Computing Welfare Losses from Data under Imperfect Competition with Heterogeneous Goods

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Abstract

We study the percentage of welfare losses (PWL) yielded by imperfect competition under product differentiation. When demand is linear and firms are identical, if prices, outputs, costs and the number of firms can be observed, PWL is arbitrary in both Cournot and Bertrand equilibrium. However, if the elasticity of demand can be estimated, under Cournot equilibrium, PWL is a function of the elasticity of demand, the number of firms and the price-marginal cost margin. In Bertrand equilibrium, PWL is a function of the cross elasticity of demand, the number of firms and the price-marginal cost margin. When firms are not identical, we provide conditions under which PWL increases with concentration. When demand is isoelastic and there are many firms, PWL can be computed from prices, outputs, costs and the number of firms. In all these cases we find that price-marginal cost margins and demand elasticities may influence PWL in a quite counterintuitive way.

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1. Introduction

One of the most robust findings of Industrial Organization is that, very often, market equilibrium yields inefficient allocations. But, how large these inefficiencies are? This topic has aroused a large empirical literature, starting from the seminal paper by Harberger (1954). In contrast, the theoretical literature is scarce and focus on the case of homogeneous products: In this case, it is well known that the percentage of welfare losses (PWL) in a Cournot Equilibrium when demand and costs are linear and firms are identical is $\frac{1}{(1+n)^2}$ where n is the number of firms. McHardy (2000) showed that when demand is quadratic, welfare losses can be 30% larger than in the linear model. Anderson and Renault (2003) calculated PWL for a more general class of demand functions. Johari and Tsitsiklis (2005) showed that if average costs are not increasing and the inverse demand function is concave, PWL is less than $\frac{1}{2n+1}$. Finally, Corchón (2008) offered formulae for PWL under free entry and heterogeneous firms. He showed that PWL can be very large even if price, marginal cost, output and number of firms can be observed. The only paper dealing with heterogeneous products is by Cable, J., Carruth, A. and Dixit, A. (1994) that study a linear duopoly model. They offer PWL formulae for several solution concepts.

In this paper we analyze PWL in two models of imperfect competition with heterogeneous products and a representative consumer with quasi-linear preferences: A model with linear demand functions, Dixit (1979), Singh and Vives (1984), and a model with isoelastic demand functions, Spence (1976).[1] Firms produce under constant average costs. Our first step is to find PWL as a function of the fundamentals, i.e. parameters in demand and cost functions. Generally, these parameters cannot be observed so our second step is to obtain PWL as a function of observable variables like price, output, number of firms, etc. When this is not possible, we will introduce items that might be estimated, like the elasticity of demand. The goal of our analysis is to study the impact of observable variables on PWL. Even though PWL can be calculated from data case by case, our approach allows to pinpoint the theoretical factors explaining PWL.

We first consider the model with linear demand. Assume that firms and demand

[1] We leave for future work models of horizontal and vertical differentiation.

functions are identical. We show that given an observation of a price, output, marginal cost and number of firms, there are parameters of the demand function that convert this observation in a Cournot or a Bertrand equilibrium and such that PWL is arbitrary (Propositions 1 and 2). This shows that PWL is unrelated with the differences among profit rates, contrarily to Harberger's dictum: "The differences among these profit rates, as between industries, give a broad indication of the extent of resource malallocation" (op. cit. p. 79). In our model all firms have the same rate of return on capital but PWL can be very high, specially if goods are complements. It seems that Harberger´s procedure picks up welfare losses stemming from the failure of markets to equalize profit rates and not welfare losses from oligopolistic misallocation, a related but different issue.

Next we show that if the elasticity of demand can be estimated, PWL in Cournot equilibrium can be computed from observables (Proposition 3). The elasticity of demand is of no help in the case of Bertrand equilibrium because it can be obtained from observables and the first order condition of profit maximization. We show that if the cross elasticity of demand can be estimated, PWL can be computed from observations (Proposition 5). Finally we study how PWL depends on these variables (Propositions 4 and 6). Some results are what we expected but there are pretty surprising effects too: When goods are substitutes, PWL is decreasing on the price-marginal cost margins in both Cournot and Bertrand equilibrium.[2] And PWL increases with the elasticity of demand in Bertrand equilibrium. Why is so? Consider two markets, A and B, and let the price-marginal cost margin be larger in A than in B. This means that the triangle that represents welfare losses is larger in A than in B. However, realized welfare is also larger in A than in B because the demand function in A is above the demand function in B. A priori, there is no good reason to expect that one effect is larger than the other. In fact, as we noticed before, when costs and demand are linear and firms are identical,

[2] The price-marginal cost margin is often referred to as the "monopoly index" (Lerner [1934]).

these two effects cancel out and PWL only depends on the number of firms.[3] The same argument goes with demand elasticity: a larger demand elasticity means less welfare losses and less realized welfare so the total effect is ambiguous.

Next we introduce heterogeneity in demand and costs in order to study the effect of concentration on PWL. Our first point is that in order to find the optimal number of firms, we need information that is seldom in the hands of the planner or the researcher. The second point concerns the relationship between concentration and welfare losses. Some papers found that the Hirschman-Herfindahl (H) index of concentration is not a good measure of welfare losses: In Daughety (1990) because more concentration may be associated with a larger output in a leader-follower equilibrium. In Farrell and Shapiro (1990), Cable et al. (1994) and Corchón (2008) because firms may be of different sizes.[4] Suppose that it is optimal to allow all firms to produce. This happens when products are poor substitutes. We show that when it is optimal to allow all firms to produce and goods are substitutes, PWL increases with H in both Cournot and Bertrand equilibrium (Proposition 7). This may be seen as a vindication of the 1992 Merger Guidelines issued by the Federal Trade Commission (FTC) where H is considered a reasonable measure of welfare losses, Coate (2005). We also show that when it is optimal to allow only one firm to produce, PWL decreases with H. This is what happened in the papers quoted above where products are perfect substitutes. Thus concentration is bad (resp. good) for welfare when goods are poor (resp. good) substitutes. This can be explained by saying that the efficient production must balance cost savings -which go in the direction of concentrating production in the most efficent firms- with consumer satisfaction where the latter may require considerable diversification of production. Where the last effect is not very large (i.e. when products are close substitutes) cost savings drives efficiency and thus concentration does not harm efficiency. But when products are poor substi-

[3] In other words, price-marginal cost margins do not control for the size of demand. Thus, a high margin might indicate either that demand is potentially very large and firms are having a good time -even if they are very reasonably competitive- or that firms are "exploiting" consumers and destroying a large part of the surplus. This is true even if actual production is known because it is a poor indicator of the efficient production.

[4] The point that minor firms may be harmful for welfare was first made by Lahiri and Ono (1988).

tutes efficient production requires output dispersion and concentration is harmful. We also show that at the value of H considered by the FTC as a threshold for a concentrated industry, PWL is large in Cournot equilibrium but may be small in Bertrand equilibrium. Moreover, as noticed before, it may be difficult to calculate the optimal number of firms. Thus a full microfoundation of the views of FTC on concentration is still ahead of us.

In Section 3 we assume that the representative consumer has preferences over differentiated goods representable by a CES utility function. We also assume that there is a large number of identical firms. This model and its variants (see, e.g. Dixit and Stiglitz [1977]) are popular in the fields of monopolistic competition, international trade, geography and economics, etc. But contrarily to these models, we assume that the number of firms is exogenous. The reason for this is that fixed costs -an essential ingredient of a model with endogenous firms- may produce large PWL (Corchón [2008]) and in this paper we want to focus on PWL produced by product heterogeneity alone. We show that PWL tends to zero when demand elasticity tends to infinite, but PWL tends to one when the degree of homogeneity of the CES function tends to one (Proposition 8). This qualifies a conjecture of Stigler (1949): "...the predictions of this standard model of imperfect competition differ only in unimportant respects from those of the theory of competition because the underlaying conditions will usually be accompanied by a very high demand elasticities for the individual firms". In this model, a high elasticity of demand makes PWL small, but given any elasticity of demand, we can obtain PWL as close to one as we wish. Next, we show that PWL can be recovered from the observation of a price, an output, a marginal cost and the number of firms (Proposition 9). However a low price-marginal cost margin does not guarantee that PWL is small: even if price tends to the marginal cost, when the number of firms is sufficiently large, PWL may exceed those in the linear model under monopoly. Moreover, when the number of firms tends to infinity, PWL is decreasing in the price-marginal cost margin (Proposition 10). Again, this is another case where price-marginal cost margins and welfare losses are not related in the way we previously thought.

Summing up, we have three main conclusions. First, our main message is positive:

obtaining PWL from data is possible in two well-known models of imperfect competition. Second, the impact on PWL of rates of returns, price-marginal cost margins or the elasticity of demand, is not what it was thought to be. Finally, we provide as partial vindication of FTC views on the role of the H index on welfare.

2. The Linear Model

In this section we assume that inverse demand is linear. In the first subsection we assume that all firms are identical which allows for clean formulae of welfare losses. In the second subsection we study the case where costs and intercepts of inverse demands are different among firms. The second part offers formulae for PWL that are used to discuss the role of concentration in oligopolistic markets. Unfortunately these formulae become somewhat blurred so we do not attempt to generalize the results obtained before.

2.1. The Symmetric Case

The market is composed of n firms. The output (resp. price) of firm i is denoted by x_i (resp. p_i). Firms are identical with a cost function cx_i. There is a representative consumer with a quadratic utility function,

$$U = \alpha \sum_{i=1}^{n} x_i - \frac{\beta}{2} \sum_{i=1}^{n} x_i^2 - \frac{\gamma}{2} \sum_{i=1}^{n} x_i \sum_{j \neq i} x_j - \sum_{i=1}^{n} p_i x_i.\ \alpha > c,\ \beta > \max\{0, \gamma, -\gamma(n-1)\}.$$

Under these assumptions, $U(\cdot)$ is concave. FOC of utility maximization yield

$$p_i = \alpha - \beta x_i - \gamma \sum_{j \neq i} x_j,\ i = 1, 2, ..., n. \tag{2.1}$$

Goods are substitutes (resp. complements) iff $\gamma > 0$ (resp. < 0). The ratio $\frac{\gamma}{\beta}$ represents the degree of product differentiation: if $\gamma = 0$ products are independent, and if $\gamma = \beta$ they are perfect substitutes.

