

A general equilibrium model of spatial economies: the case of finite locations

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Résumé

Dans cet article, nous considérons une économie spatiale de propriété privée. Nous pouvons distinguer deux différences essentielles entre notre modèle et le modèle classique d'Arrow-Debreu. Tout d'abord, chaque consommateur doit choisir à l'équilibre un unique endroit où consommer. La deuxième différence concerne les dotations initiales qui dépendent du choix de résidence du consommateur, de manière à prendre en compte principalement la dotation en travail. Nous démontrons alors l'existence d'un équilibre général.

Mots clés: Equilibre général, économies spatiales, espace mesuré d'agents, choix de localisation, dotations initiales endogènes.

Abstract

In this article, we consider a spatial private ownership economy with a finite number of locations and a finite measure space of consumers. We can distinguish two main differences between our model and the classical one of Arrow-Debreu. First, every consumer must choose, at the equilibrium, only one location where to consume. The second feature is that the initial endowment depends on the choice of location of the consumer, mainly to take into account the endowment in labor. We then establish the existence of a general equilibrium for this economy.

Keywords: General equilibrium, spatial economies, measure space of agents, locations choices, endogenous endowments.

JEL Classification: D51, R12, R20.

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1 Introduction

In the general equilibrium framework, the existence of equilibria is shown under assumptions which ensure that the demand correspondence of the consumers is convex-valued, namely, convexity assumptions on the preference relations and the consumption sets. Aumann (1965) and Hildenbrand (1970) proposed the consideration of an atomless measure space of agents in order to avoid the convexity assumption on the preference relations. In fact, an increase of the number of the economic agents has a “convexing effect” on the mean (global) demand. This allows to apply a fixed point argument and deduce the existence of the equilibrium. Here, we will not assume that the measure space of consumers is atomless but of course we make a convexity assumption of the preferences of the agents in the atoms.

The consideration of a measure space of consumers, corresponds to the idea that every individual economic agent has a negligible influence on the outcome of the collective decisions. But, economically, the idea of uncountable many agents is not satisfying. However, Debreu (1969) has shown that an economy with a measure space of agents can be obtained as a limit of economies with a finite number of consumers.

No assumption on the boundedness of the consumption sets of consumers is made. A difficulty appears in the proof of the existence result, the fact that the global demand remains in a compact set does not imply that the individual demand remains also in a compact set. In order to overcome this difficulty, we use the Fatou’s lemma in several dimensions. The version we use here, is the one due to Schmeidler (1970).

Another important aspect of the considered model is the presence of a territory, with a finite number of locations. Every consumer will have to chose at equilibrium a location where to live and consume. Then, the non-convex consumption set will add a new difficulty.

In the following section, we describe the model and we state the existence result. The proof of the existence result is given in section 3.

2 The Model and the Existence Result

2.1 The model and the equilibrium notions

We consider a private ownership economy \mathcal{E} with sets H of goods, K of locations, A of consumers, and J of producers. The sets H , K , and J are assumed to be finite. The set of consumers is assumed to be a finite complete measure space (A, \mathcal{A}, ν) , where \mathcal{A} denotes the σ -algebra of subsets of A , and ν is a σ -additive positive measure on \mathcal{A} such that $\nu(A) = 1$. An element $S \in \mathcal{A}$ is an admissible group of consumers, also called a coalition, and $\nu(S)$ can be interpreted as the fraction of the totality of the consumers which are in the coalition S . A fundamental example is the case where $A = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1])$ is the Borel σ -field on $[0, 1]$ and ν is the Lebesgue measure¹. We assume that every good $h \in H$ (physical goods and labor) is available at every location $t \in K$.

In the standard Arrow-Debreu model, every consumption and every production is a vector of $(\mathbb{R}^H)^K$ the commodity space, which is identified² with \mathbb{R}^L where $L = K \times H$. In the following, these vectors are called Arrow-Debreu actions. We adopt the standard convention that the outputs (different forms of labor..) of a consumer are counted negatively and the inputs (food, cars, houses..) are counted positively. The

¹The measure space $([0, 1], \mathcal{B}([0, 1]), \nu)$ takes into account a large class of measure spaces since every separable and non atomic measure space of total measure one is “isomorphic” to it (Halmos and Von Neumann (1942)).

²An element $x \in (\mathbb{R}^H)^K$ is a mapping $x : K \rightarrow \mathbb{R}^H$, which identified with the vector $(x(k)_h)$ in \mathbb{R}^L .

converse convention is used for the producers. A price is also a mapping $p : K \rightarrow \mathbb{R}^H$ so what we can define the value of an action x as $p \cdot_L x = \sum_{t \in K} p(t) \cdot_H x(t)$. In the following, without any risk of confusion, we will simply denote by “ \cdot ” the scalar products in \mathbb{R}^L and \mathbb{R}^H .

In the spatial economy model that we consider in this paper, it is assumed that every consumer $a \in A$ is imposed to consume the good h ($h \in H$) at a single location, called her “residence place”, which will be fixed at equilibrium. This is one of the two main differences between this model, referred to as the “spatial economy” and the Arrow-Debreu model in which consumers may consume commodities in all the locations (and not only in a single one). Thus, in this model, a spatial action x is an element $x = (x_0, x_1) \in K \times \mathbb{R}^H$, i.e., it specifies the location $x_0 \in K$ and the quantity x_1 of goods in H . The value of a consumption x at a price p is then

$$p \cdot x := p(x_0) \cdot x_1,$$

and a precise mathematical sense will be given to it later. Every consumer a is endowed at every location $t \in K$ with a consumption set $X(a, t) \subset \mathbb{R}^H$, which defines the set of all her possible consumptions at t . We then define $X(a) = \{(t, x) \in K \times \mathbb{R}^H \mid x \in X(a, t)\}$.

The tastes of consumer a are represented by a preference relation \preceq_a which is a binary relation on $X(a)$. We also define the strict preference relation \prec_a on $X(a)$ by $x \prec_a x'$ if $[x \preceq_a x'$ and not $x' \preceq_a x]$.

The second main difference with the Arrow-Debreu model is that the initial endowments of consumer a is allowed to depend on her residence $x_0(a) \in K$. One of the main justification of this more general model can be given by considering the various kinds of labor performed by the consumer a which will indeed depend on her location $x_0(a)$.

The initial endowments of the economy are then defined by a given mapping $e : A \times K \rightarrow \mathbb{R}^L$. We interpret $e(a, x_0(a))$ as the initial endowment of consumer a , conditionally to the fact that her location is $x_0(a) \in K$ (which will be fixed at equilibrium). In others words, given a mapping $a \mapsto x_0(a)$, from A to K , specifying the residence of every consumer, then the initial endowment of consumer a is $e(a, x_0(a)) \in \mathbb{R}^L$. The residence mapping $a \mapsto x_0(a)$ will be endogeneously determined at equilibrium, i.e., it will be part of the concept of spatial equilibrium given hereafter.

A consumption allocation specifies the consumption of every consumer, hence is a mapping $x : A \rightarrow K \times \mathbb{R}^H$ such that (i) for almost every (a.e.) $a \in A$, $x(a) \in X(a)$, (ii) the mapping $a \mapsto x_0(a)$ from A to K is measurable (iii) the mapping $a \mapsto x_1(a)$ from A to \mathbb{R}^H is integrable. The set of consumption allocations is denoted by \mathcal{X} . In all the paper we will assume that the mapping $a \mapsto e(a, t)$ from A to \mathbb{R}^L is integrable ($t \in K$) which with the above assumption implies that the mapping $a \mapsto e(a, x_0(a))$ from A to $K \times \mathbb{R}^H$ is integrable.

The production sector of the economy is composed of a finite set J of producers represented by their production sets $Y_j \subset \mathbb{R}^L$, $j \in J$. The producers are owned by the consumers and the ownership shares of the consumers are given by integrable real valued functions $\theta_j : A \rightarrow \mathbb{R}_+$ ($j \in J$).

The private ownership spatial economy \mathcal{E} is thus summarized by the list:

$$\mathcal{E} = \{K, \mathbb{R}^H, (A, \mathcal{A}, \nu), (X(a), \prec_a, e(a, \cdot))_{a \in A}, (Y_j, \theta_j)_{j \in J}\}.$$

The definitions of a equilibrium and a quasi-equilibrium are then:

Definition 2.1 An equilibrium (resp. quasi-equilibrium) of the economy \mathcal{E} is an element (x^*, y^*, p^*) in $\mathcal{X} \times (\mathbb{R}^L)^J \times \mathbb{R}^L$ such that :

(a) [Preference Maximization] for a.e. $a \in A$, $x^*(a)$ is a maximal element of \prec_a in the budget set

$B_a^* = \{x \in X(a) \mid p^* \cdot x \leq p^* \cdot e(a, x_0) + \sum_{j \in J} \theta_j(a) p^* \cdot y_j^*\},$
in the sense that $x^*(a) \in B_a^*$ and there is no $x \in B_a^*$ such that $x^*(a) \prec_a x$;

(resp. (a') for a.e. $a \in A$, $x^*(a) \in \beta_a^*$ and there is no $x \in \beta_a^*$ such that $x^*(a) \prec_a x$, where

$$\beta_a^* = \{x \in X(a) \mid p^* \cdot x < p^* \cdot e(a, x_0) + \sum_{j \in J} \theta_j(a) p^* \cdot y_j^*\}.$$

(b) [Profit Maximization] for every $j \in J$, $y_j^* \in Y_j$ and $p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_j$;

(c) [Market Clearing] for every $t \in K$,

$$\int_{\{a \in A \mid x_0^*(a) = t\}} x_1^*(a) d\nu(a) = \int_A e(a, x_0^*(a))(t) d\nu(a) + \sum_{j \in J} y_j^*(t).$$

One notices that every spatial equilibrium is clearly a quasi-equilibrium. The main result of this paper will show the existence of an equilibrium. This will be deduced from an existence result of a quasi-equilibrium.

Theorem 2.1 If (x^*, y^*, p^*) is a spatial equilibrium then it is weak Pareto.

