Convection in deep vertically shaken particle beds. I. General features

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It has long been known that shaking a granular bed can produce circulating convection. This is the first in a series of papers that explores convection in beds a hundred particles or more deep. In such deep beds, the pressures are high enough that the particles remain in contact with their neighbors throughout much of the vibrational period. As such, they interact elastically and convection becomes dependent on the elastic properties of the bed. Changing the particle stiffness can dramatically alter the convection, and for very soft or very hard particles, eliminate it completely. In this paper, the effect of stiffness, bed geometry, and frictional properties on the global convection properties are assessed. © 2008 American Institute of Physics. [DOI: 10.1063/1.2996134]

I. INTRODUCTION

Granular materials typically consist of solid particles in a fluid environment (usually air). Examples of such materials include sand, stones, soil, grains, etc. However when one speaks of a granular material, the interactions between the particles are generally assumed to dominate the interaction between the particles and their environment so that the effects of the fluid can be ignored.

Shaking a granular material is a part of many important industrial processes. Shakers are used to mix, to separate, and to dry granular materials. They are also used to compact granular materials into molds for sand casting. Many phenomena have been studied in such systems such as arching,\textsuperscript{1} heaping\textsuperscript{2–5} (and transitions between these phenomena by varying the vibrational acceleration),\textsuperscript{6} segregation,\textsuperscript{7–10} surface waves,\textsuperscript{11–13} convection,\textsuperscript{2, 8, 14–26} and wave propagation through the bed.\textsuperscript{25–31} Convection is one of the most intriguing of these phenomena. Under vertical vibration, the granules in the bed flow in a circulating pattern, from the top of the container, down along the walls toward the bottom, and then returning back to the top in the channel center.

In this work, we study a two-dimensional (2D) deep granular bed in a rectangular container under vertically shaking using a soft-particle computer simulation technique. The primary purpose of this phase of the study is to determine the effect of particle stiffness, particle-particle and particle-wall friction, and the dimensionless acceleration on the strength of convection as a way of understanding how vibrated convection fits into elastic description of granular flows.\textsuperscript{32–34} This work is condensed from a Ph.D. thesis\textsuperscript{35} and the reader is directed there for a more complete description.

II. COMPUTER SIMULATION

We choose a soft-particle simulation for these studies as in these situations, each particle can have multiple simultaneous particle contacts, and each contact can endure for longer than a collision time and more nearly approximate the elastic response of the bulk material. (See Campbell\textsuperscript{36} for a review of simulation methods.) We will employ the most common model for interparticle contacts, used by the originators of the discrete element method, Cundall and Strack,\textsuperscript{37} which assumes that the interaction between two particles in the normal direction is modeled as a linear spring that provides a repulsive force, connected in parallel to a viscous dashpot which partially dissipates the collision energy. These act in the direction connecting the centers of the particles for as long as they remain in contact. In the direction tangential to the contact point, the particles are connected to a frictional slider in parallel with another linear spring. As is typical, the time step $\Delta t$ for the simulation is chosen to be roughly 1/50th of a binary collision time,

\begin{equation}
\Delta t = \frac{\pi}{50 \sqrt{\frac{2k}{m} - \frac{D^2}{m}}},
\end{equation}

and thus will decrease with the particle stiffness. To obtain a reasonable value of the angle of repose while using round particles, we employ the common trick of eliminating round particles, we employ the common trick of eliminating particle rotation.\textsuperscript{35} The forces acting on each particle in the system are first determined from the contact model and then Newton’s second law is applied to calculate the acceleration of each particle. Both the velocity and position of each particle are found by integrating the accelerations in time throughout the duration of the simulation.

Here we study the dynamics of a 2D system of $N$ disk-shaped particles with diameters evenly distributed in a range of $\pm 25\%$ about a mean diameter $d_0$. They are put in a semi-infinite box (flask) of width $W$ and infinite height. The box is vibrated with a sinusoidal motion, $z(t)=A \sin(2\pi ft)$, where $A$
is the amplitude of the vibration, $f$ is the vibration frequency, and $t$ is time. The gravitational acceleration is $g$. The problem is characterized by the following parameters: $N$, $\mu_p$, $\mu_v$, $\epsilon$, $k_p/(\rho gd_0)$, $A/d_0$, $f/(g/d_0)^{0.5}$, and $\Gamma$, where $N$ is the number of particles in the container, $\mu_p$ is the surface friction coefficient for particle-particle contact, $\mu_v$ is the surface friction coefficient for particle-wall contact, $\epsilon$ is the restitution coefficient, $k_p$ is the spring stiffness in normal direction, $\rho$ is the particle density, $g$ is the gravitational acceleration, $d_0$ is the mean particle diameter, and $\Gamma=\Lambda(2\pi f)^2/g$ is the ratio of maximum acceleration to the acceleration of gravity.

