length theory for fluid turbulence, indicating that the same fluid mechanics that leads to the Bagnold dispersive stresses also excites fluid turbulence. There is not yet enough detailed information to make a judgment as to the importance of each mechanism in determining the final stresses.

Regardless of the questionable interpretation of the suspension data, a limited amount of current evidence suggests that Equation (7) may be valid for flows in which the solid particles are denser than the interstitial
fluid. Hanes & Inman (1985) report shear-cell experiments on glass beads in water, which roughly follow Bagnold’s prescriptions. Buggish & Stadler (1986) and Stadler & Buggish (1985) performed shear-cell tests on fine powders in air, which, by virtue of their small size, correspond to small Bagnold numbers; these tests indicate a behavior more akin to a Bingham fluid than a rapid granular flow, in that the powders require an initial yield stress (which may be due to strong cohesive or adhesive forces between the particles) and, after yielding, take on additional stresses that are linearly proportional to the shear rate, indicating macroviscous behavior. Finally, Zeininger & Brennen (1985) (see also Zeininger et al. 1983) have performed experiments on underwater hopper flows (which are extensional rather than shear flows) and found that the data very nicely obeyed Bagnold’s prescriptions based on an extensional Bagnold number, which indicates some universality in Bagnold’s concept.

Obviously, a theory for the flow of particles within an interstitial fluid that considers in full the interaction between the particles and the turbulence of the fluid is too complex to be handled exactly. The problem has been tackled, in a very approximate manner, theoretically by Shen & Ackermann (1982) and in a computer simulation by Shen et al. (1988). They assumed that the sole contribution of the interstitial fluid was to act as a simple drag on the particles, the major effect of which was to add a dissipation mechanism for the granular temperature that does not depend on the collision rate. As a consequence, the generated temperatures and stresses are smaller, and the stress asymptotes that occur as \( v \to 0 \) in Figure 3 (which depend on the temperature dissipation going to zero) will be considerably dampened and may not make an appearance at all. This is probably a good approximation for very dense mixtures when particle collisions provide the large majority of the momentum transfer. A much more sophisticated theory has recently been devised by Savage & Lun (1989) that takes into account the interaction of the particle motion and the fluid turbulence. The mixture is modeled as individual phase equations for the particle and fluid that are coupled through interaction terms. The particle mechanics are modeled in much the same way as in the fluid-free rapid-flow theory of Lun et al. (1984), but additional terms are added to the equation of motion to account for buoyancy, drag, and added-mass forces on the particles, and to the energy equation to describe the exchange of energy between particles across the fluid and from particle to the fluid during a collision. The fluid turbulence is modeled by an eddy-viscosity theory in which the particle-fluid interactions appear as source terms for the turbulent energy. The method depends on a few heuristically determined constants, but it is enlighteningly intuitive and shows great promise.
Solid/Fluid Behavior of Granular Systems

There are a large number of industrial situations in which a granular material may assume a fluid (flowing) or solid (nonflowing) behavior. The most significant of these are funnel flows in hoppers, for which stagnant material covers the walls, creating a funnel through which a central core of material flows. The edge of the stagnant funnel is an interface; on one side of it the material is flowing, and on the other side the material is stagnant. Similar behavior has been observed in nearly all bulk-material handling devices, in Hanes & Inman’s (1985) shear cell, and in a chute-flow computer simulation of Campbell & Brennen (1985b). The performance of any such system depends on the shape of this interface, which in turn must be determined by the forces that govern the change between solid and fluid behavior. However, it has not been possible to accurately predict either the occurrence of funnel flows or the shape of the resulting funnel. This seems like an ideal topic for rapid-granular-flow theory, although in the fluid/solid transition, long-duration contacts may be expected between particles, and these stretch the validity of rapid-flow assumptions and suggest a transition not only from fluid to solid behavior but also from rapid to quasi-static flow. [It is interesting to note, however, that Campbell & Brennen (1985b) assumed that all contacts between particles were instantaneous collisions in their computer simulation, yet they still observed a fluid-solid transition.]

Because rapid granular flows are treated as dissipative gases, the flowing/nonflowing transition may be looked upon as akin to a phase change within the material and therefore is governed by the local conditions of stress, concentration, and granular temperature through some kind of state equation. At its simplest the material assumes a solid behavior because the local granular temperature is too small, given the local stress state, to dilate the material below the maximum solid concentration at which flow can occur. Presumably, if the temperature could be raised to high enough levels, the material would dilate and assume fluid behavior. The problem is compounded because, with no velocity gradient within the solid region, there can be no shear work performed and thus no local temperature generation can occur, so that any granular temperature in the solid zone must be conducted in from the outside. [This process was observed in one of Campbell & Brennen’s (1985b) chute-flow simulations.] However, due to the nature of the local microstructure, there may be more to the phase-change process than simply some pressure-density-temperature relationship. Remember that the formation of the layered or stringlike structures within the material allows a shear flow to be maintained at densities at which the material would assume solid behavior if the particles were
randomly configured. Thus the phase change might or might not occur at a given pressure, density, and temperature, depending on the local microstructure. But, as stated earlier, the evolution of a microstructure and the mechanisms that bring it about are not well understood.

