A TWO-DIMENSIONAL DYNAMIC SIMULATION
OF SOLID FRACTURE
PART I: DESCRIPTION OF THE MODEL

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Received 8 May 1995

This paper describes a two-dimensional computer simulation of solid fracture that allows the body and the fragments to be followed well beyond the point of simple crack formation. The model is based on discrete particle computer simulations used for studying granular flows. Here, macroscopic polygonal solid are constructed by "gluing" together small elements. Depending on the stress conditions the glued bonds between the elements can respond elastically, undergo plastic failure or break, allowing cracks to propagate across the macroscopic particle along the boundaries between their microscopic constituents. In essence, this process creates a simulated material upon which breakage occurs. Several element shapes have been studied.

Keywords: Fracture; Fragmentation; Computer Simulation.

1. Introduction

Fracture is a difficult problem to handle numerically as the formation of new surface and subsequent motion of the fragments continuously change the location of the boundaries of the system; thus, it is difficult to apply boundary conditions to the model equations. The model, described herein, is an attempt to deal with this problem from a completely different point of view, by building an equivalent macroscopic solid out of microscopic polygonal elements. Each of the microscopic elements may be thought of as a collection of atoms or molecules that make up the bulk material. These elements are "glued" together in the sense that the faces of
the polygons that make up the interelement contacts can withstand a specified tensile force before they break and allow a crack to form and propagate along the interelement joint.

The model is an outgrowth of the technique of discrete particle computer simulation that have long been used to study granular flows (see the review by Campbell1). The particular technique is historically related to that proposed by Cundall & Strack2 which is known variously as the Distinct or Discrete Element Method (but always denoted DEM); however, neither name is appropriate for the simulation proposed in this paper as the elements are neither discrete or distinct, but are indistinguishable until the material has become finely fragmented. In fact, the original intention of developing this model was to simulate granular flows in which the particles are allowed to break, thus allowing study of flow induced particle attrition and fragmentation. However, that may only be one application for this type of model. In many ways, this model is similar to that developed by Colin Thornton3,4 to study the breakup of agglomerates of circular or spherical particles and that developed by Meguro & Hakuno5 to study the fracture of concrete structures. Trent6 has developed a similar model, but with a more complicated contact model, although he has only employed it in wave propagation studies in granular materials.7,8 Unlike the simulation technique described herein, all of these use circular discs or spheres that meet at point contacts. Thus a crack propagates in a staccato fashion from element to element rather than the continuous breakage permitted by the proposed model's use of continuous joints. The use of point contacts is appropriate for the cemented materials to which the above models are applied, but they may not be appropriate for the modeling of continuous solids. (Trent6-8 avoids this problem somewhat by allowing partial breakage of his point contact; i.e., as the contact is stressed, it becomes weaker, just as if a crack were propagating across it.) Hocking9 has used rectangular parallelepiped elements that fail under bending induced loadings to study the fragmentation of ice sheets; thus, continuous crack propagation is permitted although still staccato as it occurs on an element by element basis (although he eliminates this problem somewhat by constraining the rate of crack propagation).

The current model is based on the soft-particle model for arbitrary two-dimensional convex polygons developed by Hopkins10 (the methods by which contacts are detected, elements are moved, etc., may be found in that report.) Arbitrarily shaped macroscopic solid polygons (which need not be convex) are created by gluing together convex polygonal elements. Until broken, the contacts between elements are modeled as generalized springs creating something similar to the lattice models of elastic solids. These were first formulated and analyzed by Cauchy11 (as reported in Ref. 12), however, they have recently been adapted to computer models of solid fracture (e.g., Refs. 13-14). A glued joint may fail either plastically or by fracture. An approximate form of plasticity is introduced by allowing slip between element surfaces whenever the tangential force on the joint exceeds a specified failure criterion. However, if the tensile force on a portion of the contact (or
the tensile force that is implied by a moment acting on the element edge) exceeds a specified value, that portion of the contact breaks, a crack forms and, if the stress conditions permit, may continue to propagate along the joints between neighboring glued elements. Once a glued contact is broken, the formerly glued surfaces of the two elements involved behave as free surfaces.

Essentially, this process creates a simulated solid and, hence, the process is referred to as a "simulation". In this sense, it is profoundly different from finite element or other standard numerical techniques which are essentially integrations of continuum constitutive models. From a materials point of view, one might think of the inter-element joints as grain boundaries which, in a real material, define the paths along which cracks usually propagate. As such, it naturally allows inclusion of spatial inhomogeneities that are difficult to incorporate into continuum models. Like its predecessors in granular flow simulations, it is easy to trace the overall behavior of the systems, back to the assumption used in creating the simulated solid.

This paper will describe how the model is created and present an analysis of its elastic properties. Examples of its use are given in Part II. Collectively, these will indicate how successfully the model simulates an actual solid. A preliminary report of this work appears in Ref. 15.

2. Description of the Model

This model is based on a soft-particle discrete particle computer simulation, which may be thought of as the simultaneous integration of all the equations of motion of all of the elements that make up the simulation (see Ref. 1). The integration is performed by a standard, second-order technique. The simulation progresses in short time increments which constitute the stepsize of the numerical integration. However, the force to which the elements respond are largely applied by contact with neighboring elements. Thus, much of the integration procedure involves determining when contacts form and break and in resolving the forces that arise from the contact. The nature of the contact model will be amplified upon in the following.

Consider a macroscopic two-dimensional body divided into smaller microscopic elements. (The elements may have any convex polygonal shape. However, equilateral triangles were the first shape considered and thus appear in many of the following illustrations.) Let the characteristic length of the macroscopic body be \( L \) and the length of an element side be \( L_p \). The forces within the body are resolved as forces at inter-element contacts and with at contacts with walls and other bodies. Each element may experience two types of contacts which will be referred to as "glued" and "collisional". Glued contacts are the joints between elements interior to a solid body which can support tensile stresses and, thus, support the structure of the unbroken body; the glued bonds break when their tensile stress is exceeded and allow a crack to form. Collisional contacts refer to contacts between the constituent elements of the body with other bodies or walls, as well as contacts inside
the body where glued contacts have broken. Figure 1 shows examples of two bodies divided into triangular elements which undergo both collisional and glued contacts. The two contact types will be described in detail in the following sections.

![Diagram of collisional contact with wall and collisional contact between particles with glued contacts]

Fig. 1. A collision between two macroscopic particles showing the division of the particles into elementary triangles and examples of collisional and glued contacts.

2.1. **Glued contacts**

Suppose that two elements have a common glued contact, such as are shown in Fig. 2. One of the elements will be referred to as the “home” element and the other as a “neighbor”. The outward pointing normal unit vector to the contacting side of the home element will be denoted $\hat{n}_{hm}$, and the tangentially pointing unit vector will be $\hat{t}_{hm}$.