Definition 1. *A Linear Market is a list* $\{\alpha, \beta, \gamma, c, n\}$ *with* $\alpha > c$, $\beta > \max\{0, \gamma, -\gamma(n-1)\}$ *and* $n \in \mathbb{N}$.

Social welfare is defined as

$$W = \alpha \sum_{i=1}^{n} x_i - \frac{\beta}{2} \sum_{i=1}^{n} x_i^2 - \frac{\gamma}{2} \sum_{i=1}^{n} x_i \sum_{j \neq i} x_j - c \sum_{i=1}^{n} x_i. \tag{2.2}$$

The social optimum is a list of outputs that maximizes social welfare. It is easy to see that optimal outputs are all identical - denoted by x_i^o - and equal to

$$x_i^o = \frac{\alpha - c}{\beta + \gamma (n-1)}. \tag{2.3}$$

Social welfare in the optimum is

$$W^o = \frac{n (\alpha - c)^2}{2 (\beta + (n-1) \gamma)}. \tag{2.4}$$

Now we are ready to define our equilibrium concepts.

Definition 2. *A Cournot equilibrium in a linear market is a list of outputs* $(x_1^c, x_2^c, ..., x_n^c)$ *such that for each* i, x_i^c *maximizes* $\left(\alpha - \beta x_i - \gamma \sum_{j \neq i} x_j^c - c\right) x_i$.

From the FOC of profit maximization we obtain that

$$x_i^c = \frac{\alpha - c}{2\beta + \gamma (n-1)}, \ i = 1, 2, ..., n. \tag{2.5}$$

In order to define a Bertrand equilibrium we need to invert the system (2.1). Adding up these equations from 1 to n we get $\sum_{i=1}^n p_i = n\alpha - (\beta + (n-1)\gamma) \sum_{i=1}^n x_i$, or

$$\sum_{i=1}^{n} x_i = \frac{n\alpha - \sum_{i=1}^n p_i}{\beta + (n-1)\gamma},$$

which plugged into (2.1) yields

$$p_i = \alpha - (\beta - \gamma) x_i - \gamma \sum_{i=1}^{n} x_i = \alpha - (\beta - \gamma) x_i - \gamma \frac{n\alpha - \sum_{i=1}^n p_i}{\beta + \gamma(n-1)}, \text{ or}$$

$$x_i = \frac{\alpha(\beta - \gamma) - p_i(\beta + \gamma(n-2)) + \gamma \sum_{j \neq i} p_j}{(\beta - \gamma)(\beta + \gamma(n-1))} \equiv x_i^b(p_i, p_{-i}), \ i = 1, 2, ..., n, \tag{2.6}$$

where p_{-i} is a list of all prices minus p_i. Notice that given our assumptions on β and γ, $\frac{\partial x_i}{\partial p_j} < 0$ iff $\gamma < 0$. Now we can define a Bertrand equilibrium.

Definition 3. *A Bertrand equilibrium in a linear market is a list of prices* $(p_1^b, p_2^b, ..., p_n^b)$ *such that for each* i, p_i^b *maximizes* $(p_i - c)\, x_i^b(p_i, p_{-i}^b)$.

From the FOC of profit maximization we obtain that

$$p_i^b = \frac{\alpha(\beta - \gamma) + c(\beta + \gamma(n-2))}{2\beta + \gamma(n-3)},\ i = 1, 2, ..., n. \tag{2.7}$$

Let W^c be social welfare evaluated at Cournot equilibrium. Let us define the percentage of welfare losses in a Cournot equilibrium as

$$PWL^c \equiv \frac{W^o - W^c}{W^o}. \tag{2.8}$$

Lemma 1. *In a linear market the percentage of welfare losses in Cournot equilibrium is*

$$PWL^c = \frac{1}{\left(2 + (n-1)\frac{\gamma}{\beta}\right)^2} \tag{2.9}$$

Proof: From (2.2), social welfare in Cournot equilibrium can be written as $W^c = \alpha n x_i^c - \frac{\beta}{2} n x_i^{c2} - \frac{\gamma}{2} n\,(n-1)\, x_i^{c2} - c n x_i^c$. Thus, from (2.5) we obtain that

$$W^c = \frac{n\,(\alpha - c)^2\,(3\beta + (n-1)\,\gamma)}{2\,(2\beta + (n-1)\,\gamma)^2}.$$

Then,

$$PWL^c = 1 - \frac{W^c}{W^o} = \frac{1}{\left(2 + (n-1)\frac{\gamma}{\beta}\right)^2}. \blacksquare$$

Notice that PWL is decreasing in the degree of product differentiation, $\frac{\gamma}{\beta}$. Thus, minimal PWL is $\frac{1}{(n+1)^2}$ and occurs for the maximal value of $\frac{\gamma}{\beta}$, which is one, i. e. when products are perfect substitutes. When products are substitutes, maximal PWL occurs for the minimal value of $\frac{\gamma}{\beta}$ which is zero, and PWL is .25. When products are complements, maximal $PWL = 1$.[5]

Let PWL^b be the percentage of welfare losses in Bertrand equilibrium. Then,

[5] When goods are complements, PWL increases with n. This is because there is insufficient coordination among firms and the larger the number of firms, the larger the coordination problem.

Lemma 2. *In a linear market the percentage of welfare losses in Bertrand equilibrium is*

$$PWL^b = \left(\frac{1 - \frac{\gamma}{\beta}}{2 + (n-3)\frac{\gamma}{\beta}} \right)^2 \qquad (2.10)$$

Proof: From (2.7) we obtain that all firms produce the same output, x_i^b, namely

$$x_i^b = \frac{(\alpha - c)(\beta + (n-2)\gamma)}{(2\beta + (n-3)\gamma)(\beta + (n-1)\gamma)}.$$

Social welfare in Bertrand equilibrium is $W^b = \alpha n x_i^b - \frac{\beta}{2} n x_i^{b2} - \frac{\gamma}{2} n (n-1) x_i^{b2} - cnx_i^b$,

$$W^b = \frac{n(\alpha - c)^2 (3\beta + (n-4)\gamma)(\beta + (n-2)\gamma)}{2(2\beta + (n-3)\gamma)^2 (\beta + (n-1)\gamma)}.$$

Thus,

$$PWL^b = 1 - \frac{W^b}{W^o} = \left(\frac{1 - \frac{\gamma}{\beta}}{2 + (n-3)\frac{\gamma}{\beta}} \right)^2. \blacksquare$$

Note that PWL is decreasing in the degree of product differentiation $\frac{\gamma}{\beta}$. Thus, minimal PWL is zero and occurs when $\gamma = \beta$, i.e. when products are perfect substitutes. When products are substitutes, maximal PWL occurs for $\frac{\gamma}{\beta} = 0$, namely .25. When products are complements, maximal PWL is $\left(\frac{n}{n+1}\right)^2$. Clearly, if $n = 1$, $PWL^j = 0.25$, $j = c, b$, so in the remainder of the section we will assume that $n > 1$.

We are interested in the PWL yielded by imperfectly competitive markets, conditional on the values taken by certain variables that can be observed, namely market prices, outputs, marginal cost and number of firms. Formally:

Definition 4. *An observation is a list* $\{\mathfrak{p}, \mathfrak{x}_i, \mathfrak{c}, \mathfrak{n}\}$ *where* $\mathfrak{p}$ *is market price,* $\mathfrak{x}_i$ *is output of firm* i*,* $\mathfrak{c}$ $(< \mathfrak{p})$ *is the marginal cost and* $\mathfrak{n}$ *is number of firms.*

Let us relate PWL with observable variables. First we consider Cournot equilibrium.

Proposition 1. *Given an observation* $\{\mathfrak{p}, \mathfrak{x}_i, \mathfrak{c}, \mathfrak{n}\}$ *and a number* $\mathfrak{v} \in (\frac{1}{(\mathfrak{n}+1)^2}, 1)$ *there is a linear market* $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ *such that* $(\mathfrak{x}_i, \mathfrak{x}_i, ..., \mathfrak{x}_i)$ *is a Cournot equilibrium for this market,* $\mathfrak{p} = \alpha - \beta\mathfrak{x}_i - \gamma(\mathfrak{n}-1)\mathfrak{x}_i$ *and* $PWL^c = \mathfrak{v}$.

Proof: Let

$$\alpha = \mathfrak{c} + \frac{\mathfrak{p}-\mathfrak{c}}{\sqrt{\mathfrak{v}}},\ \beta = \frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i} \text{ and } \gamma = \frac{(\mathfrak{p}-\mathfrak{c})\left(1-2\sqrt{\mathfrak{v}}\right)}{(\mathfrak{n}-1)\,\mathfrak{x}_i\sqrt{\mathfrak{v}}}. \tag{2.11}$$

Clearly, $\alpha > \mathfrak{c}$ and $\beta > \max\{0, \gamma, -\gamma(\mathfrak{n}-1)\}$ since $\mathfrak{p} > \mathfrak{c}$, $\mathfrak{v} > \frac{1}{(\mathfrak{n}+1)^2}$ and $\mathfrak{v} <1$. We easily see that the linear market $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ yields an equilibrium where $x_i^c = \mathfrak{x}_i$, $i = 1, 2, ..., n$, $\mathfrak{p} = \alpha - \beta\mathfrak{x}_i - \gamma(\mathfrak{n}-1)\mathfrak{x}_i$ and $PWL^c = \mathfrak{v}$, so the proof is complete. ■

Now we turn to case of Bertrand equilibrium.