This problem has been examined from a completely different viewpoint by Johnson & Jackson (1987). They modeled a granular flow following Lun et al.'s (1984) analysis but simply added Coulomb friction stresses to the equation of motion to act as a bridge between the rapid-flow and quasi-static-flow regimes. This may better approximate the material behavior when there are long-duration contacts between particles, but also in this picture there would be no shearing in regions where the Coulombic forces could not be overcome. As the yield friction (which may be different from the friction in the flowing material) is allowed to vary with solid fraction, this analysis mirrors many of the features described in the previous paragraph.

**CONCLUDING REMARKS**

It was only a dozen years ago that Ogawa (1978) first suggested the idea of a granular temperature, and only within this decade have fluid-mechanical theories of rapid granular flow appeared in the literature. Since then, a great deal of work has been performed to demonstrate the validity of modeling granular materials as dense gases under the assumption that the granular temperature displaces the thermodynamic temperature as the driving force behind all of the transport processes. So far, the progress of granular-flow theory has largely been due to the interplay between theoretical models and computer simulations. All of this work is done under the umbrella of laboratory experiments, but unfortunately direct laboratory experimentation is relatively unsophisticated. The experimental data that do exist measure global properties—such as the stress/strain-rate behavior of the system—with very few measurements of microscopic properties such as the granular temperatures; the results are thus clouded by uncertainties about the actual material behavior inside the experimental device. Yet both theoretical analyses and computer simulations are based on material and boundary properties, and the behavior of the flow can be drastically altered by seemingly innocuous changes in the chosen models. At the same time, there is little direct experimental evidence to assess the appropriateness of the model choice. This field sorely needs experimental procedures capable of making detailed measurements of properties such as velocity, density, and granular-temperature profiles to help ease through these controversies.
ACKNOWLEDGMENTS

This review was supported by the International Fine Particle Research Institute and the National Science Foundation under grant number MEA-8352513, for which the author is truly grateful. Special thanks go to Professors Mark Richman and Hayley Shen for their comments on an early draft of this paper, and to Holly Van Campen for proofreading the manuscript.

Literature Cited


Drake, T. G. 1988. Experimental flows of
Kanatani, K. 1979. A micropolar continuum


Savage, S. B. 1987. Interparticle percolation
RAPID GRANULAR FLOWS


Werner, B. T. 1988. Computer simulation of sand surface self-organization in wind...


CONTENTS

NOTE ON THE HISTORY OF THE REYNOLDS NUMBER, N. Rott 1
ISSUES IN VISCOELASTIC FLUID MECHANICS, Morton M. Denn 13
WAVE LOADS ON OFFSHORE STRUCTURES, O. M. Faltinsen 35
RAPID GRANULAR FLOWS, Charles S. Campbell 57
AERODYNAMICS OF HUMAN-POWERED FLIGHT, Mark Drela 93
BOUNDARY LAYERS IN THE GENERAL OCEAN CIRCULATION, Glenn R. Ierley 111
PARAMETRICALLY FORCED SURFACE WAVES, John Miles and Diane Henderson 143
WAVE–MEAN FLOW INTERACTIONS IN THE EQUATORIAL OCEAN, M. J. McPhaden and P. Ripa 167
MIXING, CHAOTIC ADECTION, AND TURBULENCE, J. M. Ottino 207
PANEL METHODS IN COMPUTATIONAL FLUID DYNAMICS, J. L. Hess 255
NUMERICAL MULTPOLE AND BOUNDARY INTEGRAL EQUATION TECHNIQUES IN STOKES FLOW, Sheldon Weinbaum, Peter Ganatos, and Zong-Yi Yan 275
PLASMA TURBULENCE, P. L. Similon and R. N. Sudan 317
SCIENTIFIC METHODS IN YACHT DESIGN, Lars Larsson 349
OPTICAL RHEOMETRY, Gerald G. Fuller 387
AEROTHERMODYNAMICS AND TRANSITION IN HIGH-SPEED WIND TUNNELS AT NASA LANGLEY, I. E. Beckwith and C. G. Miller III 419
VISCOUS-FLOW PARADOXES, M. A. Goldshtik 441
LOCAL AND GLOBAL INSTABILITIES IN SPATIALLY DEVELOPING FLOWS, Patrick Huerre and Peter A. Monkewitz 473
WAKES BEHIND BLUNT BODIES, H. Oertel, Jr. 539
INDEXES
Subject Index 565
Cumulative Index of Contributing Authors, Volumes 1–22 573
Cumulative Index of Chapter Titles, Volumes 1–22 577
vii