The glued contact between the elements will respond much like a spring to an applied force. Thus, a tensile force will pull elements apart and a compressive force will cause them to overlap slightly. As drawn (although greatly exaggerated), the triangular elements in Fig. 2 are subjected to forces that are generating a shear-like response — i.e., tensile forces (surfaces are pulled apart) on a portion of the contact and compressive forces (elements overlap slightly) on the other portion. The relative displacement of some point on the side of a neighboring element with respect to the point on the home element with which it was initially coincident may be represented by the vector $\vec{S}$ connecting these two points. The vector $\vec{S}$ is chosen to point from the home element to the neighbor element. In a basis formed of unit vectors $\hat{n}_{hm}$ and $\hat{t}_{hm}$, (oriented normal and tangential to the side of the home triangle) this vector may be represented as $\vec{S} = (S^a, S^t)$.

The vector $\vec{S}$ may be thought of as an elastic fiber connecting initially coincident points on the home and neighbor elements. The elastic fiber behaves as a linear
spring and is stretched or compressed as the points on the sides of the elements are pulled apart or pressed together. This fiber has a stiffness $K_n$ in the direction normal to the surface and stiffness $K_t$ in the direction tangential to the surface. (From here on, $K_n$ and $K_t$ will be referred to as the "normal" and "tangential" stiffnesses, respectively.) The forces created by this fiber in the basis of vectors $\mathbf{n}_{hm}$ and $\mathbf{t}_{hm}$ will then be $K_n S_n$ and $K_t S_t$ and the directions of the forces exerted on the home element will be in the direction of the projection of vector $\mathbf{S}$ onto $\mathbf{n}_{hm}$ and $\mathbf{t}_{hm}$, respectively. The force applied to the neighbor element have the same magnitude, but act in the opposite direction. Note, as the values $K_n$ and $K_t$ may be different, the direction of the applied force will not, in general, coincide with the direction of vector $\mathbf{S}$. By integrating all of the forces applied to the elements by all such elementary fibers along the sides of all glued contacts, it is possible to obtain the total elastic forces $F_{ehm}$ and $F_{enb}$ applied to home and neighbor elements. Note that the total normal force will be roughly proportional to the areas covered by the extended elastic fibers.

An approximate form of plasticity is introduced into the model by allowing the sides of the elements to slip relative to one another following a yield criterion. This method is motivated by plastic flow induced in crystals by the motion of dislocations. Thus, as for molecules in a crystal, there must be well defined planes within the material, along which the elements might slip. This method must be limited to small deformations as the motion of elements along one slip plane will eventually disrupt and prohibit motion along other internal slip planes; as a general
deformation may require slip along more than one plane, there are thus limits on
the allowable plastic motions. The Coulomb law was adopted as a yield criterion:

$$|\sigma_t| \leq Coh + f\sigma_n$$

(2.1.1)

where $\sigma_t$ is tangential stress on some plane inside the body, $\sigma_n$ is normal stress
on this plane, $Coh$ is a cohesive limit and $f$ is the internal friction coefficient.
For simplicity, $\sigma_t$ and $\sigma_n$ will be determined as the total elastic-plastic tangential
and normal forces on the glued contacts divided by the length of the contact; i.e.,
any variation in the force along the contact that might cause yielding along only a
portion of the joint, will not be taken into consideration. As usual, the normal force
is taken to be positive in compression and negative in tension. In tension, $\sigma_n < 0$,
the frictional term is omitted from the calculation so that yield occurs when:

$$|\sigma_t| = Coh$$

(2.1.2)

It might be possible to add some hardening or softening criterion based on the
degree to which a joint has failed, but these have not been explored at this time.

The Coulomb criteria gives the total elastic–plastic forces $\overline{Fe_{phm}}$ and $\overline{Fe_{phb}}$ on
a yielding contact. These will have the same normal components as $\overline{Fe_{hm}}$ and $\overline{Fe_{nb}}$
but may differ from the elastic tangential components. The elastic–plastic moments,
$\overline{Me_{phm}}$ and $\overline{Me_{phb}}$, are found by integrating the contribution of the forces along each edge,
multiplied by appropriate moment arms. These moments may be regarded as scalar
values in two-dimensional problems.

As this is basically a dynamic Lagrangian model, some dissipative force should be
introduced to the system to maintain stability and suppress the artificial vibrations
induced by numerical errors. This is done by introducing a small degree of viscosity
into the contact model. This viscous force is applied to each elementary fiber,
acts in a direction parallel to the direction of the vector $\frac{d\bar{S}}{dt}$, and generates a
reaction force proportional to, and opposite in direction to $\frac{d\bar{S}}{dt}$ with a coefficient
of proportionality $K_v$. By integrating the forces over the element edges it is possible
to obtain values of viscous forces $\overline{Fv_{hm}}$ and $\overline{Fv_{nb}}$ and viscous moments $\overline{Mv_{hm}}$
and $\overline{Mv_{nb}}$ applied to home and neighbor elements. These forces are added to
elastic–plastic forces and elastic moments to obtain the total forces and moments
on each glued contact. However, the viscosity required for stability is of such small
magnitude that it dissipates only 0.02% of the collisional energy in test collisions of
single particles with walls, such as will be described in Part II.

2.2. Collisional contacts

Collisional contacts occur between surfaces of elements that are free or partially
broken. A collisional contact can withstand no tensile force and occurs only when
there is some overlapping of elements. The generated forces will have elastic, viscous
and frictional components which are modeled as described in Ref. 10. Consider the two elements shown in Fig. 3. They overlap with an area $A$ which is changing at a rate $dA/dt$. Then the absolute value of the elastic force on this contact will be $K_n A$. Similarly, the absolute value of viscous-like force will be $K_v |dA/dt|$. Although it is not formally necessary, $K_n$ and $K_v$ will generally be given the same values as for glued contacts. The viscous-like force acts in the direction opposite to the direction of change of the elastic force. Both forces act through the center of gravity of the overlapping area.

![Diagram](image)

Fig. 3. A collisional contact between triangles. The normal and tangential vectors to the contact plane are $\vec{n}_{hm}$ and $\vec{t}_{hm}$, respectively. The contact plane is represented by a dashed line.