Proof. By contradiction. If (x^*, y^*, p^*) is not a spatial equilibrium then there exists an admissible allocation (x, y) such that $x^*(a) \prec_a x(a)$ for a. e. $a \in A$. From the Equilibrium Consumption and Producer Conditions, we deduce that, for a. e. $a \in A$

$$p^*(x_0(a)) \cdot x_1(a) > p^* \cdot e(a, x_0(a)) + \sum_{j \in J} \theta_j(a) p^* \cdot y_j^* \geq p^* \cdot e(a, x_0(a)) + \sum_{j \in J} \theta_j(a) p^* \cdot y_j.$$

Integrating over A and using the fact that $\int_A \theta_j(a) = 1$ for all j , letting $A_t := \{a \in A \mid x_0(a) = t\}$, we get

$$p^* \cdot \int_A [e(a, x_0(a)) + \sum_{j \in J} \theta_j(a) y_j] = p^* \cdot \left[\int_A e(a, x_0(a)) + \sum_{j \in J} y_j \right] < \int_A p^*(x_0(a)) \cdot x_1(a)$$

$$= \sum_{t \in K} \int_{A_t} p^*(t) \cdot_H x_1(a) = \sum_{t \in K} p^*(t) \cdot_H \int_{A_t} x_1(a) = p^* \cdot_L \left(\int_{A_t} x_1(a) \right)_{t \in K},$$

which contradicts the fact that (x, y) is an admissible allocation, e.g.,

$$\left(\int_{A_t} x_1(a) \right)_{t \in K} = \int_A e(a, x_0(a)) + \sum_{j \in J} y_j.$$

2.2 The Existence Result

We let the following assumptions on the economy \mathcal{E} .

Assumption C [Consumption side]

- (i) For a.e. $a \in A$, $X(a)$ is non-empty, closed and the correspondence $a \rightarrow X(a)$ is bounded below³.
- (ii) [Measurability] The measure space (A, \mathcal{A}, ν) is atomless⁴, the preference relation correspondence $a \rightarrow \prec_a$ is measurable⁵ and the consumption set correspondence $a \rightarrow X(a)$, from A to \mathbf{R}^L , is measurable.
- (iii) For a.e. $a \in A$, the strict preference relation \prec_a is irreflexive and transitive,⁶
- (iv) [upper and lower semicontinuities] for a.e. $a \in A$, for every $x \in X(a)$, the sets $\{z \in X(a) \mid z \prec_a x\}$ and $\{z \in X(a) \mid z \succ_a x\}$ are open in $X(a)$ (for its relative topology);
- (v) [local nonsatiation] for a.e. $a \in A$, for every consumption $x \in X(a)$ and every neighborhood \mathcal{N} of x , there is $x' \in X(a) \cap \mathcal{N}$ such that $x \prec_a x'$.

We denote by $Y \subset \mathbf{R}^L$ the total production set of the economy, that is, $Y = \sum_{j \in J} Y_j$ and by $\mathbf{A}Y$ the asymptotic cone of Y ⁷.

Assumption P [Production Side]

- (i) Y is closed, convex ;
- (ii) [Free Disposal] $Y - \mathbf{R}_+^L \subset Y$;
- (iii) for every $j \in J$, the function $\theta_j : A \rightarrow \mathbf{R}_+$ is integrable and, for every $j \in J$, $\int_A \theta_j(a) d\nu(a) = 1$.
- (iv) $\mathbf{A}Y \cap \mathbf{R}_+^L = \{0\}$,

Assumption E [Endowments]

For all $t \in K$, the mapping $a \mapsto e(a, t)$ is integrable.

Assumption S [Survival] (i) For every $j \in J$, $0 \in Y_j$;

- (ii) For a.e. $a \in A$, there exists $x = (x_0, x_1) \in X(a)$ such that

$$\begin{cases} e(a, x_0)(t) \geq x_1 & \text{if } t = x_0 \\ e(a, x_0)(t) \geq 0 & \text{if } t \neq x_0 \end{cases}$$

Assumption SS [Strong Survival] For a.e. $a \in A$,

- (i) For every $j \in J$, $0 \in Y_j$;
- (ii) for every $x_0 \in K$, $X(a, x_0)$ is convex
- (iii) for every $x_0 \in K$ such that $X(a, x_0) \neq \emptyset$, there exists $x_1 \in \mathbf{R}^H$ such that $(x_0, x_1) \in X(a)$ and

$$\begin{cases} e(a, x_0)(t) \gg x_1 & \text{if } t = x_0 \\ e(a, x_0)(t) \geq 0 & \text{if } t \neq x_0 \end{cases}$$

We can now state the main result of this paper.

Theorem 2.2 *The economy \mathcal{E} admits an equilibrium (resp. quasi-equilibrium) if it satisfies Assumptions C, P, E and SS (resp. S).*

³in the sense that there is $\underline{x} \in -\mathbf{R}_{++}^H$ such that, for a.e. $a \in A$ and for every $x = (x_0, x_1) \in X(a)$, $\underline{x} \leq x_1$.

⁴An element $T \in \mathcal{A}$ is said to be an atom for the measure ν if $\nu(T) > 0$ and if for every $S \in \mathcal{A}$, $S \subset T$ one has either $\nu(S) = \nu(T)$ or $\nu(S) = 0$. The measure space (A, \mathcal{A}, ν) is said to be atomless if the measure ν has no atoms.

⁵in the sense that $\{(a, x) \in A \times (K \times \mathbf{R}^H) \mid x \in X(a)\} \in \mathcal{A} \otimes \mathcal{B}(K \times \mathbf{R}^H)$ and $\{(a, x, x') \in A \times (K \times \mathbf{R}^H) \times (K \times \mathbf{R}^H) \mid x \prec_a x'\} \in \mathcal{A} \otimes \mathcal{B}(K \times \mathbf{R}^H) \otimes \mathcal{B}(K \times \mathbf{R}^H)$.

⁶i.e., for every x, x', x'' in $X(a)$, not $x \prec_a x$ and $[x \prec_a x' \text{ and } x' \prec_a x'']$ implies $x \prec_a x''$.

⁷We recall that, for a convex set Y , one can define $\mathbf{A}Y := \{v \in \mathbf{R}^L \mid Y + v \subset Y\}$ [Theorem 8.1 of Rockafellar (1970)] and we refer to Debreu (1959) for an alternative definition in the general (nonconvex) case.

The proof of Theorem 2.2 is a consequence of the following more general result which will be prove in Section 3. We first introduce a weak version of the survival assumption.

Assumption WS [Weak Survival] For a.e. $a \in A$, there exists $(x_0, x_1) \in X(a)$, $y_j \in \overline{\text{co}} Y_j$ ($j \in J$), $q \in \mathbf{AY}$ such that for every $p \in \mathbf{R}^L$,

$$p(x_0) \cdot x = p \cdot (e(a, x_0) + \sum_{j \in J} \theta_j(a) y_j + q).$$

Theorem 2.3 (a) *The economy \mathcal{E} admits a quasi-equilibrium if it satisfies Assumptions **C**, **P**, **E** and **WS**.*

(b) *every quasi-equilibrium (x^*, y^*, p^*) is an equilibrium if Assumption **C**(v) holds and if for a.e. $a \in A$,*

(i) *for every $x_0 \in K$, $X(a, x_0)$ is convex;*

(v) *for every $x_0 \in K$ such that $X(a, x_0) \neq \emptyset$, there exists $x_1 \in \mathbf{R}^H$ such that $(x_0, x_1) \in X(a)$ and*

$$p^*(x_0) \cdot x_1 < p^* \cdot e(a, x_0) + \sum_{j \in J} \theta_j(a) \sup p^* \cdot Y_j.$$

The proof of Part (a) Theorem 2.3 is given in Section ??.

Proof of Part (b). By contraposition. Suppose that (x^*, y^*, p^*) is a quasi-equilibrium and that it is not an equilibrium. Then, there exists $E \in \mathcal{A}$ such that $\mu(E) > 0$ and such that, for a.e. $a \in E$, there exists $x^{**}(a) = (x_0^{**}(a), x_1^{**}(a)) \in B_a^*$ with $x^*(a) \prec_a x^{**}(a)$.

From the assumption of part (b), there exists $x_1(a) \in \mathbf{R}^H$ such that $x(a) = (x_0^*(a), x_1(a)) \in \beta_a^* = \{x \in X(a) \mid p^* \cdot x < p^* \cdot e(a, x_0) + \sum_{j \in J} \theta_j(a) p^* \cdot y_j^*\}$. From the above convexity assumption, using the fact that $x^{**}(a) \in B_a^*$ and $x(a) \in \beta_a^*$, one deduces that for every $\lambda \in [0, 1[$,

$$\begin{aligned} \lambda p^*(x_0^{**}(a)) \cdot x_1^{**}(a) + (1 - \lambda) p^*(x_0^*(a)) \cdot x_1(a) &< \lambda p^* \cdot e(a, x_0^{**}(a)) \\ &+ (1 - \lambda) p^* \cdot e(a, x_0^*(a)) \\ &+ \sum_{j \in J} \theta_j(a) \sup p^* \cdot Y_j \\ &= \\ &p^* \cdot e(a, x_0^{**}(a)) + \sum_{j \in J} \theta_j(a) \sup p^* \cdot Y_j \end{aligned}$$

Consequently, $\lambda x^{**}(a) + (1 - \lambda)x(a) \in \beta_a^*$. From above, $x^{**}(a)$ belongs to the set $\{x \in X(a) \mid x^*(a) \prec_a x\}$ which is open in $X(a)$ by Assumption **C**(iv), and for λ close enough to 1, one has $x^*(a) \prec_a \lambda x^{**}(a) + (1 - \lambda)x(a)$, which contradicts the fact that (x^*, y^*, p^*) is a quasi-equilibrium. \blacksquare

3 Definition of an Auxiliary Economy

We need first some mathematical results.

3.1 The embedding of $K \times \mathbf{R}^H$ in \mathbf{R}^L

For every $a \in A$, we shall now 'identify' every consumption $x = (x_0, x_1) \in X(a)$ to an element of \mathbf{R}^L as follows (we recall that $X(a) \subset K \times \mathbf{R}^H$ and that $\mathbf{R}^L = \mathbf{R}^{K \times H}$). This will allow us to work in the linear space \mathbf{R}^L instead of $K \times \mathbf{R}^H$ and will give a classical interpretation of summation of consumptions.