To measure the strength of convection, we measure the scalar parameter $J$ introduced by Taguchi. To compute $J$, the box is first divided into squares fixed in the laboratory frame of side length $L$, which is taken to be the mean diameter of particles. Then, at each time step of duration $\Delta t$, the number of particles, whose center passes in and out of a cell, is counted. Then, the cell-to-cell flow is averaged over many time steps. Thus, $J$ is computed first by taking the averaged vector $\langle J(x,y) \rangle$ defined as

$$
\langle J(x,y) \rangle = \langle n_i(x,y,t) - n_i(x,y,t-\Delta t) \rangle (x_i(t) - x_i(t-\Delta t), y_i(t) - y_i(t-\Delta t)),
$$

where $x_i$ and $y_i$ are integers describing the cell containing the $i$th particle. The average is taken over many millions of time steps spaced in time by $\Delta t$. If the $i$th particle is in the cell having coordinates $x$ and $y$, then $n_i(x,y,t)$ equals unity, otherwise it is zero. (Note that any oscillating motions, back and forth across a cell boundary cancel out and do not contribute to $J$. Only particles that cross the boundary and do not return make a contribution.) Finally, the strength of convection $J$ is defined by using the following equation:

$$
J = \left[ \sum_{x,y} J(x,y)^2 \right]^{0.5},
$$

where $J(x,y)^2$ is the scalar product of $J(x,y)$ with itself and the sum is taken over all cells in the control volume. Despite its long-time use as a measure of convection strength, $J$ is chosen for this study because it is less noisy than other measures we considered. Presumably, this is because of the fact that it actually measures the motion of particles in and out of cells and thus reflects the mass flow induced by the convection and is immune to the much larger velocities associated with the vibration of the flask.

While technically, $J$ is dimensionless, it is in some sense a velocity. Furthermore it should depend, to some degree, on the choice of cell size $L$ and time step $\Delta t$ making $L/\Delta t$ a characteristic velocity. In particular, the larger the value of $L$, the longer it takes a particle to pass through the cell and the smaller the value of $J$. If one associates an average convective speed with the motion, then it is easy to see that the time it takes for a particle to pass through the cell, and be "counted" in Eq. (2), is proportional to $L$, indicating that $J$ is proportional to $1/L$. Similarly, the smaller $\Delta t$, the smaller the distance a particle moves during that time step and a similar argument yields that $J$ is proportional to $\Delta t$. Tests performed by varying $L$ and $\Delta t$ showed that $JL/\Delta t$ scaled with $(gd_0)^{1/2}$. Thus a scaled value, $J^*$, will be used in the majority of this paper.

$$
J^* = JL[(\Delta t(gd_0)^{1/2}].
$$

This scaling effectively scales $J$ for the effects of cell size $L$, gravity $g$, time step $\Delta t$, and diameter $d_0$.

III. RESULTS

Figure 1 shows four "study cases" representing four values of the stiffness parameter $k_p/(\rho gd_0)=1400$, $14,000$, $140,000$, and $1,400,000$. These cases will be examined in detail and used as reference points in this paper. In all of these cases, the depth of bed is around 100 particle diameters. Figure 2 shows the effect of the ratio of the friction parameters. Figure 3 shows the effect of the ratio of the friction parameters. Figure 4 shows the effect of the ratio of the friction parameters.

The individual figures show the local average velocity in the bed represented by lines whose length and direction reflect the magnitude and direction of the velocity. The counter-rotating convection cells are not seen for the case of $k_p/(\rho gd_0)=1400$ [Fig. 1(a)] but can be observed in other three cases. The convection rolls are strongest and penetrate the entire depth for $k_p/(\rho gd_0)=14,000$ [Fig. 1(b)], while the rolls in case of $k_p/(\rho gd_0)=140,000$ [Fig. 1(c)] are only strong near the top of bed. However, the convection is noticeably weaker for the largest stiffness $k_p/(\rho gd_0)=1,400,000$ in Fig. 1(d) although it also penetrates the entire depth. Obviously, the particle stiffness has a complex influence on the strength of the convection.