The direction in which the elastic and viscous-like forces are applied is determined by introducing the idea of a contact line as the line lying between the points at which the edges of the impacting elements intersect. Figure 4 shows several exaggerated examples of how elements might overlap while undergoing collisional contact; within these figures, the contact line is drawn as a dotted line. The elastic and viscous forces act on the elements in a direction perpendicular to the contact line. The definition of the contact line becomes somewhat complicated if, as shown in Fig. 4(a) and Fig. 4(d), the vertex of a element passes through to the opposite side of the other. The case shown in Fig. 4(a) will be quite common for triangular elements as the positions of vertices of glued contacts will initially coincide and may be held in this position by neighboring elements, even after the contact is broken; any compression of the joint will move the centers of the elements towards one another and will, in mathematical sense, push the vertices through one another as shown. Should this happen there will be more than two intersections between
the edges of the elements and some choice must be made between the possible contact lines. The decision is made by requiring that the location of the contact line proceeds smoothly, with no abrupt jumps between possible choices. The procedure presented in Ref. 16 is used to calculate the area of overlap at each moment of time.

Fig. 4. Several examples showing the orientations of contact planes for different geometries of overlap between triangles.

A collisional contact also experiences a frictional force which acts parallel to the contact plane in the direction opposite to the relative velocity of the elements. The frictional force is yield-like and characterized by a specified friction coefficient, $\mu$, which, in general, will be different from the coefficient of internal friction, $f$, used for the plastic limit in (2.1.1). As long as the force, $F_1$, acting parallel to the contact plane is smaller than $\mu F_n$ (where $F_n$ is the corresponding force normal to the contact plane), $F_1$ will be elastic and proportional to the relative displacement of the elements along the direction of the contact plane since the beginning of contact. The corresponding elastic constant is $L_\mu K_1$. If the predicted elastic response exceeds $\mu F_n$, then the magnitude of $F_1$ is chosen to equal $\mu F_n$, and this force acts in the direction opposing the relative motion.

By applying the above procedures it is possible to calculate elastic, viscous-like and frictional forces $F_{e_{hm}}, F_{e_{nb}}, F_{v_{hm}}, F_{v_{nb}}, F_{f_{hm}}, F_{f_{nb}}$ on the home and neighbor elements and also the moments created by these forces: $M_{e_{hm}}, M_{e_{nb}}, M_{v_{hm}}, M_{v_{nb}}, M_{f_{hm}}, M_{f_{nb}}$. Body forces may be easily added to the model. The gravity force $F_{g_{hm}}$ and $F_{g_{nb}}$ are just the masses of each element multiplied by
the gravitational acceleration \( \bar{g} \), act through the center of gravity of the elements and do not create moments.

When a glued contact breaks, it undergoes transition into a collisional contact. Where and when it breaks depends on the amplitude of the tensile stress, in the direction normal to the contact, relative to the strength of the body, \( \sigma_{\text{tens}} \). This process creates the simulated material with the tensile strength \( \sigma_{\text{tens}} \). This corresponds to a work of fracture (energy released per unit length of the crack) \( \sigma_{\text{tens}}^2/(2K_n) \), which is the energy stored in the normal component of the deformation of the "elementary fiber" at the moment of breakage when the normal stress equals \( \sigma_{\text{tens}} \). Any limit on the speed of crack propagation through the glued joint can also be easily added to the model. Now, the breakage will always begin at a vertex as the tensile stresses will be largest there. But, a glued joint does not break all at once; only those portions that exceed the tensile stress criterion will be broken. Consequently, it is possible that there may be partial breakage of a joint such as that shown in Fig. 2, even though the joint is partially experiencing a compressive stress. (In the earliest realization of the simulation, the entire joint was broken instantaneously, once the average tensile stress exceeded the tensile strength of the body. However, this released all the joints elastic energy instantaneously, producing local elastic waves that were strong enough to induce further breakage. That problem was avoided by permitting partial joint breakage. However, similar problems will arise in point-contact models such as those described in Refs. 5–8.

3. Choice of Element Shapes

The first element shape we considered was the equilateral triangle as it is the simplest geometrically uniform space filling shape, that at every vertex will allow a crack to propagate nearly isotropically in any of six directions. It is also well-known in crystallography (see, for example, Ref. 17) that planar systems with hexagonal symmetry (such as equilateral triangles) are two-dimensional crystalline structures that are describable by two elastic moduli as is the case for isotropic elastic bodies. Bodies, so divided, have lines of joints between the elements that span the body, and thus permit large-scale slip and can exhibit large-scale plastic deformations.

However, problems arise because a body composed of equilateral triangles is, in many ways, too perfect a crystal. Those same lines of joints that permit large scale plastic behavior induce a large degree of directionality into the possible fracture patterns. Examples of this can be seen in Fig. 5, which shows the collision of two identical circular particles with flat plates. The interiors of these particles consist of crystals of equilateral triangles, although different element shapes must be used near the circular boundary. Figure 5(a) shows a collision in which the contact occurs directly along a joint that spans the body. As a result of the collision, the particle splits in two along the central joint, in what is often called a "median vent crack". Physically, the source of this crack may be understood if one realizes that even though the collision produces no torque about the center of mass of the particle, it does produce
a torque about any point that lies off the vertical line that intersects the contact point — in particular about the centers of mass of the two particle halves into which the particle has broken in Fig. 5(a). If the particle remained whole, those torques would be balanced by the tensile stress along the vertical line through the particle center and are thus the source of the median vent crack.

However, if the particle is rotated so that the collision occurs perpendicular to the base of the internal hexagonal crystal, such as shown in Fig. 5(b), the fracture pattern is entirely different and no median vent crack appears. Instead, two symmetric cracks form at a 60° angle, that follow the crystal pattern from the impact point across the particle. Mechanically, these two cracks fulfill the same function as the median vent crack.
Such directionality may be appropriate for modeling perfect crystals where different behavior might be expected for different geometrical relationships between the point of impact and the internal crystal planes. However, the fracture patterns for amorphous bodies should be nearly the same, regardless of direction. Such materials might be modeled using more or less randomly oriented internal surfaces. A random orientation has the further advantage that, as long as the element size is small compared to the macroscopic particle, they will have no preferred directions and the bulk material will again be describable by two elastic moduli. However, due to the lack of internal slip planes, they should exhibit no, or at best very limited, plastic behavior. To this end, we studied two organized methods of dividing the macroscopic particles into polygons of nearly uniform size, but somewhat random shape and orientation: Voronoi polygons and Delaunay triangles.