We define first the mapping $\delta : K \times (\mathbb{R}^H \setminus \{0\}) \rightarrow \mathbb{R}^L$ as follows. To every $x = (x_0, x_1) \in K \times (\mathbb{R}^H \setminus \{0\})$, we associate the vector $\delta(x) \in \mathbb{R}^L$ given by, for every $(t, h) \in K \times \mathbb{R}^H$,

$$\begin{cases} \delta(x)_{(t,h)} = 0 \text{ if } t \neq x_0, \\ \delta(x)_{(t,h)} = x_h, \text{ if } t = x_0. \end{cases}$$

Then, $\delta(x)$ is⁸ the unique element in \mathbb{R}^L such that for every $p \in \mathbb{R}^L$,

$$p \cdot x = p \cdot \delta(x) = p(x_0) \cdot x_1. \quad (1)$$

Proposition 3.1 *The mapping δ is continuous, closed and injective on $K \times (\mathbb{R}^H \setminus \{0\})$.*

Proof. It is direct consequence of relation (1) and of the fact that K is finite. \blacksquare

The definition of the mapping δ on $K \times \{0\}$ would give $\delta(x) = 0$ for every $x \in K \times \{0\}$, and then δ would not be any more injective for K larger than the singleton.

The space \mathbb{R}^L is a linear space, and hereafter will be referred to as the commodity space. We then define a new economy, equivalent to the original one.

3.2 Definition of an Auxiliary Economy

We need to construct the economy $\tilde{\mathcal{E}}$ from the initial economy \mathcal{E} in order to obtain the Aumann-Hildenbrand framework with many locations.

We first define an auxiliary mapping, φ , from $K \times (\underline{x} + \mathbb{R}_+^H)$ to \mathbb{R}^L by $\varphi(x) = \delta(x_0, x_1 - 2\underline{x})$, where \underline{x} is given in Assumption C. Since $\underline{x} \in -\mathbb{R}_{++}^L$, and with Proposition 3.1, one notices that the mapping φ is continuous and injective on $K \times (\underline{x} + \mathbb{R}_+^H)$. Moreover, from relation (1), one obtain for every $p \in \mathbb{R}^L$ the relation below, usefull in the following:

$$p \cdot x = p(x_0) \cdot (x_1 - 2\underline{x}) - p(x_0) \cdot (-2\underline{x}) = p \cdot \varphi(x) - p \cdot \varphi((x_0, 0)). \quad (2)$$

For a.e. $a \in A$, we define the consumption set:

$$\tilde{X}(a) = \{\varphi(x) \in \mathbb{R}^L \mid x = (x_0, x_1) \in X(a)\} = \varphi \circ X(a).$$

The preference relation $\tilde{\preceq}_a$ of the consumer a on $\tilde{X}(a)$ is defined, for every x, x' in $\tilde{X}(a)$, by $x \tilde{\preceq}_a x'$ if $\varphi^{-1}(x) \preceq_a \varphi^{-1}(x')$. Finally, the initial endowments are given by the mapping $\tilde{e}(a, \cdot)$ from $\varphi(K \times (\underline{x} + \mathbb{R}_+^H))$ to \mathbb{R}^L such that, for every $x \in \varphi(K \times (\underline{x} + \mathbb{R}_+^H))$,

$$\tilde{e}(a, x) = e(a, \varphi^{-1}(x)_0) + \varphi((\varphi^{-1}(x)_0, 0)).$$

3.3 The existence result

We establish now the link between the two economies, \mathcal{E} and $\tilde{\mathcal{E}}$.

Proposition 3.2 *Let $(\tilde{x}^*(\cdot), (y_j^*)_{j \in J}, p^*)$ be a quasi-equilibrium of the economy $\tilde{\mathcal{E}}$. Then $(\varphi^{-1}(\tilde{x}^*(\cdot)), (y_j^*)_{j \in J}, p^*)$ is a quasi-equilibrium of the economy \mathcal{E} .*

Proof. Let $(\tilde{x}^*(\cdot), (y_j^*)_{j \in J}, p^*)$ be a quasi-equilibrium of $\tilde{\mathcal{E}}$. For every $a \in A$, we let $x^*(a) = \varphi^{-1}(\tilde{x}^*(a))$. Since the profit maximization is obviously satisfied for the economy \mathcal{E} , one only needs to prove the 2 others conditions:

⁸Another way to define $\delta(x)$ is to consider $\delta(x)$ as a mapping from $(\mathbb{R}^H)^K$ to \mathbb{R} and such that for every $p \in (\mathbb{R}^H)^K$, $\delta(x)(p) = p(x_0) \cdot x_1$. In this case, $\delta(x)$ belongs to $(\mathbb{R}^L)^*$, the dual space of \mathbb{R}^L .

Claim 3.1 (*Preference Maximization*) For a.e. $a \in A$,

- (i) $x^*(a) \in \{x \in X(a) \mid p^* \cdot x \leq p^* \cdot (e(a, x) + \sum_{j \in J} \theta_j(a) y_j^*)\}$;
- (ii) there exists no $x \in X(a)$ such that $p^* \cdot x < p^* \cdot (e(a, x) + \sum_{j \in J} \theta_j(a) y_j^*)$ and $x^*(a) \prec_a x$.

Proof. Part (i) For every $a \in A$, since $\tilde{x}^*(a) \in \tilde{X}(a) = \varphi \circ X(a)$, then $x^*(a) \in X(a)$. Moreover, one has from relation (2),

$$p^* \cdot \tilde{x}^*(a) = p^* \cdot x^*(a) + p^* \cdot \varphi((x_0^*(a), 0)). \quad (3)$$

The definition of $\tilde{e}(a, \cdot)$ gives:

$$p^* \cdot \tilde{e}(a, \tilde{x}^*(a)) = p^* \cdot e(a, x_0^*(a)) + p^* \cdot \varphi((x_0^*(a), 0)). \quad (4)$$

From the budget constraint, $p^* \cdot \tilde{x}^*(a) \leq p^* \cdot (\tilde{e}(a, \tilde{x}^*(a)) + \sum_{j \in J} \theta_j(a) y_j^*)$, and the two previous relations, one then obtains $p^* \cdot x^*(a) \leq p^* \cdot (e(a, x_0^*(a)) + \sum_{j \in J} \theta_j(a) y_j^*)$.

We prove the second part by contraposition. Suppose there exists $x \in X(a)$ such that $p^* \cdot x < p^* \cdot (e(a, x_0(a)) + \sum_{j \in J} \theta_j(a) y_j^*)$ and $x^*(a) \prec_a x$. In the same way as before, one gets $p^* \cdot \varphi(x) < p^* \cdot (\tilde{e}(a, \varphi(x)) + \sum_{j \in J} \theta_j(a) y_j^*)$. Moreover, from the definition of $\tilde{\prec}_a$, one obtains $\tilde{x}^*(a) \tilde{\prec}_a \varphi(x)$ which contradicts the definition of $\tilde{x}^*(a)$. ■

Claim 3.2 (*Market clearing*)

$$\int_A x^*(a) d\lambda(a) = \int_A e(a, x_0^*(a)) d\lambda(a) + \sum_{j \in J} y_j^*$$

Proof. We recall first that $\int_A x^*(a) d\lambda(a)$ is the unique element in \mathbb{R}^L such that, for every $p \in \mathbb{R}^L$,

$$p \cdot \int_A x^*(a) d\nu(a) = \int_A p \cdot x^*(a) d\nu(a) = \int_A p(x_0^*(a)) \cdot x_1^*(a) d\nu(a).$$

The market clearing condition on $\tilde{\mathcal{E}}$ gives

$$\int_A \tilde{x}^*(a) d\lambda(a) = \int_A \tilde{e}(a, \tilde{x}^*(a)) d\lambda(a) + \sum_{j \in J} y_j^*. \quad (5)$$

With relations (3), (4) and (5), one obtains

$$\begin{aligned} p \cdot \int_A x^*(a) d\nu(a) &= \int_A p \cdot \tilde{x}^*(a) d\lambda(a) + \int_A p \cdot \varphi((x_0^*(a), 0)) d\lambda(a) \\ &= \int_A p \cdot [\tilde{e}(a, \tilde{x}^*(a)) + \varphi((x_0^*(a), 0))] d\lambda(a) + p \cdot \sum_{j \in J} y_j^* \\ &= p \cdot (\int_A e(a, x_0^*(a)) d\lambda(a) + \sum_{j \in J} y_j^*). \end{aligned}$$

Since the previous relation is true for every $p \in \mathbb{R}^L$, then

$$\int_A x^*(a) d\nu(a) = \int_A e(a, x_0^*(a)) d\lambda(a) + \sum_{j \in J} y_j^*. \quad \blacksquare$$

Then, the three conditions for a quasi-equilibrium are satisfied. ■

Theorem 2.3 is then a consequence of Theorem 3.1 and of Proposition 4.1.

One has the following existence result on the economy

$$\tilde{\mathcal{E}} = \{(A, \mathcal{A}, \lambda), (\tilde{X}, \tilde{\succeq}, \tilde{e}), (Y_j, \theta_j)_{j \in J}\}.$$

Theorem 3.1 *The economy $\tilde{\mathcal{E}}$ admits a quasi-equilibrium if the economy \mathcal{E} satisfies Assumptions C, P, E and S.*

4 Proof of Theorem 2.3

The proof relies on the result of Cornet-Medecin(2002). To apply it to the private ownership economy \mathcal{E} , we need to construct another private ownership economy $\tilde{\mathcal{E}}$, equivalent to the original one, and such that it satisfies the assumption of Cornet-Medecin(2002).

Proof. In order to apply the result of Cornet-Medecin (2002), we need to verify that the following Assumptions are full-filled.