To set the stage for an examination of the effects of the particle-particle contacts, it is best to understand the effect of other parameters. Figure 2 shows the effect of the ratio of the friction coefficients for particle-wall and particle-particle contacts $\mu_w/\mu_p$, on $J^*$ for $k_p/(\rho gd_0)=1,400,000$, the case that demonstrated the strongest convection in Fig. 1, and thus will most clearly shows the effect of $\mu_w/\mu_p$. At constant $\Gamma$, $J^*$ scales...
nearly linearly with $\mu_w/\mu_p$ and that convection disappears for $\mu_w/\mu_p<1$ or $\mu_w<\mu_p$. Note that while Lee,4 Erich et al.,21 and Grossman39 showed that rougher walls can produce stronger convection, it has never previously been reported that it is the ratio $\mu_w/\mu_p$ that controls the convection strength. However in these simulations, this is likely an artifact of using nonrolling particles to obtain a reasonable angle of repose; because of this assumption, $\mu_w$ controls the actual friction force in the bed at the wall and $\mu_p$ is the dominant factor in controlling the resistance to shear within the bed itself. From that point of view, convection disappears for $\mu_w/\mu_p<1$ because the walls cannot provide enough shear stress to force the granular bed to yield.

Note that Fig. 2 shows a zone marked as “no convection” even though $J$ is nonzero. We noted early during the performance of this work that $J$ could be nonzero even though there is no apparent convection. For example, Fig. 1(a) corresponds to a $J^*$ of about $1.2 \times 10^{-4}$ yet there is clearly no sign of the rolls associated with convection. The no convection line was determined by visually checking each velocity distribution individually for signs of convection. The dashed line in the figure was positioned so as to separate those points from which convection was evident from those where no convection could be observed.

Figure 3 shows the effect of $\Gamma$ on the strength of convection $J^*$ at $k_n/(\rho gd_0)=140\ 000$ for various values of $\mu_w$ and $\mu_p$. No convection is apparent for $\Gamma<2.7$ and $J^*$ reaches its maximum value at $\Gamma=4.0$. Following that maximum, the data plateau and become roughly independent of $\Gamma$ before eventually dropping off at larger $\Gamma$. Taguchi16 found that the convection only increases with $\Gamma$; but as he presented no data above $\Gamma=3.5$, he did not reach the point where $J$ begins to drop. Others12,25,26,40 have also seen that the convection strength increases with $\Gamma$, although they are not directly comparable to these results as they used different measures of the strength of convection. For example, Hsiau and Chen22 and Tai and Hsiau26 defined the convection strength in terms of the mass flow rate inside the convection rolls. Philippe and Bideau25 evaluated the convection strength through the displacement of a ring of dyed tracers initially placed at the periphery of the free surface. Metcalfe et al.40 measured the convection strength by the thickness of the convection roll.) Luding et al.17 also used Taguchi’s $J$ parameter to study a vertical vibrated system 7 particles deep and saw behavior similar to that seen here; however, they assumed a viscous interaction rather than an actual frictional interaction between particles and between particles and the wall so that their results are again not directly comparable to the current work. However almost all of those studies were performed on shallow beds, typically an order of magnitude smaller than the beds studied here, and for that reason alone, may not be directly comparable to the results in this paper.

Note that Fig. 2 indicates that $J^*$ depends nearly linearly on the friction ratio $(\mu_w/\mu_p)$. As a result, Fig. 4 shows that all the data in Fig. 3 will nearly collapse onto a single line when $J^*$ is scaled by the friction ratio $(\mu_w/\mu_p)$. Note that the collapse is almost perfect for small $\Gamma (\Gamma<4)$. For large $\Gamma$, there is still spread in the data, but the spread is much reduced compared to the unscaled data shown in Fig. 3. Figure 5 shows the combined effect of $k_n$ and $\mu_w/\mu_p$ at a single $\Gamma$.

![Figure 2](image1.png)

**FIG. 2.** The effect of ratio of the surface friction coefficients $(\mu_w/\mu_p)$ between particles $(\mu_w)$ and between particles and the wall $(\mu_p)$ on the strength of convection $J^*$ at $k_n/(\rho gd_0)=140\ 000$ and $\Gamma=4.0$.

![Figure 3](image2.png)

**FIG. 3.** The effect of the ratio of maximum acceleration to the acceleration of gravity $(\Gamma)$ on the strength of convection $J^*$ at $k_n/(\rho gd_0)=140\ 000$ and $\Gamma=4.0$.