Voronoi polygons are generated by arbitrarily dropping points onto a two-dimensional body. Assume that there are $N$ such points whose coordinates are stored in an array $B$, such that $B(i)$ represents the coordinates of the $i$th point in the series. The Voronoi polygon number $i$ is a locus of points closer to the point $B(i)$ than to any other point of the array $B$. Given two points $B(i)$ and $B(j)$, the set of points closer to $B(i)$ than $B(j)$ is just the half-plane containing $B(i)$ that is defined by the perpendicular bisector of the line linking $B(i)$ and $B(j)$. The intersections of the $N-1$ such half-planes containing $B(i)$ will be Voronoi polygon number $i$, and the set of all such polygons is a Voronoi diagram of the body. (Obviously, as there are infinitely many choices for how the points are distributed on the body, there will be infinitely many Voronoi diagrams for a given macroscopic body.) Computationally more efficient procedures for generating a Voronoi diagram than that described above, as well as a discussion of the characteristics of such diagrams, may be found in Ref. 16. For our purposes, it is desirable to place limits on the sizes of the Voronoi polygons, in particular, because the elastic properties of the body will be shown to depend on the size of the elements. Here, the points are initially dropped onto the surface so that the distance between any two of these points is larger than some value $d_{\text{min}}$, and the distance between any of these points and the boundary of the body must be larger than $0.5d_{\text{min}}$. Points are added until they are approximately evenly distributed across the body. An example of a circular particle, divided into Voronoi polygons is shown in Fig. 6 along with the fracture pattern generated by collision with a flat wall. Notice that, like the triangular material shown in Fig. 5(a), this breaks in two along a nearly vertical median vent crack.

However, the median vent crack in Fig. 6 is not clean and sharp like those for equilateral triangle elements; i.e., the crack is filled with a column of rubble. Furthermore, the bulk of the particle is filled with small cracks that have propagated a distance approximately equal to the length of an element size, but no further. Notice that only three joints meet at each vertex of the Voronoi diagram, while six joints meet at each vertex of the equilateral triangle mesh. Thus, to propagate past a vertex within the Voronoi diagram, a crack has to make a radical shift in direction. The many short aborted cracks indicate that this puts severe limitations on
the ability of a crack to propagate through the material. Therefore, it is conceptually desirable that as many inter-element joints as possible meet at each vertex in the material, so that a crack might have many choices of propagation direction.

A similar procedure can be used to divide the macroscopic body into random triangular elements known as Delaunay triangles. The procedure again begins by dropping points nearly randomly onto the surface (but with the same limits on the spacing of the points as for Voronoi polygons). But, instead of drawing polygons about those points, they are used as the vertices of triangles. In this scheme, two such points $B(i)$ and $B(j)$ are connected if and only if Voronoi polygons drawn about those points would share a common side. Such a division is called a Delaunay tessellation. An example of so dividing a circular particle and the results of its

![Image 6](image6.png)

**Fig. 6.** The division of a circular particle into Voronoi polygons and the fracture pattern resulting from a collision with a solid wall. The initial velocity is vertically downward.

![Image 7](image7.png)

**Fig. 7.** The division of a circular particle into Delaunay triangles and the fracture pattern resulting from collision with a solid wall. The initial velocity is vertically downward.
collision with a flat wall is shown in Fig. 7. The fracture pattern is asymmetric due to the randomness in the division, but lacks the short abortive cracks of the Voronoi diagram.

By now it should be apparent that the pattern, into which macroscopic particles are subdivided, affects their behavior in much the same way as does the arrangement of molecules in a real material. Equilateral triangle elements show a preference for large-scale fracture along their crystal planes. Furthermore, they permit plastic-like deformations by allowing slip between elements along those planes in much the same way as plastic deformation is envisioned as occurring between molecules in real crystals. Divisions with randomly oriented sides, such as Voronoi polygons and Delaunay triangles have no internal slip planes and thus will only show brittle behavior in much the same way as solids with amorphously distributed molecules.

4. Elastic Properties

As is apparent from the above discussion, the behavior of the bulk solid created from the microscopic elements depends on the properties (size, shape and stiffnesses) of the elements from which it was created. The goal of this section is to theoretically predict the overall elastic behavior of the macroscopic body for all of the element shapes so far discussed. All of these configurations will be describable by two elastic moduli such as Young’s modulus, \( E \), and Poisson’s ratio, \( \nu \). For a real material, such properties are usually determined by a uniaxial loading of the body inside a press with rigid walls. Similar tests (schematically illustrated in Fig. 8) will be simulated in order to determine the elastic properties of our simulated material. However, the elastic properties may also be determined by a theoretical analysis.

![Fig. 8](image)

(a) vertical orientation, (b) horizontal orientation.

There will be some confusion as to the interpretation of these results arising from taking concepts from three-dimensional elasticity and applying them to two-dimensional systems. The two-dimensional loading may be considered to occur in either a plane-stress state or a plane-strain state or somewhere in between. In a plane-stress state, it is assumed that there is no stress in the direction out of the two-
dimensional plane — but then there may be strain in that direction. In a plane-strain state, however, all of the strain is confined to the two-dimensional plane which assumes that stress is applied in the third dimension to prevent any corresponding strain. Which state is assumed will not affect the actual results of a simulation in terms of the forces and displacements of the elements, but will affect how those results are interpreted as the elastic properties, $E$ and $\nu$. The calculations below will assume plane-stress conditions to begin with and then will be reinterpreted for plane-strain conditions. For uniaxial loading of a rectangular sample under plane-stress conditions, Young's modulus $E$ is just the ratio of the force per unit length, $|\mathbf{N}|$, acting on the walls of the press to the strain in the direction of the loading and Poisson's ratio $\nu$ is ratio of the strain in the direction perpendicular to the direction of loading to the strain in the direction of loading (it is implicitly assumed that the same strain is observed in the third dimension).

4.1. Elastic properties for bodies composed of equilateral triangle elements

Consider a two-dimensional rectangular body divided into equilateral triangles such as those shown in Fig. 8. Note that it is impossible to decompose this shape (or most any other shape) into equal sized equilateral triangles. Instead, half-triangles (i.e., 30-60-90 triangles) must be employed along the boundaries. For arbitrary orientations of the triangles relative to the applied forces, these calculations are difficult as the loading induces rotation of the triangles. It is possible to make such a calculation in the limit of large $L/L_p$ in which the induced element rotations are arbitrarily close to zero. However, there are two orientations in which a uniaxial loading induces no rotation regardless of the element size. These are the "vertical orientation" (Fig. 8(a)) where lines of joints are perpendicular to the walls and the "horizontal orientation" (Fig. 8(b)) where lines of joints are parallel to the walls. The calculation will be done separately in these two configurations as the results allow an estimate of the number of elements into which a body must be subdivided in order to yield reasonable elastic properties. It will be shown that, as long as the ratio of body size, $L$, to element size, $L_p$, is large, the orientation is unimportant. It is assumed that the body is loaded by the walls in the vertical direction. Friction between the walls of the press and the body is absent. The height of the body is denoted by $L_y$, while the width is denoted by $L_x$.