Proposition 4.1 *The economy $\tilde{\mathcal{E}}$ satisfies the following assumptions.*

Assumption $\tilde{\mathbf{M}}$

- (i) *The consumption set correspondence $a \rightarrow \tilde{X}(a)$, from A to \mathbb{R}^L , is measurable;*
- (ii) *the preference relation correspondence $a \rightarrow \tilde{z}_a$ is measurable;*
- (iii) *for every $j \in J$, the function $\theta_j : A \rightarrow \mathbb{R}_+$ is integrable and $\int_A \theta_j(a) d\nu(a) = 1$.*

Assumption $\tilde{\mathbf{C}}$

- (i) *For a.e. $a \in A$, $\tilde{X}(a)$ is closed;*
- (ii) *[integrably bounded below] the correspondence $a \rightarrow \tilde{X}(a)$ is bounded below, in the sense that there is $\underline{x}' \in -\mathbb{R}_{++}^L$ such that, for a.e. $a \in A$, $\tilde{X}(a) \subset \underline{x}' + \mathbb{R}_+^L$;*
- (iii) *for a.e. $a \in A$, the preference relation \tilde{z}_a is irreflexive and transitive;*
- (iv) *[continuity] for a.e. $a \in A$, for every $x \in \tilde{X}(a)$, the sets $\{z \in \tilde{X}(a) \mid z \tilde{z}_a x\}$ and $\{z \in \tilde{X}(a) \mid x \tilde{z}_a z\}$ are open in $\tilde{X}(a)$ (for its relative topology);*
- (v) *[local nonsatiation] for a.e. $a \in A$, for every consumption $x \in \tilde{X}(a)$ and every neighborhood \mathcal{N} of x , there is $x' \in \tilde{X}(a) \cap \mathcal{N}$ such that $x \tilde{z}_a x'$.*

Assumption $\tilde{\mathbf{S}}$ *For a.e. $a \in A$,*

$$\emptyset \neq \{x \in \tilde{X}(a) \mid x - \tilde{e}(a, x) \in \sum_{j \in J} \theta_j(a) \overline{\text{co}} Y_j + \mathbf{A}Y\}.$$

Assumption $\tilde{\mathbf{E}}$ (i) *The mapping $e : A \times \mathbb{R}^L \rightarrow \mathbb{R}^L$ is a Caratheodory mapping, that is, for all $a \in A$, the mapping $x \mapsto \tilde{e}(a, x)$ is continuous on \mathbb{R}^L and for all $x \in \mathbb{R}^L$, the mapping $a \mapsto \tilde{e}(a, x)$ is measurable.*

(ii) *There exist a real number $c \in [0, 1[$, and an integrable mapping $\bar{e}(\cdot) : A \rightarrow \mathbb{R}_+$ such that*

$$|\tilde{e}_h(a, x)| \leq \bar{e}(a) + c|x_h|, \forall (a, x) \in A \times \mathbb{R}^L, \forall h \in L.$$

Proof. Assumptions $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{C}}$ are satisfied as a direct consequence of the fact that the mapping φ from $K \times (\underline{x} + \mathbb{R}_+^H)$ to \mathbb{R}^L is continuous (then measurable) and closed, and that K is a finite set.

From Assumption \mathbf{S} , for a.e. $a \in A$, there exists $x \in X(a)$, $y_j \in \overline{\text{co}} Y_j$ ($j \in J$) and $q \in \mathbf{A}Y$ such that, for every $p \in \mathbb{R}^L$,

$$p \cdot x = p(x_0) \cdot x_1 = p \cdot (e(a, x_0) + \sum_{j \in J} \theta_j(a) y_j + q). \quad (6)$$

One let $z = \varphi(x)$ and then $z \in \tilde{X}(a)$. Moreover, with relations (2) and (6), one has, for every $p \in \mathbb{R}^L$,

$$\begin{aligned} p \cdot z &= p \cdot x + p \cdot \varphi((x_0, 0)) \\ &= p \cdot (e(a, x_0) + \sum_{j \in J} \theta_j(a) y_j + q) + p \cdot \varphi((x_0, 0)) \\ &= p \cdot (\tilde{e}(a, z) + \sum_{j \in J} \theta_j(a) y_j + q) \end{aligned}$$

It then implies $z = \tilde{e}(a, z) + \sum_{j \in J} \theta_j(a) y_j + q$ and, hence, Assumption $\tilde{\mathbf{S}}$ is satisfied. ■

Thus, since Assumptions of Cornet-Medecin (2002) are satisfied, there exists a quasi-equilibrium for the auxiliary economy $\tilde{\mathcal{E}}$. ■

4.1 Equivalence of equilibria

5 Appendix

5.1 Appendix A: some theorems

We first recall some results.

Theorem A (Aumann). *Let F be a measurable correspondence from a measure space $(A, \mathcal{A}, \lambda)$ to \mathbb{R}^L with non-empty values. We have the following:*

- (i) *There exists a measurable selection of F .*
- (ii) *There exists a sequence of measurable functions (f_k) from A to \mathbb{R}^L such that for every $a \in A$, $(f_k(a))$ is dense in $F(a)$.*

Theorem B (Schmeidler (1970)). *Let (f_k) be a sequence of integrable functions from A to \mathbb{R}^L with the properties: (1) there is an integrable function g such that for all k , $g \leq f_k$; (2) $\lim_k \int_A f_k d\lambda = v$. Then there exists an integrable function $f : A \rightarrow \mathbb{R}^L$ such that (a) $\int_A f d\lambda \leq v$; (b) a.e., $f(a)$ is adherent to $(f_k(a))$.*

Theorem B' *Let C be a pointed, closed, convex cone⁹ in \mathbb{R}^L , let $\{f_k\}$ be a sequence of integrable mappings, from a measure space (A, \mathcal{A}, ν) to \mathbb{R}^L , satisfying the two following properties:*

- (1) $\lim_k \int_A f_k(a) d\nu$ exists;
- (2) *there is an integrable mapping $g : A \rightarrow \mathbb{R}_+$ such that, for all k , a.e. a , $f_k(a) \in \overline{B}(0, g(a)) + C$.*

Then there exists an integrable mapping $f : A \rightarrow \mathbb{R}^L$ such that:

- (a) $\lim_k \int_A f_k(a) d\nu - \int_A f(a) d\nu \in C$;
- (b) *for a.e. $a \in A$, $f(a)$ is adherent to $\{f_k(a)\}$.*

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Theorem C. *Let (f_k) be a sequence of integrable functions from A to \mathbb{R}^L with the properties: (1) There is an integrable function g such that for every k , $\|f_k(a)\| \leq g(a)$ for every a ; (2) $\lim_k \int_A f_k d\lambda = v$. Then there exists an integrable function f from A to \mathbb{R}^L such that: (a) $\int_A f d\lambda = v$; (b) for a.e. $a \in A$, $f(a)$ is adherent to the sequence $(f_k(a))$.*

⁹that is, such that $C \cap (-C) = \emptyset$.

Theorem D (Debreu, Gale, Nikaido). *Let φ be an upper semicontinuous correspondence from $\bar{B}(0, 1)$ to \mathbb{R}^L with nonempty, convex, compact values. If, for all $p \in \Delta := \{p \in \mathbb{R}^L \mid \|p\| = 1\}$, $\sup p \cdot \varphi(p) \leq 0$, then there exists $p^* \in \bar{B}(0, 1)$ such that $0 \in \varphi(p^*)$.*

Theorem E (Lyapounov) *Let $(\nu_i)_{i \in I}$ a finite family of atomless measures on (A, \mathcal{A}) . Then the subset of \mathbb{R}^I defined by*

$$\{(\nu_i(E))_{i \in I} \mid E \in \mathcal{A}\}$$

is closed and convex.

We give definitions and properties of Carathéodory mappings and of Nemytskii mappings which can be found in De Figueiredo [chapter 2, 1989].

Let Ω be an open subset of \mathbb{R}^ℓ . A function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a *Carathéodory function* if:

- (a) for each fixed $s \in \mathbb{R}$, the function $x \mapsto f(x, s)$ is measurable,
- (b) for almost every $x \in \Omega$, the function $s \mapsto f(x, s)$ is continuous.

Let \mathcal{D} be the set of all measurable functions $u : \Omega \rightarrow \mathbb{R}$.

Theorem 5.1 *If $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, then the function $x \mapsto f(x, u(x))$ is measurable for all $u \in \mathcal{D}$.*

Given a Carathéodory function f , we define the *Nemytskii mapping* $N_f : \mathcal{D} \rightarrow \mathcal{D}$, by $N_f(u) : x \mapsto f(x, u(x))$ for every $x \in \Omega$.

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Theorem 5.2 *Assume that Ω has a finite measure μ . Let (u_n) be a sequence in \mathcal{D} which converges in measure to $u \in \mathcal{D}$. Then $N_f u_n$ converges in measure to $N_f u$.*

Theorem 5.3 *Suppose that there are real numbers $c > 0$ and $r > 0$, a function $b \in L^q(\Omega)$, $1 \leq q \leq \infty$, such that*

$$|f(x, s)| \leq c|s|^r + b(x), \quad \forall x \in \Omega, \quad \forall s \in \mathbb{R}.$$

Then (a) N_f maps L^{qr} into L^q ; (b) N_f is continuous and bounded (that is, it maps bounded sets into bounded sets).

5.2 Appendix B: Proof of Theorem 3.1

The proof consists in several steps. The first step associates to $\tilde{\mathcal{E}}$ an auxiliary economy \mathcal{E}^k , whose consumption sets and production sets are bounded. The second step shows that we can associate to the economy \mathcal{E}^k , a list $(x^k(\cdot), y^k, (p^k)) \subset \mathcal{X} \times Y \times \mathbb{R}^L$ by a classical fixed-point argument. The third step will show that the sequence $((\int_A x^k(a) d\nu(a), (y^k), (p^k)) \subset \mathbb{R}^L \times Y \times \mathbb{R}^L$ converges to some element (x^*, y^*, p^*) and, via a version of Fatou's lemma in several dimension [cf. the Appendix], there exists $x^*(\cdot) \in \mathcal{X}$ such that $z^* := \int_A x^*(a) d\nu(a) - \int_A e(a, x^*(a)) d\nu(a) - y^* \in -\mathbf{AY}$. Noticing that $y^* + z^* \in Y$ from the definition of \mathbf{AY} , there exist $y_j^* \in Y_j$ ($j \in J$) such that $y^* + z^* = \sum_j y_j^*$ and the last step consists to show that $(x^*(\cdot), (y_j^*)_{j \in J}, p^*)$ is a quasi-equilibrium of \mathcal{E} .

5.2.1 Compactification of the economy

We first begin with a lemma the proof of which is given in the Appendix.