![Figure 4](image3.png)

**FIG. 4.** The $J^*$ data shown in Fig. 3, but scaled by dividing by the friction ratio $(\mu_w/\mu_p)$. 
the stress induced by an elastic wave varies as 

\[ J = 4.0. \]

=4, the acceleration for which convection is a maximum. Again convection is never observed when \( \mu_w \leq \mu_p \). For all cases where \( \mu_w > \mu_p \), the trends of \( J^* \) are similar. That is, \( J^* \) is small when convection is not observed and sharply increases at \( k_n/(p g d_0) = 5500 \) when convection begins. After that, \( J^* \) reaches its maximum at \( k_n/(p g d_0) = 140 000 \) [the strongest case in Fig. 1(c) corresponds to \( k_n/(p g d_0) = 140 000 \) and \( J^* \) starts to decrease after \( k_n/(p g d_0) > 140 000 \). Eventually, \( J^* \) decreases until convection can no longer be observed. Note that for \( \mu_w/\mu_p = 4.5 \), \( J^* \) plateaus to a constant after \( k_n/(p g d_0) > 140 000 \); a smaller plateau occurs at a higher \( k_n/(p g d_0) \) for \( \mu_w/\mu_p = 3 \). Even for \( \mu_w/\mu_p = 1 \), for which there is no noticeable convection, \( J^* \) still shows a clear maximum at \( k_n/(p g d_0) = 140 000 \).

Luding et al. documented a dependence of the convection strength on collision time. This is effectively a stiffness dependence as the collision time decreases with increasing stiffness. However, Luding et al. found that the convection strength increases uniformly with collision time, that is, decreases with decreasing collision time. The differences from the current results may be due to a combination of their small bed depth or their use of an ad hoc viscous surface interaction in place of friction.

A question remains as to the mechanism by which the particle stiffness can affect convection. As pure collisional interactions are relatively independent of the stiffness, an apparent stiffness dependence indicates that particles remain in contact for long periods of time. Now as the bottom of the flask oscillates up and down, its motion will be transmitted through the bed as a series of waves. As long as the particles are in contact, the waves will be transmitted elastically through the contact points. Any elastic wavespeed \( c \) is proportional to \( (E/\rho)^{1/2} \), where \( E \) is the Young’s modulus and \( \rho \) is the density of the bulk material. Bathurst and Rothenburg showed that \( E \) is directly proportional to the particle stiffness \( k_n \). Thus the wavespeed is ultimately controlled by the stiffness and is proportional to \( k_n^{1/2} \). Note that the stress induced by an elastic wave varies as \( \rho c v_p \), where \( v_p \) is the material velocity induced in the bed, i.e., the actual velocity of the granules. At least near the bottom of the bed, \( v_p \) is induced by the motion of the bottom and may, to a first approximation, be taken as the bottom wall velocity. This indicates that if the bed is deep enough, it may never lose contact with the flask bottom no matter what the acceleration \( \Gamma \) is. To lose contact, the stress in the wave at the flask bottom must be larger than the weight per unit area of the bed, and as the stress varies as \( E^{1/2} \sim k_n^{1/2} \), the softer the material, the more likely the bed is to remain in contact with the flask bottom.

Figure 6 shows time histories of the stress at the bottom of the flask for the four study cases in Fig. 1 \([k_n/(p g d_0) = 1400, 14 000, 140 000, \text{ and } 1 400 000] \) all for \( \Gamma = 4 \). The plot at the bottom of each subfigure shows the corresponding position of the oscillating flask bottom. Note that for the stiffest case, \([k_n/(p g d_0) = 1400] \) in Fig. 6(a), the normal stress at the bottom never goes to zero, indicating that the bed remains in contact with the bottom despite the 4-\( g \) vibration. Note also that while the stress primarily oscillates along with the flask bottom, it does not exactly follow the bottom’s sinusoidal motion but follows a pattern that roughly repeats every two periods.

For the stiffer particle cases \( k_n/(p g d_0) = 140 000, 140 000, \text{ and } 1 400 000 \) [Figs. 6(b) and 6(d)], the normal stress at the bottom wall does fall to zero during a portion of the cycle indicating when the bed loses contact with the bottom. Here, the stress does not follow the motion of the bottom but, in all cases, repeats over roughly three periods of the flask’s oscillation. While the normal stress is zero, the bed should be in free fall except possibly near the vertical sidewalls where there may be frictional forces on the near-wall particles. Although the box is oscillated with the same amplitude and frequency as for the stiffest particle case in Fig. 6(a), the stiffer the particles, the higher the elastic wavespeed \( c \), the more stress \( \sim \rho c v_p \) that can be supported by the elastic waves. Hence increasing the stiffness increases the force applied to the bed by the flask. The larger the force, the higher the bed is lofted by the bottom. Thus it should not be surprising that increasing the stiffness increases the period of time that the bed has lost contact with the flask bottom.