Proposition 2. *Given an observation* $\{\mathfrak{p}, \mathfrak{x}_i, \mathfrak{c}, \mathfrak{n}\}$ *and a number* $\mathfrak{v} \in \left(0, \left(\frac{\mathfrak{n}}{\mathfrak{n}+1}\right)^2\right)$ *there is a linear market* $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ *such that* $(\mathfrak{p}, \mathfrak{p}, ..., \mathfrak{p})$ *is a Bertrand equilibrium for this market,* $\mathfrak{x}_i = x_i^b(\mathfrak{p}, \mathfrak{p}_{-i})$ *where* $\mathfrak{p}_{-i}$ *is a list of* $\mathfrak{n}-1$ *identical* $\mathfrak{p}$*, and* $PWL^b = \mathfrak{v}$*.*

Proof: Let

$$\alpha = \mathfrak{c} + \frac{\mathfrak{p}-\mathfrak{c}}{\sqrt{\mathfrak{v}}},\ \beta = \frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}\frac{\left(\sqrt{\mathfrak{v}}-1\right)\left(1+\sqrt{\mathfrak{v}}\,(\mathfrak{n}-3)\right)}{\mathfrak{v}+\mathfrak{n}\left(\mathfrak{v}-\sqrt{\mathfrak{v}}\right)} \text{ and } \gamma = \frac{\mathfrak{c}-\mathfrak{p}}{\mathfrak{x}_i}\frac{1-3\sqrt{\mathfrak{v}}+2\mathfrak{v}}{\mathfrak{v}+\mathfrak{n}\left(\mathfrak{v}-\sqrt{\mathfrak{v}}\right)}.$$

It is easy to check that $0 < \mathfrak{c} < \alpha$ and $\beta > \max\{0, \gamma, -\gamma(\mathfrak{n}-1)\}$. The Linear market $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ yields a Bertrand equilibrium where $p_i^b = \mathfrak{p}$ and $x_i^b = \mathfrak{x}_i$ with $PWL^b = \mathfrak{v}$, that completes the proof. ■

Propositions 1 and 2 show that observable variables put very little restrictions on PWL. In particular, neither price-marginal cost margins nor profit rates have any relationship with PWL. Let us look for restrictions that can have a bite on PWL.[6] Suppose that the demand elasticity, denoted by ε, is observable. From (2.6)

$$\varepsilon \equiv -\frac{\partial x_i}{\partial p_i}\frac{\mathfrak{p}}{\mathfrak{x}_i} = \frac{\beta + \gamma(\mathfrak{n}-2)}{(\beta-\gamma)(\beta+\gamma(\mathfrak{n}-1))}\frac{\mathfrak{p}}{\mathfrak{x}_i}. \tag{2.12}$$

Let us introduce a new piece of notation, namely $\mathfrak{T} \equiv \varepsilon\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{p}}$. Now we have the following result.

[6] If goods are substitutes, the maximum PWL in both Cournot and Bertrand occurs when $\gamma \simeq 0$, namely $PWL \simeq .25$ which corresponds to PWL under monopoly.

Proposition 3. *Given an observation* $\{\mathfrak{p}, \mathfrak{x}_i, \mathfrak{c}, \mathfrak{n}, \varepsilon\}$ *such that* $\mathfrak{T} \equiv \varepsilon \frac{\mathsf{p}-\mathsf{c}}{\mathsf{p}} \geq 1$ *and an information that goods are substitutes or complements, there is a linear market* $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ *such that* $(\mathfrak{x}_i, \mathfrak{x}_i, ..., \mathfrak{x}_i)$ *is a Cournot equilibrium for this market,* $\mathfrak{p} = \alpha - \beta \mathfrak{x}_i - \gamma (\mathfrak{n}-1) \mathfrak{x}_i$ *and*

$$PWL^c = \frac{1}{\left(2 + \frac{(\mathsf{T}-1)(\mathsf{n}-2) \pm \sqrt{(\mathsf{T}-1)\left(\mathsf{n}^2\mathsf{T}-(\mathsf{n}-2)^2\right)}}{2\mathsf{T}}\right)^2} \tag{2.13}$$

with sign "+" (resp. sign "-") corresponding to the case of substitutes (resp. complements).

Proof: Let us consider the case of substitutes first. Let

$$\begin{aligned} \alpha &= \mathfrak{c} + \frac{\mathfrak{p}}{2\varepsilon}\left(\mathfrak{T}(\mathfrak{n}+2) - (\mathfrak{n}-2) + \sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T} - (\mathfrak{n}-2)^2\right)}\right) \\ \beta &= \frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i} \\ \gamma &= \frac{(\mathfrak{T}-1)(\mathfrak{n}-2) + \sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T} - (\mathfrak{n}-2)^2\right)}}{2\mathfrak{T}(\mathfrak{n}-1)} \frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}. \end{aligned}$$

Clearly, $\beta > 0$. We need to show that $0 < \frac{\gamma}{\beta} < 1$ and $\alpha > \mathfrak{c}$. Note that for $\mathfrak{T} \geq 1$ the square root is defined in the real numbers and $\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T} - (\mathfrak{n}-2)^2\right)} \geq (\mathfrak{T}-1)(\mathfrak{n}-2)$ because if not, we would have $\mathfrak{n}^2\mathfrak{T} < (\mathfrak{n}-2)^2\mathfrak{T}$ which is impossible. Then the condition $0 < \frac{\gamma}{\beta} < 1$ amounts to

$$0 < \frac{(\mathfrak{T}-1)(\mathfrak{n}-2) + \sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T} - (\mathfrak{n}-2)^2\right)}}{2\mathfrak{T}(\mathfrak{n}-1)} < 1 \Longrightarrow 4\mathfrak{T}(\mathfrak{n}-1)^2 > 0,$$

that always holds for $\mathfrak{T} \in [1, \infty)$. The condition $\alpha > \mathfrak{c}$ amounts to $(\mathfrak{T}-1)(\mathfrak{n}-2) + \sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T} - (\mathfrak{n}-2)^2\right)} + 4\mathfrak{T} > 0$, that holds for $\mathfrak{T} \in [1, \infty)$. Now we need to prove that the linear market $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ yields a Cournot equilibrium where $x_i^c = \mathfrak{x}_i$ and $\mathfrak{p} = \alpha - \beta\mathfrak{x}_i - \gamma(\mathfrak{n}-1)\mathfrak{x}_i$. First,

$$x_i^c = \frac{\alpha - \mathfrak{c}}{2\beta + \gamma(\mathfrak{n}-1)} = \frac{\frac{\mathsf{p}}{2\varepsilon}\left(\mathfrak{T}(\mathfrak{n}+2) - (\mathfrak{n}-2) + \sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T} - (\mathfrak{n}-2)^2\right)}\right)}{2\frac{\mathsf{p}-\mathsf{c}}{\mathsf{x}_i} + \frac{(\mathsf{T}-1)(\mathsf{n}-2) + \sqrt{(\mathsf{T}-1)\left(\mathsf{n}^2\mathsf{T}-(\mathsf{n}-2)^2\right)}}{2\mathsf{T}} \frac{\mathsf{p}-\mathsf{c}}{\mathsf{x}_i}} = \mathfrak{x}_i.$$

Then,

$$\alpha-\beta\mathfrak{x}_i-\gamma(\mathfrak{n}-1)\mathfrak{x}_i=\mathfrak{c}+\frac{\mathfrak{p}}{2\varepsilon}\left(\mathfrak{T}(\mathfrak{n}+2)-(\mathfrak{n}-2)+\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}\right)-$$

$$\mathfrak{x}_i\left(\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}+\frac{(\mathfrak{T}-1)(\mathfrak{n}-2)+\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}}{2\mathfrak{T}}\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}\right)=\mathfrak{p}.$$

So we have shown in the case of substitutes that there exists a linear market $\{\alpha,\beta,\gamma,\mathfrak{c},\mathfrak{n}\}$ that yields a Cournot equilibrium where $x_i^c=\mathfrak{x}_i$ and $\mathfrak{p}=\alpha-\beta\mathfrak{x}_i-\gamma(\mathfrak{n}-1)\mathfrak{x}_i$. Then it is straightforward to find PWL^c plugging the values of β and γ in (2.9).

Now we consider the case of complements. Let

$$\begin{aligned}\alpha &= \mathfrak{c}+\frac{\mathfrak{p}}{2\varepsilon}\left(\mathfrak{T}(\mathfrak{n}+2)-(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}\right)\\ \beta &= \frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}\\ \gamma &= \frac{(\mathfrak{T}-1)(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}}{2\mathfrak{T}(\mathfrak{n}-1)}\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}.\end{aligned}$$

We need to show that $-\frac{1}{\mathfrak{n}-1}<\frac{\gamma}{\beta}<0$ and $\alpha>\mathfrak{c}$. The former condition amounts to

$$-\frac{1}{\mathfrak{n}-1}<\frac{(\mathfrak{T}-1)(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}}{2\mathfrak{T}(\mathfrak{n}-1)}<0\Longrightarrow 4\mathfrak{T}>0,$$

that holds for $\mathfrak{T}\in[1,\infty)$. The latter condition amounts to $\mathfrak{T}(\mathfrak{n}+2)-(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}>0\Rightarrow 4\mathfrak{T}(3+\mathfrak{T}+\mathfrak{n}(\mathfrak{T}-1))>0$, that holds for $\mathfrak{T}\in[1,\infty)$. Let us show now that the linear market $\{\alpha,\beta,\gamma,\mathfrak{c},\mathfrak{n}\}$ yields a Cournot equilibrium where $x_i^c=\mathfrak{x}_i$ and $\mathfrak{p}=\alpha-\beta\mathfrak{x}_i-\gamma(\mathfrak{n}-1)\mathfrak{x}_i$:

$$x_i^c=\frac{\alpha-\mathfrak{c}}{2\beta+\gamma(\mathfrak{n}-1)}=\frac{\frac{\mathfrak{p}}{2\varepsilon}\left(\mathfrak{T}(\mathfrak{n}+2)-(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}\right)}{2\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}+\frac{(\mathfrak{T}-1)(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}}{2\mathfrak{T}}\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}}=\mathfrak{x}_i,$$

$$\alpha-\beta\mathfrak{x}_i-\gamma(\mathfrak{n}-1)\mathfrak{x}_i=\mathfrak{c}+\frac{\mathfrak{p}}{2\varepsilon}\left(\mathfrak{T}(\mathfrak{n}+2)-(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}\right)-$$

$$\mathfrak{x}_i\left(\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}+\frac{(\mathfrak{T}-1)(\mathfrak{n}-2)-\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}}{2\mathfrak{T}}\frac{\mathfrak{p}-\mathfrak{c}}{\mathfrak{x}_i}\right)=\mathfrak{p}.$$