Lemma 5.1 *There exist integrable mappings $\hat{x} : A \rightarrow \mathbb{R}^L$, $\hat{q} : A \rightarrow \mathbf{AY}$, and measurable mappings $\hat{y}_j : A \rightarrow \overline{\text{co}} Y_j$ ($j \in J$) such that, for a.e. $a \in A$, $\hat{x}(a) \in X(a)$ and $\hat{x}(a) = e(a, \hat{x}(a)) + \sum_{j \in J} \theta_j(a) \hat{y}_j(a) + \hat{q}(a)$.*

For every integer k , we define the truncated economy \mathcal{E}^k as follows. For every consumer a , we let

$$X^k(a) := X(a) \cap \overline{B}(0, \max\{k, \|\hat{x}(a)\|\}),$$

where $\hat{x}(\cdot)$ is defined as in Lemma 5.1, and we notice that the set $X^k(a)$ is nonempty (since $\hat{x}(a) \in X^k(a)$) and is integrably bounded. For every $(a, p) \in A \times \mathbb{R}^L$, and every (consumption-dependent) wealth function $w : X^k(a) \rightarrow \mathbb{R}$, we define the following budget sets :

$$B^k(a, p, w) := \{x \in X^k(a) \mid p \cdot x \leq w(x)\}$$

$$\beta^k(a, p, w) := \{x \in X^k(a) \mid p \cdot x < w(x)\}$$

and the "quasi-demand" $D^k(a, p, w)$ of consumer a :

$$D^k(a, p, w) := \{x \in B^k(a, p, w) \mid \text{there is no } x' \in X^k(a), p \cdot x' < w(x') \text{ and } x \prec_a x'\}.$$

For every j , we let $y_j \in Y_j$ (which is nonempty by Assumption **S**), and for every integer k , we define the production sets economy \mathcal{E}^k as follows

$$Y_j^k := \overline{\text{co}} Y_j \cap \overline{B}(y_j, k) \quad (j \in J), \quad \text{and } Y^k := Y \cap \overline{B}(\sum_{j \in J} y_j, k \# J),$$

the profit function $\pi_j^k : \mathbb{R}^L \rightarrow \mathbb{R}$ ($j \in J$) :

$$\pi_j^k(p) := \sup\{p \cdot y \mid y \in Y_j^k\}$$

and the total supply S^k of the economy \mathcal{E}^k , which is a correspondence from \mathbb{R}^L to Y^k defined by:

$$S^k(p) := \{y \in Y^k \mid p \cdot y \geq p \cdot y', \forall y' \in Y^k\}.$$

The endowment $e^k(a, x)$ of consumer a is now defined as follows. We let

$$e^k(a, x) := e(a, x) + \sum_j \theta_j(a) [\hat{y}_j(a) - y_j] \varepsilon^k(a), \quad (7)$$

where $\varepsilon^k : A \rightarrow [0, 1]$ is an integrable function such that¹⁰

for a.e. a , $\lim_{k \rightarrow \infty} \varepsilon^k(a, x) = 0$, and $y_j^k(a) := \hat{y}_j(a) - \varepsilon^k(a) [\hat{y}_j(a) - y_j] \in \text{co} Y_j \cap \overline{B}(y_j, k)$. (8)

¹⁰The proof of the assertion goes as follows. For every $a \in A$, we let $\varepsilon^k(a) := \max_{j \in J} \{\varepsilon_j^k(a)\}$ where, for every $j \in J$, $\varepsilon_j^k(a) := \begin{cases} 0 & \text{if } y_j = \hat{y}_j(a) \\ \max\{0, (\|\hat{y}_j(a) - y_j\| - k) / \|\hat{y}_j(a) - y_j\|\} & \text{if not} \end{cases}$. The function ε^k is easily shown to be integrable since it is measurable (from the measurability of the mappings \hat{y}_j by Lemma 3.1) and with values in $[0, 1]$.

For every $p \in \mathbb{R}^L$, the wealth distribution of the consumer a is defined by the function $w^k(a, p) : X^k(a) \rightarrow \mathbb{R}$:

$$w^k(a, p)(x) := p \cdot e^k(a, x) + \sum_j \theta_j(a) \pi_j^k(p) + (1 - \|p\|),$$

We now define the excess demand Z^k of the economy \mathcal{E}^k , by:

$$Z^k(p) := \left\{ \int_A [x(a) - e^k(a, x(a))] d\nu(a) \mid x(a) \in D^k(a, p, w^k(a, p)) \right\} - S^k(p).$$

The main property of the excess demand is given by the following theorem. In the following we denote by $B := \bar{B}(0, 1)$, the closed unit Euclidean ball of \mathbb{R}^L centered at 0, and by ∂B , the closed unit Euclidean sphere centered at 0.

Theorem 5.4 *The correspondence Z^k from $B \cap \mathbf{AY}^\circ$ to \mathbb{R}^L is upper semi-continuous with non-empty, convex, compact values and satisfies*

Walras' law for all $p \in \partial B \cap \mathbf{AY}^\circ$, $\sup p \cdot Z^k(p) \leq 0$.

Proof. *Step 1* The fact that Z^k is upper semi-continuous with non-empty, convex, compact values is a direct consequence of the two following claims, recalling that the sum of correspondences which are u.s.c. with non-empty, convex, compact values, is also an u.s.c. correspondence with non-empty, convex, compact values.

Claim 5.1 *For $j \in J$, the function $\pi_j^k : \mathbb{R}^L \rightarrow \mathbb{R}$ is continuous and the correspondence S^k , from \mathbb{R}^L to \mathbb{R}^L , is upper semi-continuous with nonempty, convex, compact values.*

Proof. It is a direct consequence of the maximum theorem (Berge (1959)) and the fact that the set Y_j^k ($j \in J$) is nonempty, convex and compact. ■

Claim 5.2 *The correspondence D^k from $B \cap \mathbf{AY}^\circ$ to \mathbb{R}^L is u.s.c. with nonempty, convex, compact values.*

Proof. It is a consequence of Theorem E in the Appendix A and we just have to show that (i) the correspondence $a \rightarrow X^k(a)$ is integrably bounded, (ii) w^k is a Caratheodory function, i.e., for every $(p, x) \in \mathbb{R}^L \times \mathbb{R}^L$, the function $a \rightarrow w^k(a, p)(x)$ is measurable and, for every $a \in A$, the function $(p, x) \rightarrow w^k(a, p)(x)$ is continuous, and (iii) $B \cap \mathbf{AY}^\circ \subset \{p \in \mathbb{R}^L \mid \exists x \in X^k(a), p \cdot x \leq w^k(a, p)(x)\}$.

Assertion (i) is a consequence of the definition $X^k(a) := X(a) \cap \bar{B}(0, \max\{k, \|\hat{x}(a)\|\})$ and of the fact that the function $a \rightarrow \max\{k, \|\hat{x}(a)\|\}$ is integrable (since $\hat{x} : A \rightarrow \mathbb{R}^L$ is integrable by Lemma 3.1). Assertion (ii) is a consequence of the continuity of π_j^k by Claim 3.1, of the integrability of $e^k : A \rightarrow \mathbb{R}^L$ from above, and of the integrability of $\theta_j : A \rightarrow \mathbb{R}_+$ by the Measurability Assumption M. Finally, Assertion (iii) is a consequence of Lemma 3.1, recalling that, from our definition of $X^k(a)$ one has $\hat{x}(a) \in X^k(a)$. Hence, from Lemma 3.1, our definition of $e^k(a, x)$, for every $p \in B \cap \mathbf{AY}^\circ$, one has

$$\begin{aligned} w^k(a, p)(\hat{x}(a)) &:= p \cdot e^k(a, \hat{x}(a)) + \sum_j \theta_j(a) \pi_j^k(p) + (1 - \|p\|) \\ &\geq p \cdot e^k(a, \hat{x}(a)) + \sum_j \theta_j(a) p \cdot y_j^k(a) + 0 \end{aligned}$$

$$\begin{aligned}
&= p \cdot e(a, \hat{x}(a)) + \sum_j \theta_j(a) p \cdot \hat{y}_j(a) \\
&= p \cdot \hat{x}(a) - p \cdot \hat{q}(a) \geq p \cdot \hat{x}(a) + 0.
\end{aligned}$$

Step 2 We now check that Z^k satisfies Walras' law. Indeed, for all $p \in \partial B \cap \mathbf{A}Y^\circ$, let $z = \int_A [x(a) - e^k(a, x(a))] d\nu(a) - y$ with $x : A \rightarrow \mathbb{R}^L$ integrable such that $x(a) \in D^k(a, p, w^k(a, p))$ for a.e. $a \in A$, and $y \in S^k(p)$. From the definition of the correspondence D^k , one has, for a.e. $a \in A$, and every $p \in \partial B \cap \mathbf{A}Y^\circ$ (hence $\|p\| = 1$)

$$p \cdot x(a) \leq w^k(a, p)(x(a)) = p \cdot e^k(a, x(a)) + \sum_j \theta_j(a) \pi_j^k(p) + 0.$$

Summing up these inequalities, and using the assumption that $\int_A \theta_j(a) d\nu(a) = 1$, for every j , we obtain

$$p \cdot \int_A [x(a) - e^k(a, x(a))] d\nu(a) \leq \sum_j \pi_j^k(p) \leq \max p \cdot Y^k = p \cdot y,$$

which implies that $p \cdot z \leq 0$. This shows that Z^k satisfies Walras' law. \blacksquare

5.2.2 The fixed point argument

We first recall the following theorem which is a direct consequence of Kakutani's Theorem¹¹ and gives the existence of a critical point for a correspondence which satisfies a condition on the boundary of the ball. This is the key tool to get the existence of equilibria.

Theorem 5.5 *Let Z be an upper semicontinuous correspondence from $B \cap \mathbb{R}_+^L$ to \mathbb{R}^L with nonempty, convex, compact values, satisfying*

Walras' law *for all $p \in \partial B \cap \mathbb{R}_+^L$, $\sup p \cdot Z(p) \leq 0$.*

Then there exists $p^ \in B \cap \mathbb{R}_+^L$ such that $Z(p^*) \cap -\mathbb{R}_+^L \neq \emptyset$.*

The proof of the result is left to the reader.

In view of Theorem 3.1, the assumptions of the above fixed-point theorem are by $Z := Z^k$, the excess demand correspondence of the economy \mathcal{E}^k . Consequently, from the above fixed point theorem, for k large enough, there exists $p^k \in B \cap \mathbf{A}Y^\circ$ such that $Z^k(p^k) \cap -\mathbf{A}Y \neq \emptyset$. In other words, there exists a sequence $((x^k(\cdot)), (y^k), (z^k), (p^k)) \subset \mathcal{X} \times Y^k \times -\mathbf{A}Y \times [B \cap \mathbf{A}Y^\circ]$ such that:

$$\text{for a.e. } a \in A, x^k(a) \in D^k(a, p^k, w^k(a, p^k)); \quad (9)$$

$$y^k \in S^k(p^k); \quad (10)$$

$$\int_A x^k(a) d\nu(a) - \int_A e^k(a, x^k(a)) d\nu(a) - y^k = z^k. \quad (11)$$

¹¹One easily deduces Kakutani's Theorem from Theorem 5.5. Indeed, let F be an upper semicontinuous correspondence from B to B with nonempty, convex, compact values. Let us now consider the correspondence Z defined by $Z(x) = F(x) - \{x\}$. If $x \in \partial B$, then for all $z \in Z(x)$, there exists $y \in F(x)$ such that $z = y - x$. Consequently, $x \cdot z = x \cdot y - x \cdot x = x \cdot y - r^2$. Since $y \in F(x) \subset B$, one has $|x \cdot y| \leq \|x\| \|y\| \leq r^2$. Thus, $x \cdot z \leq 0$. Consequently, Z satisfies the assumptions of Theorem 3.1 which implies that there exists $x^* \in B$ such that $0 \in Z(x^*)$. Clearly, x^* is a fixed point of F . \blacksquare

5.2.3 The limit argument

We now prove that the sequence $((\int_A e^k(a, x^k(a))d\nu(a), \int_A x^k(a)d\nu(a), y^k, z^k))$ is bounded.