Note that for \( k_n/(p g d_0) = 14 000 \) case shown in Fig. 6(b), the stress drops to zero for only brief periods during the three cycle period, while for \( k_n/(p g d_0) = 1 400 000 \), the bed is out of contact for the majority of each cycle.

As no convection is observed for the stiffest case, \( k_n/(p g d_0) = 1400 \) (which never loses contact with the bottom), these results suggest that there is some connection between the loss of contact and convection. That conclusion makes a certain amount of sense. As the loss of contact implies that the bed is in free fall, the induced stresses in the bed are small or zero during that period. If the forces on the contacts are small, then there can be little internal frictional resistance to the convective motions. However the relationship is certainly not that simple because the strongest convection is found not for the stiffest but also for the second stiffest case, \( k_n/(p g d_0) = 140 000 \), which also remains in contact with the bottom for longer times than the stiffest case \( k_n/(p g d_0) = 1 400 000 \).
Many simulations were performed to create a map showing the conditions under which the bed remains in contact or lifts off from the flask’s bottom. We propose that the boundary between the two types of behavior is determined by \( \Gamma \) and a new dimensionless parameter \( 2 \pi f H / \sqrt{k_n / \rho} \) which can be interpreted as the time it takes an elastic wave (wavespeed \( \sim \sqrt{k_n / \rho} \)) to traverse the depth of the bed (i.e., \( \sim H / \sqrt{k_n / \rho} \)) multiplied by the oscillation frequency \( f \). Note that the softer the particles, the larger \( 2 \pi f H / \sqrt{k_n / \rho} \); hence, the bed is likely to remain in contact for large \( 2 \pi f H / \sqrt{k_n / \rho} \). Similarly, the bed cannot lose contact unless the acceleration exceeds \( 1 - g \) (\( \Gamma = 1 \)). The results are shown in Fig. 7. The data are separated into two regions with two straight lines: \( \Gamma = 1 \) and \( \Gamma = 4(2 \pi f H / \sqrt{k_n / \rho}) - 4 \) with the lift-off region lying above and to the left of the dividing lines.

IV. SUMMARY AND CONCLUSION

This paper used a soft-particle computer simulation to examine the effects of various particle properties, in particular, the particle stiffness on the convection in a deep 2D vertically shaken bed. The stiffness \( k_n \) has a complex influence on the strength of convection. There is no convection observed for soft particles \( [k_n / (\rho g d_0) < 5500] \). The convective strength generally increases with stiffness, but the maximum convection was observed at an intermediate stiffness \( (k_n / (\rho g d_0) = 140000) \) and further increasing the stiffness reduces the convective strength. Besides the particle stiffness, the friction between container wall and particle \( (\mu_w) \) and between particles \( (\mu_p) \), and the dimensionless acceleration

\[
\Gamma = \frac{4(2 \pi f H / \sqrt{k_n / \rho}) - 4}{\sqrt{k_n / \rho}}
\]

FIG. 7. Flowmap showing the separation between the In-contact and lift-off regions for \( \mu_w/\mu_p = 4.5 \).
(1) also affect the strength of convection. Convection is only observed when $\mu_\text{v}/\mu_\text{p} > 1$ and is weaker for the smaller values of $\mu_\text{v}/\mu_\text{p}$. In fact, the convection was found to be nearly linearly dependent on the ratio $\mu_\text{v}/\mu_\text{p}$. Convection rolls never appear in the bed if $\Gamma$ is less than 2.7 and the strength of convection initially increases with $\Gamma$ but reaches the strongest flow when $\Gamma$ is around 4.0 and decreases thereafter.

The stiffness also affects the stress associated with an elastic wave generated by the interaction of the bed with the flask’s bottom. The bed will lose contact with the bottom only if the flask acceleration exceeds $1 - g$ ($\Gamma > 1$) and if the stress generated in the wave is sufficient to support the weight of the bed. As the wave stress varies with the square root of the Young’s modulus, it is directly proportional to $(k_n)^{1/2}$. Thus, the larger the stiffness, the longer the bed remains out of contact with the bottom. Furthermore, the convection rolls are only found in the bed when the bed loses contact with the flask’s bottom presumably because the internal stresses are relaxed when the bed loses contact, freeing the particles to follow the convective motion.