So we have proved in the case of complements that there exists a linear market $\{\alpha,\beta,\gamma,\mathfrak{c},\mathfrak{n}\}$ that yields a Cournot equilibrium where $x_i^c=\mathfrak{x}_i$ and $\mathfrak{p}=\alpha-\beta\mathfrak{x}_i-\gamma(\mathfrak{n}-1)\mathfrak{x}_i$. Then it is straightforward to find PWL^c plugging the values of β and γ in (2.9). ■

According to Proposition 3 we can calculate PWL in a Cournot equilibrium in (2.13) from three variables -the number of firms, the elasticity of demand and the price-marginal cost ratio- plus the information that goods are complements or substitutes which gives us the sign of $\frac{\gamma}{\beta}$. Let us now study how PWL depends on $\mathfrak{n}$ and $\mathfrak{T}$.

Proposition 4. *When goods are substitutes (resp. complements), PWL^c is decreasing (resp. increasing) on $\mathfrak{n}$, the elasticity of demand and the price-marginal costs margins.*

Proof: First, we consider the case of substitutes, that is $\frac{\gamma}{\beta}=\frac{(\mathsf{T}-1)(\mathsf{n}-2)+\sqrt{(\mathsf{T}-1)\left(\mathsf{n}^2\mathsf{T}-(\mathsf{n}-2)^2\right)}}{2\mathsf{T}(\mathsf{n}-1)}$.

$$\frac{\partial PWL^c}{\partial\mathfrak{n}}=-\frac{8\mathfrak{T}^2\left(\mathfrak{T}-1+\frac{(2+\mathsf{n}(\mathsf{T}-1))(\mathsf{T}-1)}{\sqrt{(\mathsf{T}-1)\left(\mathsf{n}^2\mathsf{T}-(\mathsf{n}-2)^2\right)}}\right)}{\left(2+\mathfrak{n}(\mathfrak{T}-1)+2\mathfrak{T}+\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}\right)^3}<0,$$

so an increase in the number of firms decreases PWL^c, that is what intuition suggests. To continue, we compute $\frac{\partial\left(\frac{\gamma}{\beta}\right)}{\partial\mathsf{T}}$.

$$\frac{\partial\left(\frac{\gamma}{\beta}\right)}{\partial\mathfrak{T}}=\frac{\mathfrak{n}(4+\mathfrak{n}(\mathfrak{T}-1)-2\mathfrak{T})-2(2-\mathfrak{T})+(\mathfrak{n}-2)\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}}{2(\mathfrak{n}-1)\mathfrak{T}^2\sqrt{(\mathfrak{T}-1)\left(\mathfrak{n}^2\mathfrak{T}-(\mathfrak{n}-2)^2\right)}}$$

which is positive, so PWL^c decreases with $\mathfrak{T}$ when goods are substitutes.

Second, we study the case of complements, that is $\frac{\gamma}{\beta}=\frac{(\mathsf{T}-1)(\mathsf{n}-2)-\sqrt{(\mathsf{T}-1)\left(\mathsf{n}^2\mathsf{T}-(\mathsf{n}-2)^2\right)}}{2\mathsf{T}(\mathsf{n}-1)}$.

$$\frac{\partial PWL^c}{\partial \mathfrak{n}}=-\frac{8\mathfrak{T}^2\left(1-\mathfrak{T}+\frac{(2+\mathsf{n}(\mathsf{T}-1))(\mathsf{T}-1)}{\sqrt{(\mathsf{T}-1)\left(\mathsf{n}^2\mathsf{T}-(\mathsf{n}-2)^2\right)}}\right)}{\left(-2-\mathfrak{n}\left(\mathfrak{T}-1\right)-2\mathfrak{T}+\sqrt{\left(\mathfrak{T}-1\right)\left(\mathfrak{n}^2\mathfrak{T}-\left(\mathfrak{n}-2\right)^2\right)}\right)^3}>0.$$

Therefore, when the goods are complements PWL^c is increasing in the number of firms $\mathfrak{n}$. Next, we calculate $\frac{\partial\left(\frac{\gamma}{\beta}\right)}{\partial \mathsf{T}}$ which amounts to

$$\frac{\partial\left(\frac{\gamma}{\beta}\right)}{\partial\mathfrak{T}}=\frac{\mathfrak{n}\left(-4-\mathfrak{n}\left(\mathfrak{T}-1\right)+2\mathfrak{T}\right)-2\left(\mathfrak{T}-2\right)+(n-2)\sqrt{\left(\mathfrak{T}-1\right)\left(\mathfrak{n}^2\mathfrak{T}-\left(\mathfrak{n}-2\right)^2\right)}}{2\left(\mathfrak{n}-1\right)\mathfrak{T}^2\sqrt{\left(\mathfrak{T}-1\right)\left(\mathfrak{n}^2\mathfrak{T}-\left(\mathfrak{n}-2\right)^2\right)}}$$

which is negative, so PWL^c increases with $\mathfrak{T}$. ■

In Proposition 4 the sign of the effect of the number of firms and demand elasticity is what we expected: The more competition -i.e. the higher $\mathfrak{n}$ or ε- is good (resp. bad) when goods are substitutes (complements). However the effect of price-marginal cost margins runs counter to our intuition. As we remarked in the introduction this is because this margin affects to both welfare losses and realized welfare.

We now consider Bertrand equilibrium. In this case, FOC condition of profit maximization can be written as $p_i=\varepsilon(p_i-c)$. Thus the observation of ε does not add any new information once p_i and c are observed. A way out to this problem is provided if the cross elasticity of demand $\frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$, denote by ρ, is observable, as shown next.

Proposition 5. *Given an observation* $\{\mathfrak{p},\mathfrak{x}_i,\mathfrak{c},\mathfrak{n},\rho\}$ *such that* $\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}>\max\{\rho(\mathfrak{n}-1),-\rho\}$, *there is a linear market* $\{\alpha,\beta,\gamma,\mathfrak{c},\mathfrak{n}\}$ *such that* $(\mathfrak{p},\mathfrak{p},...,\mathfrak{p})$ *is a Bertrand equilibrium for this market,* $\mathfrak{x}_i=x_i^b(\mathfrak{p},\mathfrak{p}_{-i})$ *where* $\mathfrak{p}_{-i}$ *is a list of* $\mathfrak{n}-1$ *identical* $\mathfrak{p}$*, and*

$$PWL^b=\left(\frac{\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathfrak{n}-1)}{2\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathfrak{n}-1)}\right)^2. \tag{2.14}$$

Proof: Let

$$\alpha = \mathfrak{p} + \frac{\mathfrak{p}}{\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)}$$

$$\beta = \frac{\mathfrak{p}\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-2)\right)}{\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} + \rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)\right)}$$

$$\gamma = \frac{\mathfrak{p}\rho}{\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} + \rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)\right)}.$$

It is easy to prove that $\alpha > \mathfrak{c}$ and $\beta > \max\{0, \gamma, -\gamma(\mathfrak{n}-1)\}$ for $\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} > \max\{\rho(\mathfrak{n}-1), -\rho\}$. Let us show that the linear market $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ yields a Bertrand equilibrium where $p_i^b = \mathfrak{p}$ and $x_i^b(\mathfrak{p}, \mathfrak{p}_{-i}) = \mathfrak{x}_i$:

$$p_i^b = \frac{\alpha(\beta-\gamma) + \mathfrak{c}(\beta+\gamma(\mathfrak{n}-2))}{2\beta+\gamma(\mathfrak{n}-3)} = \frac{1}{2\frac{\mathsf{p}\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathsf{n}-2)\right)}{\mathsf{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}+\rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathsf{n}-1)\right)} + \frac{\mathsf{p}\rho(\mathsf{n}-3)}{\mathsf{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}+\rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathsf{n}-1)\right)}} *$$

$$(\left(\mathfrak{p} + \frac{\mathfrak{p}}{\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)}\right)\left(\frac{\mathfrak{p}\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-2)\right)}{\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} + \rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)\right)} - \frac{\mathfrak{p}\rho}{\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} + \rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)\right)}\right) +$$

$$\mathfrak{c}\left(\frac{\mathfrak{p}\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-2)\right)}{\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} + \rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)\right)} + \frac{\mathfrak{p}\rho(\mathfrak{n}-2)}{\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} + \rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} - \rho(\mathfrak{n}-1)\right)}\right)) = \mathfrak{p},$$

$$x_i^b(\mathfrak{p}, \mathfrak{p}_{-i}) = \frac{\alpha(\beta-\gamma) - \mathfrak{p}(\beta+\gamma(\mathfrak{n}-2)) + \gamma(\mathfrak{n}-1)\mathfrak{p}}{(\beta-\gamma)(\beta+\gamma(n-1))} = \frac{\alpha-\mathfrak{p}}{\beta+\gamma(\mathfrak{n}-1)} =$$

$$\frac{\frac{\mathsf{p}}{\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathsf{n}-1)}}{\frac{\mathsf{p}\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathsf{n}-2)\right)}{\mathsf{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}+\rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathsf{n}-1)\right)} + \frac{\mathsf{p}\rho(\mathsf{n}-1)}{\mathsf{x}_i\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}+\rho\right)\left(\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}}-\rho(\mathsf{n}-1)\right)}} = \mathfrak{x}_i.$$

Thus given an observation $\{\mathfrak{p}, \mathfrak{x}_i, \mathfrak{c}, \mathfrak{n}, \rho\}$ such that $\frac{\mathsf{p}}{\mathsf{p}-\mathsf{c}} > \max\{\rho(\mathfrak{n}-1), -\rho\}$ there is a linear market $\{\alpha, \beta, \gamma, \mathfrak{c}, \mathfrak{n}\}$ such that $(\mathfrak{p}, \mathfrak{p}, ..., \mathfrak{p})$ is a Bertrand equilibrium for this market, $\mathfrak{x}_i = x_i^b(\mathfrak{p}, \mathfrak{p}_{-i})$ where $\mathfrak{p}_{-i}$ is a list of $\mathfrak{n}-1$ identical $\mathfrak{p}$. Then it is straightforward to find PWL^b plugging the values of β and γ in (2.10). ■

The formula (2.14) allows to calculate PWL in a Bertrand equilibrium from three magnitudes alone: the number of firms, the price-marginal cost margins (or, alternatively, the elasticity of demand) and the cross elasticity of demand. Let us analyze the impact of a change in observable variables on PWL^b.