First, consider the vertical orientation, such as that shown in Fig. 8(a). To eliminate any end effects, assume that the body in large in extent in the direction parallel to the driving walls. Assume that there is no stress inside the body before loading. Then, let the distance between the walls of the press be decreased by a value $dy$. Because of the symmetry of the problem there can be no moments on any of the triangles inside the body and, as there is no force acting on the body in the horizontal direction, there will be no forces or deformations on the glued contacts between the vertical sides of triangles.
For each triangle inside the body, denote the absolute value of the normal force acting on those surfaces that are inclined with respect to the vertical direction as $F_n$ and the absolute value of the tangential force acting on these sides as $F_t$ (Fig. 9). Because of the symmetry of the problem, all the forces on the triangles must balance. In particular, for the horizontal direction:

$$F_t \sin(60^\circ) = F_n \sin(30^\circ)$$  \hspace{1cm} (4.1.1)

or:

$$F_t \sqrt{3} = F_n.$$  \hspace{1cm} (4.1.2)

Consider two triangles that have a glued contact as a common edge. Let the tangential displacement of the side of one triangle with respect to the side of its glued counterpart be $\delta_t$ and the corresponding normal displacement be $\delta_n$. Since these displacements are proportional to the forces $F_t$ and $F_n$:

$$\delta_n = F_n / (L_p K_n)$$  \hspace{1cm} (4.1.3)

and

$$\delta_t = F_t / (L_p K_t).$$  \hspace{1cm} (4.1.4)

Equation (4.1.2) gives:

$$\delta_t = K_n \delta_n / (\sqrt{3} K_t).$$  \hspace{1cm} (4.1.5)

A vertical displacement $\delta_y$ of center of mass of one triangle with respect to that of its glued neighbor gives:

$$\delta_y = \delta_n \cos(30^\circ) + \delta_t \cos(60^\circ)$$  \hspace{1cm} (4.1.6)

Fig. 9. The forces on a triangular elements in vertical orientation which is experiencing uniaxial compression in vertical direction.
and, from Eq. (4.1.5),
\[ \delta_y = \frac{\delta_n}{[2\sqrt{3}/(3 + K_n/K_t)]}. \] 
(4.1.7)

Since each vertical strip of triangle contains \(2L_y/L_p\) glued contacts, the total displacement, \(dy\), may be expressed as:
\[ dy = 2(L_y/L_p)\delta_y = \frac{2(L_y/L_p)\delta_n}{[2\sqrt{3}/(3 + K_n/K_t)]}. \] 
(4.1.8)

Let \(\sigma_y\) be the vertical stress created by the displacement, \(dy\). Since there is no stress in the horizontal direction and there is no shear stress, the normal stress, \(\sigma_n\), on the surface with outward pointing normal vector, \(\vec{n}\), is:
\[ \sigma_n = \sigma_{ij}n_in_j \] 
(4.1.9)

and taking into account that the normal stress on the inclined side of the triangle equals \(F_n/L_p\):
\[ F_n = L_p\sigma_y[\sin(60^\circ)]^2 = \frac{3}{4}L_p\sigma_y \] 
(4.1.10)

or, from expression (4.1.3):
\[ \delta_n = \frac{3}{4}K_n \sigma_y \] 
(4.1.11)

or:
\[ \sigma_y = \frac{4}{3}K_n\delta_n \] 
(4.1.12)

Using expressions (4.1.8) and (4.1.12) one obtains:
\[ \sigma_y = \left[ \frac{4K_nL_p}{\sqrt{3}(3 + K_n/K_t)} \right] \left( \frac{dy}{L_y} \right). \] 
(4.1.13)

Since \(dy/L_y\) is the vertical deformation of each element, the expression in square brackets is the elastic modulus of this body:
\[ E = \frac{(4K_nL_p)K_t/K_n}{\sqrt{3}(3K_t/K_n + 1)}. \] 
(4.1.14)

Obviously the same result would be obtained for a tensile, rather than a compressive, loading.

The second material constant, required to completely describe the material's elastic properties is Poisson's ratio \(\nu\). For the case of uniaxial compression, \(\nu\) equals the ratio of the body's horizontal and vertical strains. As discussed above, there will be no relative displacement between triangles having glued contacts oriented in the vertical direction. Thus any horizontal deformation \(\delta_x\), must arise on the inclined surfaces so:
\[ \delta_x = \delta_t \sin(60^\circ) - \delta_n \sin(30^\circ) = \delta_t \frac{\sqrt{3}}{2} - \frac{1}{2} \delta_n. \] 
(4.1.15)
Since the total number of inclined contacts that cross the body in the horizontal direction equals \( L_x/[L_p \sin(60^\circ)] \) or \( 2L_x/(\sqrt{3}L_p) \), the total horizontal displacement \( dx \) equals:

\[
dx = \frac{2L_x}{\sqrt{3}L_p} \delta_e
\]  
(4.1.16)

\[
dx = \frac{2L_x}{\sqrt{3}L_p} \left( \frac{\sqrt{3} \delta_t}{2} - \frac{\delta_n}{2} \right)
\]  
(4.1.17)

Substituting for \( \delta_t \) from formula (4.1.5):

\[
dx = \frac{L_x}{\sqrt{3}L_p} (K_n/K_t - 1) \delta_n
\]  
(4.1.18)

and substituting for \( \delta_t \) from (4.1.8) into formula (4.1.18) yields:

\[
dx = \left( \frac{L_x}{L_y} \frac{1 - K_t/K_n}{3K_t/K_n + 1} \right) dy
\]  
(4.1.19)

\[
ex = \left( \frac{1 - K_t/K_n}{3K_t/K_n + 1} \right) \left( \frac{dy}{L_y} \right)
\]  
(4.1.20)

Since \( dx/L_x \) is the horizontal strain of the body as a whole and \( dy/L_y \) is the vertical strain of the body, the expression in parentheses in formula (4.1.20) is Poisson’s ratio \( \nu \):

\[
\nu = \left( \frac{1 - K_t/K_n}{3K_t/K_n + 1} \right)
\]  
(4.1.21)

This ratio is the same for both compression and tension of the body. Note that the possible range of values is \(-\frac{1}{2} < \nu < 1\).