We first notice that, if $x : A \rightarrow \mathbb{R}^L$ is a consumption allocation (hence $x(a) \in X(a)$ for a.e. $a \in A$) one has¹²

$$\text{for a.e. } a \in A, \quad e(a, x(a)) \leq cx(a) + \bar{e}(a)1_L - 2c\underline{x}(a), \quad (12)$$

where $\bar{e}(a)$ is defined in Assumption **E**, and $1_L = (1, \dots, 1) \in \mathbb{R}^L$.

We let $\bar{e} := \int_A \bar{e}(a)d\nu(a)1_L$ and $\underline{x} := \int_A \underline{x}(a)d\nu(a)$ and from the definition of e^k (Assertions (7) and (8)) and Assertions (11) and (12) we deduce that, for k large enough,

$$\begin{aligned} - \int_A e^k(a, x(a))d\nu(a) + c \int_A x(a)d\nu(a) + \bar{e} - 2c\underline{x} &\geq \alpha^k := \int_A e(a, x(a)) - e^k(a, x(a))d\nu(a) \\ &\in \bar{B}(0, 1) + \mathbb{R}_+^L \end{aligned}$$

The last assertion being a consequence of the fact that $\lim_{k \rightarrow \infty} \alpha^k = 0$ which is a direct consequence of Lebesgue's theorem and Assertions (7) and (8). Consequently, the sequence $((\int_A e^k(a, x^k(a))d\nu(a), \int_A x^k(a)d\nu(a), y^k, z^k))$ belongs to the set

$$M := \{(u, x, y, z) \in \mathbb{R}^L \times [\underline{x} + \mathbb{R}_+^L] \times Y \times -\mathbf{A}Y \mid -u + cx + \bar{e} - 2c\underline{x} \in \bar{B}(0, 1) + \mathbb{R}_+^L \text{ and } x = u + y + z\},$$

which is clearly convex. Furthermore it is bounded from the following Claim.

Claim 5.3 *The set M is bounded.*

Proof. Indeed, by a standard argument [cf. Debreu (1959)], one shows that $\mathbf{A}M \subset \{(u, x, y, z) \in \mathbb{R}^L \times \mathbb{R}_+^L \times \mathbf{A}Y \times -\mathbf{A}Y \mid u \leq cx \text{ and } x = u + y + z\}$. This, together with the fact that $0 \leq c < 1$ (by Assumption **E**) imply that $y \geq (1 - c)x - z$, hence $y \in \mathbb{R}_+^L \cap \mathbf{A}Y = \{0\}$ (by Assumption **P**). But $z \geq (1 - c)x$ implies that $x = z = 0$. Thus $u = 0$. Hence, $\mathbf{A}M = \{0\}$, which implies that the set M is bounded. ■

Consequently, without any loss of generality, we can assume that the bounded sequence $((\int_A e^k(a, x^k(a))d\nu(a), \int_A x^k(a)d\nu(a), (y^k), (z^k), p^k)) \subset M \times [B \cap \mathbf{A}Y^\circ]$ (a closed set) converges to some element $(e^*, x^*, y^*, z^*, p^*) \in M \times [B \cap \mathbf{A}Y^\circ]$.

Applying Fatou's Lemma (Theorem B in the Appendix), and noticing that, if $x : A \rightarrow \mathbb{R}^L$ is a consumption allocation (hence $x(a) \in X(a)$ for a.e. $a \in A$) one has¹³

$$\text{for a.e. } a \in A, \quad (x(a), x(a) - e(a, x(a))) \geq (\underline{x}(a), -\bar{e}(a) + 2\underline{x}(a)),$$

¹²Indeed, for every $h \in L$, recalling that $c \in [0, 1[$, one gets from Assumption **E** that

$$\begin{aligned} e_h(a, x(a)) &\leq \bar{e}(a) + c|x_h(a)| \\ &\leq cx_h(a) + \bar{e}(a) + c[|x_h(a)| - x_h(a)] \leq cx_h(a) + \bar{e}(a) - 2c\underline{x}_h(a). \end{aligned}$$

¹³where $\bar{e}(a), \underline{x}(a)$ are defined in Assumptions **C** and **E**. Indeed, the fact that $x(a) \geq \underline{x}(a)$ is a consequence of the fact that $X(a)$ is integrably bounded below (Assumption **C**). We now prove the second inequality. Indeed, for every $h \in L$, recalling that $c \in [0, 1[$, one gets from Assumption **E** that

$$e_h(a, x(a)) \leq \bar{e}(a) + c|x_h(a)| \leq x_h(a) + \bar{e}(a) + c|x_h(a)| - x_h(a),$$

and, one checks that, if $x_h(a) \geq 0$ one has $c|x_h(a)| - x_h(a) = (c - 1)x_h(a) \leq (c - 1)\underline{x}_h(a) \leq -2\underline{x}_h(a)$, and if $x_h(a) \leq 0$ one has $c|x_h(a)| - x_h(a) = -(c + 1)x_h(a) \leq -2\underline{x}_h(a)$. ■

there exist two integrable mappings $x^*(\cdot) : A \rightarrow \mathbb{R}^L$, and $\varphi^*(\cdot) : A \rightarrow \mathbb{R}^L$ such that

$$z^{**} := \int_A x^*(a) d\nu(a) - \lim_k \int_A x^k(a) d\nu(a) \in -\mathbb{R}_+^L \quad (13)$$

$$z^{***} := \int_A \varphi^*(a) d\nu(a) - \lim_k \int_A (x^k(a) - e^k(a, x^k(a))) d\nu(a) \in -\mathbb{R}_+^L. \quad (14)$$

for a.e. $a \in A$, $(x^*(a), \varphi^*(a))$ is adherent to the sequence $((x^k(a), (x^k(a) - e^k(a, x^k(a))))$. (15)

From Assertions (9), (10) and (11) we deduce that

$$\int_A \varphi^*(a) d\nu(a) = y^* + z^* + z^{***} \quad (16)$$

From above, and the Free Disposal Assumption (in \mathbf{P}), one gets

$$y^* + z^* + z^{***} \in Y - \mathbb{R}_+^L \subset Y = \sum_j Y_j.$$

Consequently, there exists $y_j^* \in Y_j$ ($j \in J$) such that $y^* + z^* + z^{***} = \sum_j y_j^*$,

Finally, from Assertion (15), and the continuity of the mapping $x \rightarrow e(a, x)$ one deduces that

$$\text{for a.e. } a \in A, \varphi^*(a) = x^*(a) - e(a, x^*(a)).$$

Consequently, from above, the market clearing condition holds, i.e.

$$\int_A x^*(a) d\nu(a) = \int_A e(a, x^*(a)) d\nu(a) + \sum_j y_j^*.$$

The following claims will show that $(x^*(\cdot), (y_j^*), p^*)$ is a quasi-equilibrium of \mathcal{E} .

Claim 5.4 (i) $\sup p^* \cdot Y = p^* \cdot y^*$;

(ii) $\liminf_k \pi_j^k(p^k) = \sup p^* \cdot Y_j$ and is finite.

Proof. We first show that

$$\sup p^* \cdot Y_j \leq \liminf_k \pi_j^k(p^k) \text{ for every } j \quad (17)$$

Indeed, let $y_j \in Y_j$, then for k large enough, $y_j \in Y_j^k$, hence $p^k \cdot y_j \leq \pi_j^k(p^k) := \sup p^k \cdot Y_j^k$. Taking the limit we get $p^* \cdot y_j = \liminf_k p^k \cdot y_j \leq \liminf_k \pi_j^k(p^k)$. Taking the supremum over Y_j , we get (17).

From (17), and the fact that $p^k \cdot y^k = \sup p^k \cdot Y^k$ (by Assertion (10)) we get

$$\begin{aligned} \sup p^* \cdot Y &= \sum_j \sup p^* \cdot Y_j \\ &\leq \sum_j \liminf_k \pi_j^k(p^k) \leq \liminf_k \sum_j \pi_j^k(p^k) \\ &\leq \liminf_k p^k \cdot y^k = p^* \cdot y^* \leq \sup p^* \cdot Y. \end{aligned}$$

This ends the proof of Part (i) of the claim. Finally, Part (ii) is a consequence of (17) and the fact that we have shown above that

$$\sum_j \sup p^* \cdot Y_j = \sum_j \liminf_k \pi_j^k(p^k).$$

In the following, we let

$$w(a, p)(x) := p \cdot e(a, x) + \sum_j \theta_j(a) \sup p \cdot Y_j + (1 - \|p\|),$$

and we recall that :

$$\begin{aligned} w^k(a, p)(x) &= p \cdot e^k(a, x) + \sum_j \theta_j(a) \pi_j^k(p) + (1 - \|p\|) \\ &= p \cdot e(a, x) + \sum_j \theta_j(a) p \cdot [\hat{y}_j(a) - y_j^k(a)] + \sum_j \theta_j(a) \pi_j^k(p) + (1 - \|p\|) \end{aligned}$$

Claim 5.5 For a.e. $a \in A$,

- (i) $x^*(a) \in B^*(a) := \{x \in X(a) \mid p^* \cdot x \leq w(a, p^*)(x)\}$;
- (ii) there exists no $x \in X(a)$ such that $p^* \cdot x < w(a, p^*)(x)$ and $x^*(a) \prec_a x$.

Proof. Part (i) We first recall that, from Assertion (15), for a.e. $a \in A$, $x^*(a)$ is adherent to the sequence $(x^k(a))$. Since $X(a)$ is closed (by Assumption **C**), we deduce that $x^*(a) \in X(a)$. Furthermore, from Assertion (9), $x^k(a) \in D^k(a, p^k, w^k(a, p^k))$ hence $p^k \cdot x^k(a) \leq w^k(a, p^k)(x^k(a))$. Taking the limit, when $k \rightarrow \infty$, recalling that (p^k) converges to p^* , that the mapping $x \rightarrow e(a, x)$ is continuous, and using Claim 5.4, we get $p^* \cdot x^*(a) \leq w(a, p^*)(x^*(a))$. Hence $x^*(a) \in B^*(a)$.