Proposition 6. *When goods are substitutes (resp. complements) PWL^b is decreasing (increasing) in the number of firms and price-marginal cost margins and it is increasing (resp. decreasing) in the elasticity of demand. PWL^b is decreasing in the cross elasticity of demand.*

Proof: From (2.14) we get

$$\frac{\partial PWL^b}{\partial \mathfrak{n}} = -\frac{2\varepsilon\rho\left(\varepsilon - \rho(\mathfrak{n}-1)\right)}{\left(2\varepsilon - \rho(\mathfrak{n}-1)\right)^3} < 0 \Leftrightarrow \rho > 0,$$

$$\frac{\partial PWL^b}{\partial \varepsilon} = \frac{2\left(\mathfrak{n}-1\right)\rho\left(\varepsilon - \rho(\mathfrak{n}-1)\right)}{\left(2\varepsilon - \rho(\mathfrak{n}-1)\right)^3} > 0 \Leftrightarrow \rho > 0,$$

$$\frac{\partial PWL^b}{\partial \rho} = -\frac{2\left(\mathfrak{n}-1\right)\varepsilon\left(\varepsilon - \rho(\mathfrak{n}-1)\right)}{\left(2\varepsilon - \rho(\mathfrak{n}-1)\right)^3} < 0.$$

From these formulae the proposition follows ■

Proposition 6 confirms our intuitions about the role of the number of firms and the cross elasticity of demand on welfare losses, namely that when goods are substitutes (resp. complements) an increase in the number of firms decreases (resp. increases) PWL and an increase in the cross elasticity of demand decreases PWL both for substitutes and for complements. But, again, the impact of the price-marginal cost margin goes contrarily to what intuition suggests: It is negative (resp. positive) for substitutes (resp. complements). It is also remarkable that demand elasticity affects PWL in a counterintuitive way. Again we have to bear in mind that demand elasticity affects to both welfare losses and realized welfare.

2.2. Heterogeneous Firms

The purpose of the subsection is to evaluate the influence of concentration on PWL. As you will see the formulae relating PWL with the fundamentals are pretty messy so we do not attempt to generalize the results of the previous section.

We extend the model presented before to the case where firms are heterogeneous on two counts. On the one hand marginal costs, denoted by c_i for firm i, may be different across firms. On the other hand the parameter α, denoted by α_i for firm i, may be different across firms.[7] Assume $\alpha_i > c_i$ for all i. The utility function of the representative consumer is now

$$U = \sum_{i=1}^{n} \alpha_i x_i - \frac{\beta}{2} \sum_{i=1}^{n} x_i^2 - \frac{\gamma}{2} \sum_{i=1}^{n} x_i \sum_{j \neq i} x_j - \sum_{i=1}^{n} p_i x_i, \ \beta > \max\{0, \gamma, -\gamma(n-1)\} \quad (2.15)$$

The restrictions below guarantee that the outputs of all firms are positive in Cournot and Bertrand equilibrium.

$$2\beta + \gamma (n-1) > \frac{\gamma \sum_{i=1}^{n} (\alpha_i - c_i)}{\alpha_i - c_i}, \ i = 1, 2, ..., n. \quad (2.16)$$

$$\frac{(\beta + \gamma (n-1)) (2\beta + \gamma (n-3))}{\beta + \gamma (n-2)} > \frac{\gamma \sum_{i=1}^{n} (\alpha_i - c_i)}{\alpha_i - c_i} \ i = 1, 2, ..., n. \quad (2.17)$$

Under our assumptions, $U(\cdot)$ is concave. FOC of utility maximization yields

$$p_i = \alpha_i - \beta x_i - \gamma \sum_{j \neq i} x_j, \ i = 1, 2, ..., n.$$

Social welfare reads now

$$W = \sum_{i=1}^{n} \alpha_i x_i - \frac{\beta}{2} \sum_{i=1}^{n} x_i^2 - \frac{\gamma}{2} \sum_{i=1}^{n} x_i \sum_{j \neq i} x_j - \sum_{i=1}^{n} c_i x_i. \quad (2.18)$$

To evaluate social welfare in the optimum is not straightforward because it depends on the number of active firms in the optimum. For the time being, let us assume that it is optimal that m firms are active. Then optimal outputs -denoted by x_i^o - equal to

$$x_i^o = \frac{\alpha_i - c_i}{\beta - \gamma} - \frac{\gamma \sum_{i=1}^{m} (\alpha_i - c_i)}{(\beta + \gamma (m-1))(\beta - \gamma)}, \ i = 1, 2, ..., m \quad (2.19)$$

and aggregate output in the optimum, denoted by x^o, equals to

$$x^o = \sum_{i=1}^{m} x_i^o = \frac{\sum_{i=1}^{m} (\alpha_i - c_i)}{\beta + \gamma (m-1)}. \quad (2.20)$$

[7] This model has been used, among others, by Häckner (2000) and Hsu and Wang (2005).

We now find the optimal number of firms m. Let us rank firms according with the value of $\alpha_i - c_i$. Without loss of generality assume that $\alpha_v - c_v \geq \alpha_{v+1} - c_{v+1}$, $v = 1, 2, ..., n-1$. Clearly if firm v produces a positive output in the optimum, Firms $v-1, v-2$, etc. also produce a positive output in the optimum. Suppose that it is optimal that Firms 1 to $k-1$ produce a positive output. Now evaluate $\frac{\partial W}{\partial x_k}$ in (2.18) at $x_k = 0$ and $x_j = x_j^o$, $j = 1, ..., k-1$ according to (2.19), and we obtain that

$$\frac{\partial W}{\partial x_k} = \alpha_k - c_k - \gamma \sum_{j=1}^{k-1} x_j^o. \tag{2.21}$$

If $\frac{\partial W}{\partial x_k} \leq 0$, clearly, $x_k^o = 0$. If $\frac{\partial W}{\partial x_k} > 0$, firm k must produce a positive output in the optimum. The problem with this algorithm is that it requires knowledge of all the parameters defining a market. Two particular cases are worth to analyze, though. First, when $\beta(\alpha_2 - c_2) \leq \gamma(\alpha_1 - c_1)$, only Firm 1 will produce a positive output in the optimum since from (2.19) and (2.21), $\frac{\partial W}{\partial x_2} = \alpha_2 - c_2 - \gamma \frac{\alpha_1 - c_1}{\beta} \leq 0$. Second, when

$$(\alpha_n - c_n)(\beta + \gamma (n-2)) > \gamma \sum_{i=1}^{n-1} (\alpha_i - c_i) \tag{2.22}$$

the number of active firms is the same in the optimum and in the Cournot equilibrium, because from (2.19) and (2.21), $\frac{\partial W}{\partial x_n} = \alpha_n - c_n - \gamma \sum_{j=1}^{n-1} x_j^o > 0$. Notice that when goods are substitutes the conditions in (2.16) and (2.17) are implied by (2.22). If goods are complements, (2.16) implies (2.17) and (2.22).

In this framework, a Cournot equilibrium is a list of outputs $(x_1^c, x_2^c, ..., x_n^c)$ such that for each i, x_i^c maximizes $\left(\alpha_i - \beta x_i - \gamma \sum_{j \neq i} x_j^c - c_i\right) x_i$. From the FOC of profit maximization we obtain that

$$x_i^c = \frac{\alpha_i - c_i}{2\beta - \gamma} - \frac{\gamma}{2\beta - \gamma} \frac{\sum_{i=1}^{n} (\alpha_i - c_i)}{2\beta + \gamma (n-1)}$$

and aggregate output at Cournot equilibrium reads

$$x^c = \sum_{i=1}^{n} x_i^c = \frac{\sum_{i=1}^{n} (\alpha_i - c_i)}{2\beta + \gamma (n-1)}.$$

In order to compute a Bertrand equilibrium we first write the demand for firm i:

$$x_i = \frac{\alpha_i \left(\beta + \gamma \left(n-2\right)\right) - p_i \left(\beta + \gamma \left(n-2\right)\right) - \gamma \sum_{j \neq i} (\alpha_j - p_j)}{\left(\beta - \gamma\right) \left(\beta + \gamma \left(n-1\right)\right)}.$$

A Bertrand equilibrium is a list $(p_1^b, p_2^b, ..., p_n^b)$ such that for all i p_i^b maximizes

$$\left(\frac{\alpha_i \left(\beta + \gamma \left(n-2\right)\right) - p_i \left(\beta + \gamma \left(n-2\right)\right) - \gamma \sum_{j \neq i} (\alpha_j - p_j^b)}{\left(\beta - \gamma\right) \left(\beta + \gamma \left(n-1\right)\right)} \right) \left(p_i - c_i\right).$$

Then,

$$x_i^b = \frac{\beta + \gamma \left(n-2\right)}{\left(\beta - \gamma\right) \left(2\beta + \gamma \left(2n-3\right)\right)} \left(\alpha_i - c_i - \gamma \frac{\beta + \gamma \left(n-2\right)}{\left(2\beta + \gamma \left(n-3\right)\right) \left(\beta + \gamma \left(n-1\right)\right)} \sum_{i=1}^{n} \left(\alpha_i - c_i\right) \right),$$

and aggregate output at Bertrand equilibrium reads

$$x^b = \frac{\beta + \gamma \left(n-2\right)}{\left(2\beta + \gamma \left(n-3\right)\right) \left(\beta + \gamma \left(n-1\right)\right)} \sum_{i=1}^{n} \left(\alpha_i - c_i\right).$$

Next, we link PWL with the Hirschman-Herfindahl index of concentration. Let s_i^j be the market share of firm i in Cournot equilibrium ($j = c$), Bertrand equilibrium ($j = b$) or in the optimum ($j = o$). We define the Hirschman-Herfindahl index of concentration in Cournot equilibrium, Bertrand equilibrium or the optimum as $H^j \equiv \sum_{i=1}^{n} (s_i^j)^2$, $j = c, b, o$.