Now consider the body divided into triangles with horizontally oriented contact surfaces (Fig. 8(b)). In this case there are no moments on the triangles far from the free boundaries and a similar analysis can be used to obtain values of \( E \) and \( \nu \), although the values will be slightly different from those given by (4.1.14) and (4.1.21). For any triangle inside the body, let the absolute value of the normal force acting on the horizontal side be \( F_{h_n} \) (there is no tangential force on this contact) and the absolute values of the normal and tangential forces acting on the surfaces inclined with respect to horizontal be \( F_{i_n} \) and \( F_{i_t} \) (due to the symmetry of the problem, these values will be the same for all inclined surfaces, regardless of the orientation). This is shown schematically in Fig. 10. From the conditions of equilibrium for the triangle in the vertical direction:

\[
F_{h_n} = 2[F_{i_t} \cos(30^\circ) + F_{i_n} \cos(60^\circ)]
\]  
(4.1.22)

or:

\[
F_{h_n} = \sqrt{3}F_{i_t} + F_{i_n}
\]  
(4.1.23)
Using formula (4.1.9) and taking into account that, for this case, \( \sigma_y = Fh_n/L_p \)

\[
Fi_n = Fh_n[\cos(60^\circ)]^2 = Fh_n/4.
\] (4.1.24)

Let the absolute values of the normal and tangential displacements of the center of gravity of the triangle experiencing a glued contact with the reference triangle (relative to the center of gravity of the reference triangle) be \( \delta i_n \) and \( \delta i_t \), and let the relative normal displacement of the center of gravity of the triangles having contact with the reference triangle along a horizontal side (with respect to center of gravity of reference triangle) be \( \delta h_n \). Taking into account that

\[
\delta i_n = Fi_n/(L_p K_n)
\] (4.1.25)

\[
\delta i_t = Fi_t/(L_p K_t)
\] (4.1.26)

\[
\delta h_n = Fh_n/(L_p K_n)
\] (4.1.27)

and from expressions (4.1.23) and (4.1.24):

\[
\delta i_n = \delta h_n/4
\] (4.1.28)

\[
\delta i_t = \frac{\sqrt{3}}{4} (K_n/K_t) \delta h_n.
\] (4.1.29)

Since the number of horizontal joints in the y-direction is \( (L_y/(\sqrt{3}L_p/2)) - 1 \), the number of inclined joints is \( L_y/(\sqrt{3}L_p/2) \) and the relative vertical displacement of triangles having glued contacts inclined with respect to the horizontal is \( \delta i_n \cos(60^\circ) + \delta i_t \cos(30^\circ) \), the total vertical displacement \( dy \) is:

\[
dy = \left( \frac{2L_y}{\sqrt{3}L_p} - 1 \right) \delta h_n + \frac{2L_y}{\sqrt{3}L_p} \left( \frac{\delta i_n}{2} + \frac{\sqrt{3} \delta i_t}{2} \right)
\] (4.1.30)
or, substituting the expressions for $\delta i_n$ and $\delta i_t$ from formulae (4.1.28) and (4.1.29):

$$dy = \frac{L_y}{\sqrt{3}L_p} \left( \frac{9}{4} + \frac{3}{4} \frac{K_n}{K_t} - \frac{\sqrt{3}L_p}{L_y} \right) \delta h_n.$$  \hspace{1cm} (4.1.31)

Since $\sigma_y = K_n \delta h_n$, from expression (4.1.31) we have:

$$\sigma_y = \frac{(4K_nL_p)K_t/K_n}{\sqrt{3}(1 + \frac{K_t}{K_n}(3 - \frac{4L_p}{\sqrt{3}L_y}))} \left( \frac{dy}{L_y} \right),$$  \hspace{1cm} (4.1.32)

so the elastic modulus in this case is given by:

$$E = \frac{(4K_nL_p)K_t/K_n}{\sqrt{3} \left[ 1 + \frac{K_t}{K_n}\left( 3 - \frac{4L_p}{\sqrt{3}L_y} \right) \right]}$$  \hspace{1cm} (4.1.33)

which is nearly the same as (4.1.14) except for a term of order $L_p/L_y$ in the denominator.

Poisson’s ratio may be found in much the same way as before. For a relative horizontal displacement, $\delta x$, of triangles sharing a glued contact along their inclined sides:

$$\delta x = \delta i_t \sin(30^\circ) - \delta i_n \sin(60^\circ)$$  \hspace{1cm} (4.1.34)

and, from formulae (4.1.28) and (4.1.29),

$$\delta x = \frac{\sqrt{3}}{8} (K_n/K_t - 1) \delta h_n.$$  \hspace{1cm} (4.1.35)

Since total horizontal displacement $dx$ equals $(2L_x/L_p)\delta x$ and using formula (4.1.31) one obtains:

$$\frac{dx}{L_x} = \frac{dy}{L_y},$$  \hspace{1cm} (4.1.36)

so then expression for Poisson’s ratio is:

$$\nu = \frac{1 - K_t/K_n}{1 + \frac{K_t}{K_n}\left( 3 - \frac{4L_p}{\sqrt{3}L_y} \right)}.$$  \hspace{1cm} (4.1.37)

This expression also differs from (4.1.21) only by a term of order $L_p/L_y$ in the denominator. Thus, it is possible to use (4.1.14) and (4.1.21) as a general representation of the Young’s modulus and Poisson’s ratio for the macroscopic material, as long as the subdivision is sufficiently fine. This might present problems as after fracturing, the size of the fragments, relative to the size of the triangles will be proportionally reduced, causing a change in the apparent elastic properties. One potential solution (which we have yet to implement) would be to dynamically subdividing the fragments further as the simulation proceeds. (Remember that any equilateral triangle may be broken into 4 smaller equilateral triangles.)
These predictions for Young’s modulus $E$ and Poisson’s ratio $\nu$ may be tested using a computer simulation of the compression of a rectangular body in a rigid press with frictionless walls. The computational experiments have been performed in the following way: Initially the walls of the press are in contact with the sample, but are not yet exerting any force upon it. At $t = 0$ they start to move inwards. At the start, their velocity is zero, but they accelerate at a constant rate, $a/(4K_nL_p/\rho_p) = 2 \times 10^{-5}$, (where $a$ is the acceleration) until the velocity, $V$, of the walls reaches $V/\sqrt{4K_nL_p/\rho_p} = 1.25 \times 10^{-4}$, after which the velocity is held constant. (For these values, the body length scale of the problem, $L$, is chosen to be $L_x$ in the case of horizontal orientation and $L_y$ in the case of vertical orientation.) This pattern of loading was chosen to avoid elastic waves that would be induced in the body if the boundary elements were instantaneously accelerated to a constant velocity.

To further avoid artificial vibrations and maintain stability of the system, a small amount of artificial viscosity, $Kv/(\sqrt{4K_nL_p}\rho_p) = 10^{-2}$, has been used for these calculations. Such a small viscosity does not affect these quasistatic results, but it is able to suppress the amplitude of artificial vibrations to about 7 orders of magnitude below the induced elastic forces. To approximate rigid boundaries and rigorously apply the required strain, the stiffness of the collisional contact with the press walls has been chosen to be 3 orders of magnitude larger than $K_n$. The tensile strength and plastic limits were set arbitrarily high so avoid any plastic deformation during this set of simulations.