We prove the second part by contraposition. Suppose there exists $x \in X(a)$ such that $p^* \cdot x < w(a, p^*)(x)$ and $x^*(a) \prec_a x$. For k large enough, one has $x \in X^k(a)$. From Claim 5.4, one deduces that $w(a, p^*)(x) = \liminf_k w^k(a, p^k)(x)$. Thus, for k large enough, we have $p^k \cdot x < w^k(a, p^k)(x)$, hence $x \in \beta^k(a, p^k, w^k(a, p^k))$. From the Continuity Assumption in **C**, the set $\{x' \in X(a) \mid x' \prec_a x\}$ is open in $X(a)$ (for its relative topology) and contains $x^*(a)$, hence for k large enough, $x^k(a) \prec_a x$. But this, together with the fact that $x \in \beta^k(a, p^k, w^k(a, p^k))$ (from above) contradicts that $x^k(a) \in D^k(a, p^k, w^k(a, p^k))$ (by the fixed-point assertion (1)).

Claim 5.6 (i) $\|p^*\| = 1$;

- (ii) for every $j \in J$ $\sup p^* \cdot Y_j = p^* \cdot y_j^*$.

Proof From Claim 5.5 and the local non-satiation of the preferences of the consumers (in Assumption **C**), we deduce that ¹⁴

$$\text{for a.e. } a \in A, \quad p^* \cdot x^*(a) = p^* \cdot e(a, x^*(a)) + \sum_j \theta_j(a) \sup p^* \cdot Y_j + (1 - \|p^*\|).$$

From the above equality, making the sum over A , and recalling that $\int_A [x^*(a) - e(a, x^*(a))] d\nu(a) - \sum_j y_j^* = 0$ with $y_j^* \in Y_j^*$ ($j \in J$) one gets

$$\begin{aligned} 0 &= p^* \cdot \int_A [x^*(a) - e(a, x^*(a))] d\nu(a) - \sum_j p^* \cdot y_j^* \\ &\geq p^* \cdot \int_A [x^*(a) - e(a, x^*(a))] d\nu(a) - \sum_j \sup p^* \cdot Y_j = (1 - \|p^*\|) \geq 0 \end{aligned}$$

¹⁴Indeed, if $p^* \cdot x^*(a) < w(a, p^*)(x^*)$, from the local non-satiation assumption, there exists a sequence $(x^n(a)) \subset X(a)$ converging to $x^*(a)$ such that, for all n , $x^* \prec_a x^n(a)$. Thus, for n large enough, one has $p^* \cdot x^n(a) < w(a, p^*)(x^n(a))$, (recalling that the mapping $x \rightarrow e(a, x)$ is continuous). From Claim 5.5, we deduce that $x^n(a) \preceq_a x^*(a)$, a contradiction with $x^* \prec_a x^n(a)$ (from above).

This shows that $\|p^*\| = 1$ and $\sum_j \sup p^* \cdot Y_j = \sum_j \sup p^* \cdot y_j^*$. But this last equality and the fact that, for every j , $\sup p^* \cdot Y_j \geq p^* \cdot y_j^*$, implies that the last inequalities are in fact equalities. This ends the proof of Claim 5.6. \blacksquare

5.3 Appendix C

The distribution wealth $w(a, p) : \mathbb{R}^L \rightarrow \mathbb{R}$ can be considered as a mapping $(a, p, x) \rightarrow w(a, p)(x)$, from $A \times \mathbb{R}^L \times \mathbb{R}^L \rightarrow \mathbb{R}$ and it is said to be a Caratheodory function if, for every $(p, x) \in \mathbb{R}^L \times \mathbb{R}^L$, the function $a \rightarrow w(a, p)(x)$ is measurable and, for every $a \in A$, the function $(x, p) \rightarrow w(a, p)(x)$ is continuous.

We recall that the individual quasi-demand and the mean quasi-demand are defined by:

$$D(a, p, w) := \{x \in B(a, p, w(a, p)) \mid \forall x' \in X(a), x' \succ_a x \Rightarrow p \cdot x' \geq w(a, p)(x')\},$$

$$D(p) := \int_A D(a, p, w(a, p)) d\nu(a),$$

and we let

$$P := \{p \in \mathbb{R}^L \mid \text{a.e } a \in A, \exists x \in X(a) p \cdot x \leq w(a, p)(x)\}.$$

Theorem 5.6 *Let $[\mathbb{R}^L, (A, \mathcal{A}, \nu), (X(a), \prec_a, e(a))_{a \in A}]$ satisfy assumptions **M** and **C**, assume in addition that the correspondence $a \rightarrow X(a)$ is integrably bounded, and let the distribution wealth $w : A \times \mathbb{R}^L \rightarrow \mathbb{R}$ be a Caratheodory function.*

Then the correspondence D , from P to \mathbb{R}^L , is u.s.c. with nonempty and convex values and takes its values in a fixed compact set, in the sense that, there exists a compact subset $K \subset \mathbb{R}^L$ such that, for every $p \in P$, $D(p) \subset K$.

Proof of Theorem. Since, for $p \in P$, $D(a, p, w(a, p)) \subset B(0, r(a))$ with $r : A \rightarrow \mathbb{R}_+$ integrable, to prove that $D(p)$ is nonempty, it is sufficient to show that $D(\cdot, p, w(\cdot, p))$ has a measurable selection. In view of Theorem A (i), we need to prove that $D(\cdot, p, w(\cdot, p))$ has non-empty values and is ν -measurable:

$$\text{for a.e. } a \in A, D(a, p, w(a, p)) \neq \emptyset,$$

$$GD = \{(a, x) \in A \times \mathbb{R}^L \mid x \in D(a, p, w(a, p))\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^L).$$

Claim 5.7 *For a.e. $a \in A$, and every $p \in P$, then $D(a, p, w(a, p)) \neq \emptyset$.*

Proof. Indeed the quasi-demand set $D(a, p, w(a, p))$ always contains the demand set

$$\Delta(a, p, w(a, p)) := \{x \in B(a, p, w(a, p)) \mid \text{there is no } x' \in B(a, p, w(a, p)), x \prec_a x'\}.$$

Noticing that, if $w \geq \min p \cdot X(a)$, the budget set $\tilde{K} := B(a, p, w(a, p))$ is nonempty and compact, the nonemptyness of $\Delta(a, p, w(a, p))$ (hence of $D(a, p, w(a, p))$) is then proved by a standard argument¹⁵. \blacksquare

¹⁵We prove it by contraposition. Suppose that, for every $x^* \in \tilde{K}$, there exists $x \in \tilde{K}$, $x^* \prec_a x$, then $\tilde{K} = \cup_{x \in \tilde{K}} V_x$, where $V_x = \{x^* \in \tilde{K} \mid x^* \prec_a x\}$ is open in \tilde{K} by the continuity assumption in **C**. Since \tilde{K}

Claim 5.8 For $p \in P$, the correspondence $D(\cdot, p, w(\cdot, p))$ is ν -measurable.

Proof. Let $p \in P$. We consider the partition (A_1, A_2) of A where $A_1 = \{a \in A \mid \beta(a, p, w(a, p)) = \emptyset\}$ and $A_2 = \{a \in A \mid \beta(a, p, w(a, p)) \neq \emptyset\}$. With Assumptions C and E , the correspondences $\beta(\cdot, p, w(\cdot, p))$ and $B(\cdot, p, w(\cdot, p))$ are measurable.

Applying theorem A (ii) to the restriction of the correspondence $\beta(\cdot, p, w(\cdot, p))$ to A_2 , we obtain the existence of a sequence of measurable functions (f_n) from A_2 to \mathbb{R}^L such that for every $a \in A_2$, $(f_n(a))$ is dense in $\beta(a, p, w(a, p))$. We define the correspondences ξ_n from A_2 to \mathbb{R}^L by the following:

$$\xi_n(a) = \{x \in B(a, p, w(a, p)) \mid f_n(a) \not\prec_a x\}$$

We claim that: $D(a, p, w(a, p)) = \bigcap_{n=1}^{\infty} \xi_n(a)$ for every $a \in A_2$.

Clearly, for every n , $D(a, p, w(a, p)) \subset \xi_n(a)$. Conversely, let $x \in \bigcap_{n=1}^{\infty} \xi_n(a)$ and $x \notin D(a, p, w(a, p))$. Then, the set $U = P_a(x) \cap B(a, p, w(a, p))$ is nonempty and from Assumption C, U is open relative to $B(a, p, w(a, p))$. Since the sequence $(f_n(a))$ is dense in $\beta(a, p, w(a, p))$, we deduce that for some n_0 , $f_{n_0}(a) \in U$. This means that $f_{n_0}(a) \in P_a(x)$ which is in contradiction with $x \in \xi_{n_0}(a)$.

Thus, we have $GD(\cdot, p, w(\cdot, p)) = (\bigcap_{n=1}^{\infty} G\xi_n) \cup (GB(\cdot, p, w(\cdot, p)) \cap (A_1 \times \mathbb{R}^L))$. In order to finish the proof, it suffices to show that $G\xi_n$ is measurable for every n . But this is a direct consequence of the fact that the preference distribution, the function f_n and the budget sets are measurable. \blacksquare

The correspondence D takes its values in a fixed compact since $D(p) \subset B(0, \int_A r(a) d\nu(a))$ for every $p \in P$.

Claim 5.9 For all $p \in P$, $D(p)$ is convex.

Proof: To show that $I = \int_A D(a, p, w(a, p)) d\nu(a)$ is convex, we first remark that if $f : A \rightarrow \mathbb{R}$ is an in

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an atomless measure if ν is atomless.