Lemma 3. *When firms are heterogeneous, the percentage of welfare losses in Cournot equilibrium is*

$$PWL^c = 1 - \left(\frac{1 + \frac{\gamma}{\beta} \left(m-1\right)}{m \frac{\gamma}{\beta} + \left(2 - \frac{\gamma}{\beta}\right) \sum_{i=1}^{m} s_i^c} \right)^2 \frac{H^c \left(3 - \frac{\gamma}{\beta}\right) + \frac{\gamma}{\beta}}{H^o \left(1 - \frac{\gamma}{\beta}\right) + \frac{\gamma}{\beta}}. \tag{2.23}$$

Proof: Let W^c be social welfare evaluated at Cournot equilibrium that reads

$$W^c = \sum_{i=1}^{n} \left(\alpha_i - c_i\right) x_i^c - \frac{\beta}{2} \sum_{i=1}^{n} x_i^{c2} - \frac{\gamma}{2} \sum_{i=1}^{n} x_i^c \sum_{j \neq i} x_j^c. \tag{2.24}$$

Let us analyze (2.24) term by term.

$$\sum_{i=1}^{n} (\alpha_i - c_i)\, x_i^c = H^c\,(2\beta - \gamma)\, x^{c2} + \gamma x^{c2}.$$

$$\frac{\beta}{2}\sum_{i=1}^{n} x_i^{c2} = \frac{\beta}{2} H^c x^{c2}.$$

$$\frac{\gamma}{2}\sum_{i=1}^{n} x_i^c \sum_{j\neq i} x_j^c = \frac{\gamma}{2}\sum_{i=1}^{n} x_i^c\,(x^c - x_i^c) = \frac{\gamma}{2}\left(x^{c2} - \sum_{i=1}^{n} x_i^{c2}\right) = \frac{\gamma}{2}\left(x^{c2} - H^c x^{c2}\right).$$

Therefore,

$$W^c = H^c\,(2\beta - \gamma)\, x^{c2} + \gamma x^{c2} - \frac{\beta}{2} H^c x^{c2} - \frac{\gamma}{2}\left(x^{c2} - H^c x^{c2}\right) = \frac{3\beta - \gamma}{2} H^c x^{c2} + \frac{\gamma}{2} x^{c2}.$$

Using the definition of H^o, social welfare in the optimum is

$$W^o = \frac{\beta - \gamma}{2} H^o x^{o2} + \frac{\gamma}{2} x^{o2}.$$

Plugging the values of W^c and W^o in PWL^c we obtain

$$PWL^c = 1 - \frac{W^c}{W^o} = 1 - \left(\frac{x^c}{x^o}\right)^2 \frac{H^c\,(3\beta - \gamma) + \gamma}{H^o\,(\beta - \gamma) + \gamma},$$

and plugging the values of x^c and x^o we obtain (2.23). ■

Note that PWL^c here depends on the degree of product differentiation $\frac{\gamma}{\beta}$, the number of active firms in the optimum m, the sum of the market shares of the m largest firms $\sum_{i=1}^{m} s_i^c$ and the Hirschman-Herfindahl indexes of concentration evaluated at the Cournot equilibrium and in the optimum, H^c and H^o. When $m = 1$, we have that

$$PWL^c(m = 1) = 1 - \frac{H^c\left(3 - \frac{\gamma}{\beta}\right) + \frac{\gamma}{\beta}}{\left(\frac{\gamma}{\beta} + \left(2 - \frac{\gamma}{\beta}\right) s_1^c\right)^2},$$

which is decreasing in H^c. In the polar case where $m = n$, (i.e. the number of active firms is the same in the optimum and in Cournot equilibrium), after lengthy calculations we arrive at the following:

$$PWL^c(m = n) = \frac{H^c\left(1 + (n-1)\frac{\gamma}{\beta}\right) - \frac{\gamma}{\beta}}{H^c\left(2 - \frac{\gamma}{\beta}\right)^2\left(1 + (n-1)\frac{\gamma}{\beta}\right) + \left(\frac{\gamma}{\beta}\right)^2\left(n - 2 - (n-1)\frac{\gamma}{\beta}\right)}, \quad (2.25)$$

where

$$H^c(m=n) = \frac{1}{(2\beta-\gamma)^2}\sum_{i=1}^{n}\left(\frac{(\alpha_i-c_i)\,(2\beta+\gamma\,(n-1))}{\sum_{j=1}^{n}(\alpha_j-c_j)}-\gamma\right)^2,$$

$$H^o(m=n) = \frac{1}{(\beta-\gamma)^2}\sum_{i=1}^{n}\left(\frac{(\alpha_i-c_i)\,(\beta+\gamma\,(n-1))}{\sum_{j=1}^{n}(\alpha_j-c_j)}-\gamma\right)^2.$$

If all firms are identical, $H^c = \frac{1}{n}$ and $PWL^c(m=n) = \frac{1}{\left(2+(n-1)\frac{\gamma}{\beta}\right)^2}$, that is what we found in Lemma 1. Notice that H^c and $\frac{\gamma}{\beta}$ are less than one, so for reasonable values of n it makes sense to evaluate (2.25) as if n were a large number. In this case (2.25) simplifies to

$$PWL^c(m=n,\ n \text{ large}) = \frac{H^c}{H^c\left(2-\frac{\gamma}{\beta}\right)^2+\frac{\gamma}{\beta}\left(1-\frac{\gamma}{\beta}\right)}.$$

Computing

$$\frac{\partial PWL^c(m=n,\ n \text{ large})}{\partial\frac{\gamma}{\beta}} = -\frac{H^c\left(1-2\frac{\gamma}{\beta}-2H^c(2-\frac{\gamma}{\beta})\right)}{\left(H^c\left(2-\frac{\gamma}{\beta}\right)^2+\frac{\gamma}{\beta}\left(1-\frac{\gamma}{\beta}\right)\right)^2}$$

which is negative for $\frac{\gamma}{\beta}\in(0,\frac{1-4H^c}{2(1-H^c)})$ and positive for $\frac{\gamma}{\beta}\in(\frac{1-4H^c}{2(1-H^c)},1)$. So the minimum of $PWL^c(m=n,\ n \text{ large})$ occurs at $\frac{\gamma}{\beta}=\frac{1-4H^c}{2(1-H^c)}$. When $H^c = 0.18$, which the FTC considers the threshold for a concentrated industry, the minimal PWL^c is 0.241967 which is a large lower bound.

Now we consider welfare losses in Bertrand equilibrium.

Lemma 4. *In Bertrand equilibrium with heterogeneous firms*

$$PWL^b = 1-\left(\frac{\left(1+\frac{\gamma}{\beta}(n-2)\right)\left(1+\frac{\gamma}{\beta}(m-1)\right)}{m\frac{\gamma}{\beta}\left(1+\frac{\gamma}{\beta}(n-2)\right)+\left(1-\frac{\gamma}{\beta}\right)\left(2+\frac{\gamma}{\beta}(2n-3)\right)\sum_{i=1}^{m}s_i^b}\right)^2 * \quad (2.26)$$
$$\frac{H^b\left(1-\frac{\gamma}{\beta}\right)\left(3+\frac{\gamma}{\beta}(3n-4)\right)+\frac{\gamma}{\beta}\left(1+\frac{\gamma}{\beta}(n-2)\right)}{\left(H^o\left(1-\frac{\gamma}{\beta}\right)+\frac{\gamma}{\beta}\right)\left(1+\frac{\gamma}{\beta}(n-2)\right)}.$$

Proof: Social welfare in Bertrand equilibrium, denoted by W^b reads,

$$W^b = \frac{(\beta-\gamma)(3\beta+\gamma(3n-4))}{2(\beta+\gamma(n-2))} H^b x^{b2} + \frac{\gamma}{2} x^{b2}.$$

Let PWL^b be the percentage of welfare losses in Bertrand equilibrium.

$$PWL^b = 1 - \frac{W^b}{W^o} = 1 - \left(\frac{x^b}{x^o}\right)^2 \frac{H^b(\beta-\gamma)(3\beta+\gamma(3n-4)) + \gamma(\beta+\gamma(n-2))}{(H^o(\beta-\gamma)+\gamma)(\beta+\gamma(n-2))}.$$

So, plugging the values of x^b and x^o, we obtain the formula above. ■

Thus, PWL^b depends on the degree of product differentiation $\frac{\gamma}{\beta}$, the number of active firms in the optimum m and in Bertrand equilibrium n, the sum of the marker shares of m largest firms $\sum_{i=1}^{m} s_i^b$ and the Hirschman-Herfindahl indexes of concentration H^b and H^o evaluated, respectively, in Bertrand equilibrium and in the optimum.