To obtain values of $E$ and $\nu$, the histories of the total force applied to the body, the wall displacements and the horizontal size of the body were recorded. From these values, it is possible to plot the vertical stress in the body versus the vertical strain and the horizontal strain of the body versus vertical strain. The slopes of these lines yields $E$ and $\nu$, respectively. Five sizes of elementary triangles $L_p$ were used: $L_p/L = 1/4$, $L_p/L = 1/8$, $L_p/L = 1/12$, $L_p/L = 1/16$ and $L_p/L = 1/20$. The integration time step, $dt$, for these problems was chosen to satisfy the Courant criteria for the stability of the numerical calculation, $dt < L_p/C_i$, where $C_i$ is the wavespeed of longitudinal elastic waves under plane-stress conditions:

$$C_i = \sqrt{\frac{E}{\rho_p(1 - \nu^2)}}.$$  

(4.1.38)

The results of the calculation are presented in Fig. 11(a) for $\nu$ and on Fig. 11(b) for $E$. The lines give the predictions given by formulae (4.1.14) and (4.1.21). It is possible to see that accuracy of predictions is very high and all deviations due to the additional $L_p/L$ terms in formulae (4.1.33) and (4.1.37) vanish with decreasing size of the elementary triangles. For the values of $K_i/K_n$ that correspond to realistic Poisson ratios, $0 < \nu < 0.5$, ($0.2 < K_i/K_n < 1.0$), $L_p/L = 1/8$ appears to give good agreement between the values of $E$ and $\nu$ obtained from the different orientations. This difference is larger for the exotic cases of negative Poisson’s ratios and for Poisson’s ratio close to one. Should the study of such materials be necessary, they may be accurately represented using a finer subdivision of the macroscopic elements.
Fig. 11. Predicted and simulated values of the elastic properties of bodies divided into equilateral triangle elements. (a) Young's Modulus, (b) Poisson's ratio. The values predicted by formulae (4.1.14) and (4.1.21) are shown by solid lines. The dashed lines represent the predictions for the horizontal orientation given by (4.1.33) and (4.1.37) for various values of \( L/L_p\).

We have shown, given a small enough size of the elementary triangles, that the material behaves elastically, at least for two orientations of the body. As mentioned before, it is possible to perform a similar calculation for arbitrary orientations in the limit \( L_p/L \to 0 \). However, remember that, through a coordinate rotation, a
general stress state can be decomposed into normal stresses in those two directions and a shear stress. Thus, if the composite is to behave as a true elastic material then the shear modulus $G$ must be related to $E$ and $\nu$ through the expression:

$$G = \frac{E}{2(1 + \nu)}. \quad (4.1.39)$$

This is difficult to prove theoretically as the rotation of triangles (except in the limit $L_p/L \to 0$) must be taken into account, but may be easily checked through the simulation. If the simulations show that $G$ is a well defined quantity that may be predicted from the elastic moduli given in (4.1.14) and (4.1.21), i.e.,

$$G = \frac{\sqrt{3}K_t/K_n}{4(1 + K_t/K_n)}, \quad (4.1.40)$$

then the composite body behaves as an elastic solid. As such simulations will furthermore demonstrate that the composite body behaves elastically even under shear, it is more informative than another lengthy calculation. The simulations are similar to those used to find $E$ and $\nu$ except that the body is sheared, rather than compressed. The results are shown on Fig. 12. To avoid errors induced by the free edges, the width of the sample for this simulation has been chosen to be ten times larger than the height. In all calculations, the body was sheared to 5% strain. It is possible to see from Fig. 12 that the value of $G$ is predicted with the high degree of accuracy by the formula (4.1.40). Thus, this model appears to accurately represent an elastic solid, at least in quasistatic loading.

Fig. 12. Predicted and simulated values of shear modulus $G$. The solid line shows the predictions of (4.1.39), where $E$ and $\nu$ are determined using formulae (4.1.14) and (4.1.21).
Note that in accordance with expressions (4.1.21) and (4.1.37) it is possible to obtain Poisson’s ratios in the range from −1/3 to 1, so that all common Poisson’s ratios, which are usually between 0.1 and 0.5, may be described by this model. The values of \( \nu > 0.5 \) may be somewhat troubling as such materials exhibit volumetric expansion under hydrostatic compression. No such materials are known in nature and are prohibited by thermodynamic arguments. (That is, a hydrostatic compression will cause the body to expand and do work against the applied force.) But these unrealistic Poisson’s ratio greater than 0.5 are permissible under conditions of plane-stress. Since zero stress is assumed in the third direction, any control over the volume is lost, the volume expansion performs no work and is, thus, not excluded thermodynamically.

It is relatively easy to find the corresponding results for plane-strain conditions. Simple manipulations of the relationships between stress and strain under plane-strain conditions yield:

\[
\frac{\epsilon_{xx}}{\epsilon_{yy}} = \frac{\nu_{\text{plane-strain}}}{1 - \nu_{\text{plane-strain}}}. \tag{4.1.41}
\]

As \( \nu_{\text{plane-stress}} = \epsilon_{xx}/\epsilon_{yy} \), one can write a simple relationship between \( \nu_{\text{plane-stress}} \) and \( \nu_{\text{plane-strain}} \):

\[
\nu_{\text{plane-strain}} = \frac{\nu_{\text{plane-stress}}}{1 + \nu_{\text{plane-stress}}}. \tag{4.1.42}
\]

or, in the limit of large \( L/L_p \):

\[
\nu_{\text{plane-strain}} = \frac{1}{2}\left(1 - \frac{K_t}{K_n}\right). \tag{4.1.43}
\]

Notice that \( \nu_{\text{plane-strain}} \) varies between \(-0.5 < \nu_{\text{plane-strain}} < 0.5\) which avoids the thermodynamic questions associated with \( \nu_{\text{plane-stress}} \). The equivalent value of \( E_{\text{plane-strain}} \) may similarly be calculated. Defining, as is consistent with the last section, \( E_{\text{plane-stress}} = \tau_{yy}/\epsilon_{yy} \), one finds:

\[
E_{\text{plane-strain}} = E_{\text{plane-stress}}[1 - (\nu_{\text{plane-strain}})^2]. \tag{4.1.44}
\]

or in the limit of large \( L/L_p \):

\[
E_{\text{plane-strain}} = \frac{K_nL_p(K_t/K_n)(K_t/K_n + 3)}{\sqrt{3}(K_t/K_n + 1)^2}. \tag{4.1.45}
\]