As before let us consider two special cases. First, when in the optimum only Firm 1 is used by the planner. Then, $m = 1$ and

$$PWL^b(m = 1) = 1 - \left(\frac{1+\frac{\gamma}{\beta}(n-2)}{\frac{\gamma}{\beta}\left(1+\frac{\gamma}{\beta}(n-2)\right)+\left(1-\frac{\gamma}{\beta}\right)\left(2+\frac{\gamma}{\beta}(2n-3)\right)s_1^b}\right)^2 * \frac{H^b\left(1-\frac{\gamma}{\beta}\right)\left(3+\frac{\gamma}{\beta}(3n-4)\right)+\frac{\gamma}{\beta}\left(1+\frac{\gamma}{\beta}(n-2)\right)}{1+\frac{\gamma}{\beta}(n-2)}$$

that for $\beta \simeq \gamma$ becomes $PWL^b(m = 1) = 0$, that is what one expects for Bertrand equilibrium in the case of product homogeneity. Notice that $PWL^b(m = 1)$ is decreasing in H^b.

Second, when the number of active firms is the same in the optimum and in Bertrand equilibrium, after lengthy calculations, we obtain that

$$PWL^b(m = n) = \frac{\left(H^b\left(1+\frac{\gamma}{\beta}(n-1)\right)-\frac{\gamma}{\beta}\right)\left(1-\frac{\gamma}{\beta}\right)\left(1+\frac{\gamma}{\beta}(n-1)\right)}{H^b\left(1-\frac{\gamma}{\beta}\right)\left(2+\frac{\gamma}{\beta}(2n-3)\right)^2+\left(\frac{\gamma}{\beta}\right)^2\left(n-2+\frac{\gamma}{\beta}(3+(n-3)n)\right)}. \tag{2.27}$$

If all firms are identical, $H^b = \frac{1}{n}$ and $PWL^b(m = n) = \left(\frac{1-\frac{\gamma}{\beta}}{2+\frac{\gamma}{\beta}(n-3)}\right)^2$ as in Lemma 2.

Finally, when n is large, (2.27) simplifies to

$$PWL^b(m = n,\, n \text{ large}) = \frac{H^b(1 - \frac{\gamma}{\beta})}{\frac{\gamma}{\beta} + 4H^b\left(1 - \frac{\gamma}{\beta}\right)},$$

which is decreasing in the degree of product differentiation $\frac{\gamma}{\beta}$. Its maximal value is 0.25 (for $\frac{\gamma}{\beta} = 0$). For $H^b = 0.18$, $PWL^b(m = n,\, n \text{ large}) = \frac{0.18 - 0.18z}{0.28z + 0.72}$ which for values of $\frac{\gamma}{\beta}$ larger than 0.75 is less than 4.8%. So in this case a high concentration does not imply large welfare losses.

From (2.25) and (2.27) we obtain the following result:

Proposition 7. *If goods are substitutes (resp. complements), $PWL^j(m = n)$ is increasing (resp. decreasing) in H^j, $j = c, b$.*

Proof: Computing $\frac{\partial PWL^c}{\partial H^c}(m = n)$,

$$\frac{\frac{\gamma}{\beta}\left(1 - \frac{\gamma}{\beta}\right)\left(1 + \frac{\gamma}{\beta}(n-1)\right)\left(4 + \frac{\gamma}{\beta}(n-2)\right)}{\left(4H^c\left(1 + \frac{\gamma}{\beta}(n-2)\right) + \left(\frac{\gamma}{\beta}\right)^3(H^c - 1)(n-1) + \left(\frac{\gamma}{\beta}\right)^2(n - 2 + H^c(5 - 4n))\right)^2},$$

that is positive if $\frac{\gamma}{\beta} > 0$ and negative if $\frac{\gamma}{\beta} < 0$. Also, $\frac{\partial PWL^b}{\partial H^b}(m = n)$ equals to

$$\frac{\frac{\gamma}{\beta}\left(1 - \frac{\gamma}{\beta}\right)\left(1 + \frac{\gamma}{\beta}(n-1)\right)\left(1 + \frac{\gamma}{\beta}(n-2)\right)\left(4 + 5\frac{\gamma}{\beta}(n-2) + \left(\frac{\gamma}{\beta}\right)^2(6 + (n-6)n)\right)}{\left(4H^b\left(1 + 2\frac{\gamma}{\beta}(n-2)\right) + \left(\frac{\gamma}{\beta}\right)^3\left(3 - H^b(3 - 2n)^2 + (n-3)n\right) + \left(\frac{\gamma}{\beta}\right)^2(n - 2 + H^b(21 + 4(n-5)n))\right)^2},$$

that is positive if $\frac{\gamma}{\beta} > 0$ and negative if $\frac{\gamma}{\beta} < 0$. ■

Thus, for $m = n$ and goods are substitutes, PWL increases with H, contrarily to what happens when $m = 1$ in both Cournot and Bertrand equilibrium. This is because, as we explained in the Introduction the condition $m = n$ (resp. $m = 1$) is related to goods being poor (resp. good) substitutes. But when goods are complements (and $m = n$) PWL decreases with concentration.

3. A Model of a Large Group

In this section we consider that the market for a differentiated good is supplied by a large number of firms. You may think of goods like restaurants, wine, beer, etc. We will not consider entry and fixed costs because as it was shown in Corchón (2008), entry and fixed cost might produce very high PWL. In this paper we want to study the impact of product differentiation alone on PWL so we discard both fixed costs and entry that are likely to bias our estimates of PWL. As we will see this model is capable of yielding very high PWL. The model can be interpreted as a monopolistic competition model in which the long run aspects are not considered. In this framework, the relative size of firms is not an important issue so we will assume that all firms are identical. Also the issue of price versus quantity competition is not very important so, by convenience, we will assume that firms compete in quantities.

The utility function of the representative consumer reads

$$U = \left(\sum_{i=1}^{n} x_i^{\alpha} \right)^{\frac{r}{\alpha}} - \sum_{i=1}^{n} p_i x_i, \ \ \alpha, r \in (0,1),$$

see Spence (1976). The inverse demand function of firm i is

$$p_i = r \left(\sum_{i=1}^{n} x_i^{\alpha} \right)^{\frac{r}{\alpha}-1} x_i^{\alpha-1}.$$

Definition 5. *A CES Market is a list* $\{\alpha, r, c, n\}$ *with* $\alpha, r \in (0,1)$, $c > 0$, *and* $n \in \mathbb{N}$.

Profit function for firm i is $\pi_i = r\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{r}{\alpha}-1} x_i^{\alpha} - c x_i$. Because there is a large number of firms, each firm takes $\sum_{i=1}^{n} x_i^{\alpha}$ as given. The elasticity of demand, denoted by ϵ, is defined as the inverse of the elasticity of inverse demand, namely

$$\epsilon = \frac{1}{1-\alpha}. \tag{3.1}$$

Thus when $\alpha \to 1$ the elasticity of demand becomes infinite. Now we have the following preliminary result.

Lemma 5. *In a CES market*

$$PWL^s = 1 - \alpha^{\frac{1}{1-r}} \frac{\frac{1}{\alpha} - r}{1-r}. \tag{3.2}$$

Proof: First order condition of profit maximization for firm i is:

$$r\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{r}{\alpha}-1} \alpha x_i^{\alpha-1} - c = 0. \tag{3.3}$$

Left-hand side of (3.3) is decreasing in x_i so second order condition holds. In a symmetric equilibrium where all firms produce the same output, denoted by x_i^*, we have that:

$$x_i^* = \left(\frac{r\alpha}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{1}{1-r}}, \; p^* = \frac{c}{\alpha} \text{ and } U^* = n^{\frac{r}{\alpha}}\left(\frac{r\alpha}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{r}{1-r}}. \tag{3.4}$$

In equilibrium, social welfare is

$$W^* = n^{\frac{r}{\alpha}}\left(\frac{r\alpha}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{r}{1-r}} - nc\left(\frac{r\alpha}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{1}{1-r}}.$$

In the optimal allocation price equals marginal cost and so,

$$r\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{r}{\alpha}-1} x_i^{\alpha-1} = c.$$

From this we get,

$$x_i^o = \left(\frac{r}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{1}{1-r}} \text{ and } W^o = n^{\frac{r}{\alpha}}\left(\frac{r}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{r}{1-r}} - nc\left(\frac{r}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{1}{1-r}}, \tag{3.5}$$

there x_i^o and W^o stand for output and social welfare in the optimum. W^o is increasing in n, so in the full optimum the planner would choose a number of firms equal to n.

Consequently, the percentage of welfare losses is:

$$\begin{aligned} PWL^s &= 1 - \frac{W^*}{W^o} = 1 - \frac{n^{\frac{r}{\alpha}}\left(\frac{r\alpha}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{r}{1-r}} - nc\left(\frac{r\alpha}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{1}{1-r}}}{n^{\frac{r}{\alpha}}\left(\frac{r}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{r}{1-r}} - nc\left(\frac{r}{cn^{1-\frac{r}{\alpha}}}\right)^{\frac{1}{1-r}}} = \\ &= 1 - \alpha^{\frac{1}{1-r}}\frac{\frac{1}{\alpha} - r}{1-r}. \; \blacksquare \end{aligned}$$

At a first glance it is surprising that PWL^s does not depend on the number of firms n. However, we have assumed that the number of firms is large. Thus, (3.2) can be understood as the limit formula when n is large. The following properties of PWL^s are easily proved:

Proposition 8. *i) PWL^s is decreasing in α.*

ii) $\lim_{\alpha\to 1} PWL^s = 0$ and $\lim_{\alpha\to 0} PWL^s = 1$.

iii) PWL^s is increasing in r.

iv) $\lim_{r\to 1} PWL^s = 1$ and $\lim_{r\to 0} PWL^s = 0$.