Note that under plane-strain conditions, the longitudinal wavespeed is:

\[
C_l = \sqrt[3]{\frac{E_{\text{plane-strain}}(1 - \nu_{\text{plane-strain}})}{\rho_p(1 + \nu_{\text{plane-strain}})(1 - 2\nu_{\text{plane-strain})}}. \tag{4.1.46}
\]

Note that if (4.1.43) and (4.1.45) are substituted for \( E \) and \( \nu \) respectively, one finds exactly the same wavespeed as would be calculated from (4.1.38) using \( E_{\text{plane-stress}} \) and \( \nu_{\text{plane-stress}} \).
4.2. Elastic properties for bodies composed of Voronoi polygons and Delaunay triangles

Bathurst & Rothenberg\textsuperscript{18} describe a way of the determining of the elastic properties of a set of arbitrarily packed elastic discs with linear elastic contacts. The method provides a prediction of the Poisson ratio, $\nu$, and a prediction of Young's modulus $E$ up to an undetermined multiplicative constant. It is easy to see from the derivations presented in Ref. 18 (which assumes that forces are randomly distributed spatially about their particles) that the same logic applies to a material constructed of Voronoi polygons. Those formulae for the Young's modulus $E$ and Poisson's ratio $\nu$ for the case of plane-stress are:

$$E_{\text{plane-stress}} = \frac{Cg(4K_nL_p)(1 + K_t/K_n)}{(3 + K_t/K_n)}$$  \hspace{1cm} (4.2.1)

$$\nu_{\text{plane-stress}} = \frac{(1 - K_t/K_n)}{(3 + K_t/K_n)}$$  \hspace{1cm} (4.2.2)

where $Cg$ an undetermined multiplicative constant that depends on the averaged geometrical characteristics of the glued contacts. Here, $L_p$ is an averaged linear scale for the Voronoi polygons, which is defined as

$$\sqrt{\frac{\text{(Area of Body)}}{\text{(Number of Polygons)}}}$$

For plane-strain the corresponding formulae will be:

$$E_{\text{plane-strain}} = \frac{Cg(4K_nL_p)(5 - K_t/K_n)(K_t/K_n + 1)}{16}$$  \hspace{1cm} (4.2.3)

$$\nu_{\text{plane-strain}} = \frac{(1 - K_t/K_n)}{4}$$  \hspace{1cm} (4.2.4)

To evaluate these expressions for $E$ and $\nu$, simulations were performed to measure the elastic moduli of a rectangular body subdivided into 100, 250 and 1000 Voronoi polygons. These calculations were interpreted assuming plane-stress conditions. The undetermined constant, $Cg$, used to compute the theoretical line in the figures, was computed from a fit to the results of the simulation with the body subdivided into 250 polygons and was found nearly equal to 0.5. The results are shown in Fig. 13(a) (Young's modulus) and Fig. 13(b) (Poisson's ratio). It is easy to see from these figures that the theory presented in Ref. 18 describes the situation rather well. Note that the formulae for the Young's modulus, (4.2.1) and (4.2.4), differ substantially from their counterparts for triangles, largely because the numerator is not proportional to $K_t/K_n$ and, thus, $E$ does not go to zero as $K_t/K_n$ goes to zero. This is consistent with the simulation results.

Since the orientation of the sides of Delaunay triangles are also arbitrary, it is natural to expect that the formulas applicable to Voronoi polygons will be also hold
Fig. 13. Predicted and simulated values of the elastic properties of bodies divided into Voronoi polygons. (a) Young’s Modulus, (b) Poisson’s ratio. The values predicted by formulae (4.2.1) and (4.2.2) are shown by solid lines.

...here (up to the value of the undetermined constant, $Cg$). However, computational results presented on Fig. 14(a) and Fig. 14(b) show that Poisson’s ratio and Young’s modulus of this material are perfectly described by the formula for $E$, (4.1.14) and (4.1.45), and $\nu$, (4.1.21) and (4.1.43), for equilateral triangles, where the appropriate element length, $L_p$, is just the average length of the side of the Delaunay triangles.
Fig. 14. Predicted and simulated values of the elastic properties of bodies divided into Delaunay triangles. (a) Young's Modulus, (b) Poisson's ratio. The values predicted by formulae (4.1.14) and (4.1.21) are shown by solid lines.

This result may be unexpected, but it is possible to understand after a look at Fig. 7, which shows that the division of the material into Delaunay triangles looks qualitatively similar to a division into equilateral triangles; in particular, each point used for the subdivision is a vertex for six nearly equilateral triangles. Still, it is somewhat surprising how close the equilateral triangle results predict the behavior
of Delaunay triangles. It is still not clear to us why the results presented in Ref. 18 apply to Voronoi polygons, but not to Delaunay triangles (or, for that matter, do not apply to equilateral triangles).

5. Conclusions

In this paper, we have described a computer model that approximates the elastic behavior, internal plastic failure and breakage of two-dimensional particles. The model is based on the techniques of discrete particle computer simulation. But here, macroscopic polygonal particles are built out of many microscopic constituent elements. Thus, in a sense, the defromation and fracture occurs on a simulated material. Several element shapes have been considered including equilateral triangles, Voronoi polygons and Delaunay triangles. The elastic properties, $E$ and $\nu$, of the assembled material may be predicted in terms of properties of the constituent particles. Plastic deformation of the bulk material is accomplished by allowing slip between the elements. Thus, only materials with well defined internal slip lines, such as equilateral triangles, permit plastic deformation. The bonds between elements can withstand only a specified tensile stress and will break, either whole, or in part, whenever that stress is exceeded. Thus, the bulk material can fracture down to the size of its constituent elements. (Naturally, a crack can at best, follow the closest approximation to its natural path allowed by the shape of the elements.) The major utility of this model is that it allows the material to be followed well beyond the initial fracture and to follow the motion and further breakage of the fragments. It is clear that the shape of the elements can have a strong effect on the results as cracks can only form in the joints between elements. This is apparent in the crystal-like fracture of a system composed of equilateral triangle elements and in the many aborted cracks that form in systems built of Voronoi polygons. (Other examples will be given in Part II.) However, the manner in which plasticity is introduced requires that there be well defined crystalline slip planes, such as those present in the equilateral triangle systems. This model is potentially very versatile. For example, the tensile yield stress may be varied throughout the sample to model inhomogeneous solids. In fact, composites may be modeled by including lines of unusually strong contacts to represent fibers. Such possibilities will be explored in the future.

Acknowledgments

This work was supported by a grant from the International Fine Particle Research Institute. Mark Hopkins was supported by NASA under grant 578-31-15-02 and the Office of Naval Research under grant N000114-93-MP-24008. Special thanks to Holly Campbell for proofreading the manuscript.

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