The explanation of ii) is that when α is close to one (resp. zero), product is close to be homogeneous (resp. very differentiated), and welfare losses are small (resp. large), see (3.1). The explanation of iii) is that when r increases (resp. decreases) the gap between the optimal and the equilibrium output increases (resp. decreases) too, see (3.4) and (3.5). It follows from ii) and iv) that it is possible to have a market where the elasticity of demand is close to infinite (i.e. α close to 1) and PWL is as close to 1 as we wish.[8] In brief, elasticity of demand is only a partial measure of PWL in this model.

Let us relate PWL^s with observable variables as defined in Definition 4 in the previous section. Notice that the first order conditions of profit maximization imply that $\epsilon = \frac{p}{p-c}$ so in this framework, as in the Bertrand case in the previous section, knowledge of the elasticity of demand is of no help. We will assume that $\mathfrak{c}(\ln\mathfrak{n}+\ln\mathfrak{p}) < \mathfrak{p}\ln\mathfrak{n}$, that will ensure that $r < 1$.

In our construction, the function $ProductLn\,(t)$ will play a prominent role. This function, called the Lambert W function, gives the solution for w in $t = we^w$ and has the following properties:[9]

i) $ProductLn\,(t) \in \mathbb{R}$ for $t \in \left[-\frac{1}{e}, \infty\right)$;

ii) $ProductLn\left(-\frac{1}{e}\right) = -1$;

iii) $\lim_{t\to\infty} ProductLn\,(t) = \infty$;

iv) $ProductLn\,(0) = 0$.

v) $ProductLn\,(t)$ is increasing in $t \in \left[-\frac{1}{e}, \infty\right)$.

[8] Even if $\alpha = r$, $\lim_{\alpha\to 1} PWL^s = 0.2642$, a large number.

[9] See http://en.wikipedia.org/wiki/Lambert's_W_function.

vi) $e^{aProductLn(t)}\left(ProductLn\left(t\right)\right)^{a}=t^{a}$.

Now we have our main result in this section:

Proposition 9. *Given an observation* $\{\mathfrak{p},\mathfrak{x}_i,\mathfrak{c},\mathfrak{n}\}$ *there is a CES market* $\{\alpha,r,\mathfrak{c},\mathfrak{n}\}$ *such that* $(\mathfrak{p},\mathfrak{x}_i)$ *is an Equilibrium for this market, and*

$$\begin{aligned} PWL^{s} &= 1-\left(\frac{\mathfrak{c}}{\mathfrak{p}}\right)^{\frac{1}{1-r}}\frac{\frac{\mathsf{p}}{\mathsf{c}}-r}{1-r} \qquad (3.6)\\ \textit{with } r &= \frac{ProductLn\left(\mathfrak{n}\mathfrak{p}\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{c}}\ln\mathfrak{n}+\ln\mathfrak{x}_i\right)\right)}{\frac{\mathsf{p}}{\mathsf{c}}\ln\mathfrak{n}+\ln\mathfrak{x}_i}. \end{aligned}$$

Proof: Let α and r be such that

$$\begin{aligned} \left(\frac{r\alpha}{\mathfrak{c}\mathfrak{n}^{1-\frac{r}{\alpha}}}\right)^{\frac{1}{1-r}} &= \mathfrak{x}_i\\ \frac{\mathfrak{c}}{\alpha} &= \mathfrak{p} \end{aligned}$$

The previous equations yield

$$\begin{aligned} \alpha &= \frac{\mathfrak{c}}{\mathfrak{p}}\\ r &= \frac{ProductLn\left(\mathfrak{n}\mathfrak{p}\mathfrak{x}_i\left(\frac{\mathsf{p}}{\mathsf{c}}\ln\mathfrak{n}+\ln\mathfrak{x}_i\right)\right)}{\frac{\mathsf{p}}{\mathsf{c}}\ln\mathfrak{n}+\ln\mathfrak{x}_i}. \end{aligned}$$

It is straightforward to check that $0<\alpha<1$ and $0<r<1$ (using the condition $\frac{\mathsf{c}}{\mathsf{p}}<\frac{\ln\mathsf{n}}{\ln\mathsf{n}+\ln\mathsf{p}}$). Then by construction the CES market $\{\alpha,r,\mathfrak{c},\mathfrak{n}\}$ yields an Equilibrium where $p^{*}=\mathfrak{p}$ and $x_i^{*}=\mathfrak{x}_i$. Plugging in α and r in (3.2) we get the formula for PWL^{s} as a function of an observation $\{\mathfrak{p},\mathfrak{x}_i,\mathfrak{c},\mathfrak{n}\}$. ■

An important consequence of Proposition 9 is that, given an observation, there is a unique value of PWL^{s}. The reason for that is that in this case, the number of parameters to be "recovered" equals the number of data. We have analyzed the case where the cost function is cx_i^{γ} with $\gamma\geq 1$ (available under request). We show that PWL is undetermined and maximal under constant returns to scale.

Next, we analyze the properties of PWL^{s} in (3.6):

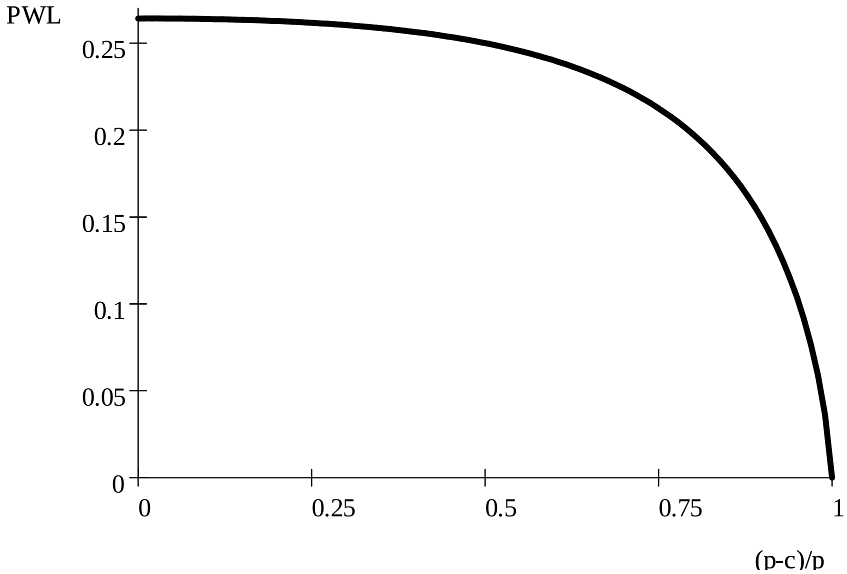

Figure 3.1: $\lim_{n\to\infty} PWL$ as a function of price-marginal cost margin $\frac{p-c}{p}$

Proposition 10. *The percentage of welfare losses in the CES model is such that:*
i) $\lim_{n\to\infty} PWL^s = 1 - \left(\frac{c}{p}\right)^{\frac{1}{1-\frac{c}{p}}} \left(\frac{p}{c}+1\right)$.
ii) $\lim_{\frac{c}{p}\to 1} PWL^s = 0$.
iii) $\lim_{\frac{c}{p}\to 1} \left(\lim_{n\to\infty} PWL^s\right) = 1 - \frac{2}{e} \simeq 0.2642$.

Note that for a finite number of firms that are pricing at the marginal cost, PWL^s is close to zero. However, with infinite number of firms that are pricing at the marginal cost, PWL^s is quite high. In fact, it can be argued that the formula in i) above is the one that should be used since we assumed that n was large. In this case, PWL^s is decreasing with the price-marginal cost margin, $\frac{p-c}{p}$, and looks like in Figure 3.1.[10]

[10] When n is not large, we have an example, available under request, showing that PWL is not monotonic in the price-marginal cost margin.

4. Conclusion

In this paper we studied the relationship of observable variables with welfare losses, taking the behavior of firms as given[11]. The models presented in this paper have been selected by their impact in the profession.[12] The main message of this paper is positive in the sense that this is a feasible endeavour in the models considered in this paper.[13] We also have uncovered several facts that contradict our intuition about how rates of returns, demand elasticities or price-marginal cost margins affect welfare losses. We remark that we are not against the use of price-marginal costs margins or elasticities as indicators of welfare losses (such variables are widely used in issues like mergers, detection of cartelized behavior, predation or abusive practices). Our point is that such use must take into account the actual role played by these variables.

We end this paper by giving some hints on how data and elasticities may help us to discriminate among these models. The clearest case is Bertrand equilibrium. A necessary condition of this equilibrium to be supported by the data is that for all i, $p_i = \varepsilon(p_i - c_i)$ (irrespective of the market being linear or not). If the elasticity of demand cannot be estimated, Proposition 2 says that any observation can be interpreted as a Bertrand equilibrium. The case for the CES model relies on two assumptions. On the one hand, the elasticity of demand must be constant. On the other hand, the cross elasticity of demand (calculated as $\frac{1}{\frac{\partial p_i}{\partial x_j}\frac{x_j}{p_i}}$) should be very high (it amounts to $\frac{n}{r-\alpha}$). Finally, Cournot equilibrium has also implications. Let $\xi \equiv -\frac{\partial p_i}{\partial x_i}\frac{x_i}{p_i}$ be the elasticity of the inverse demand function. The elasticity ξ can be obtained by inverting the system of demand functions. For instance in the symmetric case with $n = 2$, it is easy to prove that $\xi = \frac{\varepsilon}{\varepsilon^2 - \rho^2}$. Thus, from FOC of profit maximization $\xi p_i = p_i - c_i$.

[11] See Sutton (1998) for an approach where the only source of variation across firms is the degree of competitiveness.

[12] The papers by Dixit, Singh and Vives and Spence obtained, respectively, 341 citations, 422 citations and 639 citations in Google Scholar.

[13] But this is no guarantee that this exercise will always produce positive results. In some cases the model may be unable to determine PWL from observables, see Corchón (2008), Proposition 1 or our comments in the Introduction on the optimal number of firms when firms are of different sizes.

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