Let ς_1, ς_2 in I and $0 < \lambda < 1$. There exist $x^1(\cdot), x^2(\cdot)$ in $S^1_{D(\cdot, p, w(\cdot, p))}$ such that $\varsigma_1 = \int_C x^1(a) d\nu(a)$ and $\varsigma_2 = \int_C x^2(a) d\nu(a)$. Let

$$\Xi = \left\{ \left(\int_E x^1(a) d\nu(a), \int_E x^2(a) d\nu(a) \right), E \in \mathcal{A}, E \subset C \right\}.$$

From the above remark and the Lyapounov theorem, Ξ is a convex subset of \mathbb{R}^{2L} . Since $(0, 0)$ and $(\varsigma_1, \varsigma_2)$ are elements of Ξ , there exists $E \subset C, \mathcal{E} \in \mathcal{A}$ such that

$$(\lambda\varsigma_1, \lambda\varsigma_2) = \left(\int_E x^1(a) d\nu(a), \int_E x^2(a) d\nu(a) \right).$$

is compact, there exists a finite subset $\{x_i \mid i \in N\} \subset \tilde{K}$ (with $N := \{1, \dots, n\}$) such that $\tilde{K} = \bigcup_{i \in N} V_{x_i}$. We now claim that the finite set $\{x_i \mid i \in N\}$ admits a maximal element for \prec_a , that is, there exists $i \in N$ such that for every $j \in N$, not $x_i \prec_a x_j$. Indeed, if such a maximal element does not exist, for every $i \in N$, there exists $\sigma(i) \in N$ such that $x_i \prec_a x_{\sigma(i)}$. The mappings $\sigma : N \rightarrow N$ clearly admits a cycle, that is, for some i and some integer k one has $i = \sigma^k(i)$ (the composition of σ with itself k times). The transitivity of \prec_a implies that $x_i \prec_a x_{\sigma^k(i)} = x_i$ which contradicts the irreflexivity of \prec_a .

We now end the proof by considering a maximal element x_i of \prec_a in the set $\{x_i \mid i \in N\}$. The element $x_i \in \tilde{K}$ belongs to one of the sets V_{x_j} ($j \in N$) since it is a covering of \tilde{K} , that is, $x_i \prec_a x_j$ for some $j \in J$. But this is in contradiction with the maximality of x_i . This ends the proof of the assertion.

We obtain that $\lambda_{\varsigma_1} + (1 - \lambda)\varsigma_2 = \int_C x(a) d\nu(a)$, where $x(\cdot)$ is defined by $x(a) = x^1(a)$ if $a \in E$ and $x(a) = x^2(a)$ otherwise. Thus $\lambda_{\varsigma_1} + (1 - \lambda)\varsigma_2 \in I$. ■

In order to show the closeness of the graph of D , we first show that $D(a, \cdot, w(a, \cdot))$ has a closed graph. Let $(q^\nu, z^\nu) \in GD(a, \cdot, w(a, \cdot))$ a sequence converging to (q, z) such that $q \in P$. Clearly, $z \in B(a, q, w(a, q))$. Hence, if $z \notin D(a, q, w(a, q))$, then there exists $\zeta \in P_a(z)$ such that $q \cdot \zeta < w(a, q)(\zeta)$. Since $\{z' \in X(a) \mid z' \prec_a \zeta\}$ is open in $X(a)$ and since $w(a, \cdot)(\zeta)$ is continuous, one has for ν large enough $\zeta \in P_a(z^\nu)$ and $q^\nu \cdot \zeta < w(a, q^\nu)(\zeta)$. But this contradicts the fact that $z^\nu \in D(a, q^\nu, w(a, q^\nu))$.

Now consider a sequence $(p_n, x_n) \in GD$ converging to (p, x) . From the definition of D , for every n , there exists $x_n(\cdot) \in S_D^1(p_n)$ such that $x_n = \int_A x_n(a) d\nu(a)$. Applying theorem C, there exists an integrable function $x(\cdot)$ from A to \mathbb{R}^L such that $x = \int_A x(a) d\nu(a)$ and for a.e. a , $x(a)$ is adherent to $(x_n(a))$. Now, since for each fixed a , the correspondence $D(a, p, w(a, \cdot))$ has a closed graph, we have that $x(a) \in D(a, p, w(a, p))$ and thus $x \in D(p)$. ■

Proof of Lemma 3.1. From the Survival Assumption **S**, one deduces that

$$M(a) := \{(x, (y_j), \alpha) \in X(a) \times \prod_j \overline{\text{co}} Y_j \times -\mathbf{A}Y \mid x = e(a, x) + \sum_{j \in J} \theta_j(a) y_j + \alpha\} \neq \emptyset.$$

Furthermore, from the Measurability Assumption **M**, the correspondence $a \rightarrow M(a)$ is easily shown to be measurable. Hence, from Aumann's theorem (Theorem A in the Appendix), the correspondence admits a measurable selection $(\hat{x}, (\hat{y}_j), \hat{q}) : A \rightarrow \mathbb{R}^L \times (\mathbb{R}^L)^J \times \mathbb{R}^L$.

We now consider the measurable mapping $f : A \rightarrow \mathbb{R}^K$, which is defined as follows. For every a , $f(a)$ is the vector with coordinates $\{\hat{x}_h(a), \hat{y}_{jh}(a), \hat{q}_h(a), \underline{x}_h(a), e_h(a), \theta_j(a) \mid j \in J, h \in L\}$ and the dimension K of \mathbb{R}^K can be found by simple calculus. We can write $f = (f_1, f_2)$ with $f_1 = (f_k) : A \rightarrow \mathbb{R}^{K_1}$, $f_2 = (f_k) : A \rightarrow \mathbb{R}^{K_2}$ ($K = K_1 + K_2$) such that for every $k \in K_2$, f_k is integrable.

From above and Assumption **M**, the mapping f is clearly measurable, and one shows that there exists a sequence (f^n) of measurable step functions, with values in \mathbb{R}^K , satisfying the following four properties :

- (i) for a.e. $a \in A$, for every n , $f^n(a) = f(a^n)$ for some $a^n \in A$,
- (ii) for a.e. $a \in A$, $f(a) = \lim_{n \rightarrow \infty} f^n(a)$,
- (iii) $\int_A f_k(a) d\nu(a) = \lim_{n \rightarrow \infty} \int_A f_k^n(a) d\nu(a)$ for every $k \in K_2$,
- (iv) for every ν , for a.e. $a \in A$, $\|f_2^n(a)\| \leq 1 + \|f_2(a)\|^{16}$.

We have shown above that, for almost every $a \in A$, $\hat{x}(a) = \hat{e}(a, \hat{x}(a)) + \sum_j \theta_j(a) \hat{y}_j(a) + \hat{q}(a)$, which clearly implies that $x^n(a) = e^n(a, x^n(a)) + \sum_j \theta_j^n(a) y_j^n(a) + q^n(a)$. Taking the integral on A we get

$$\int_A x^n(a) d\nu(a) = \int_A e^n(a, x^n(a)) d\nu(a) + \bar{y}^n + \int_A q^n(a) d\nu(a),$$

¹⁶For the precise definition, we refer to Martins Da Rocha (2001). To every integer ν , let $\{B_i^n \mid i \in I_\nu\}$ be a finite partition of R^K such that one of the B_i^n is the complementary (in R^k) of the ball $B(0, 2^{-n})$, and each of the other B_i^n is a Borelian set whose diameter is less than 2^{-n} . Then choose $a_i^n \in A_i^n := f^{-1}(B_i^n)$ (a measurable subset of A) and define the measurable step function $f^n := \sum_{i \in I_\nu} f(a_i^n) \chi_{A_i^n}$, where $\chi_{A_i^n}$ is the characteristic function of A_i^n . Then one easily checks that the first two properties (i) and (ii) are satisfied. To prove the third one, we impose in the above choice of a_i^n the additional assumption that $\|f_2(a_i^n)\| \leq 1 + \inf\{\|f_2(a)\| \mid a \in A_i^n\}$. Consequently, for a.e. $a \in A$, $\|f_2^n(a)\| \leq 1 + \|f_2(a)\|$, an integrable function. The third property is then a consequence of Lebesgue's dominated convergence theorem.

where

$$\begin{aligned}
\bar{y}^n &:= \int_A \sum_{j \in J} \theta_j^n(a) y_j^n(a) = \sum_{i \in I^n} \sum_{j \in J} \nu(A_i^n) \theta_j(a_i^n) \hat{y}_j(a_i^n) \\
&= \sum_{j \in J} \sum_{i \in I^n} \nu(A_i^n) \theta_j(a_i^n) \hat{y}_j(a_i^n) \\
&= \left[\sum_{j \in J} (1 - \bar{\theta}_j^n) y_j + \bar{\theta}_j^n \sum_{i \in I^n} [(1/\bar{\theta}_j^n) \nu(A_i^n) \theta_j(a_i^n)] \hat{y}_j(a_i^n) \right] - \sum_{j \in J} (1 - \bar{\theta}_j^n) y_j
\end{aligned}$$

where y_j is a given element in Y_j and $\bar{\theta}_j^n := \sum_{i \in A_i^n} \nu(A_i^n) \theta_j(a_i^n) = \int_A \theta_j^n d\nu(a)$ for every $j \in J$. Thus $\bar{\theta}_j^n \rightarrow \int_A \theta_j = 1$ when $n \rightarrow \infty$. Then for n large enough, one has

$$\sum_{j \in J} (1 - \bar{\theta}_j^n) y_j + \bar{\theta}_j^n \sum_{i \in I^n} [(1/\bar{\theta}_j^n) \nu(A_i^n) \theta_j(a_i^n)] \hat{y}_j(a_i^n) \in Y + B(0, 1/2),$$

and then, $\bar{y}^n \in Y + B(0, 1)$. We also recall that $\int_A e^n(a) \rightarrow \int_A e(a) := e$.

We now prove that the sequence $(\int_A e^n(a, x^n(a)) d\nu(a), \int_A x^n(a) d\nu(a), \bar{y}^n, \int_A q^k(a) d\nu(a))$ is bounded. Since equation (6) is still valid, one has with Assumptions **C** and **E**,

$$\text{for a.e. } a \in A, \quad e^n(a, x^n(a)) \leq cx^n(a) + \bar{e}^n(a) 1_L - 2c\underline{x}^n(a), \quad (18)$$

By definition of f_2 and by property (iv), one obtains

$$\begin{aligned}
\left\| \int_A (\bar{e}^n(a) 1_L - 2c\underline{x}^n(a)) d\nu(a) \right\| &\leq \int_A (|\bar{e}^n(a)| \sqrt{L} + 2c \|\underline{x}^n(a)\|) d\nu(a) \\
&\leq (\sqrt{L} + 2c) \int_A (1 + \|f_2(a)\|) d\nu(a)
\end{aligned}$$

We let $T := \int_A (1 + \|f_2(a)\|) d\nu(a) 1_L$. Then, we deduce that,

$$-\int_A e^n(a, x^n(a)) d\nu(a) + c \int_A x^n(a) d\nu(a) + (\sqrt{L} + 2c)T \geq 0$$

Moreover, for every ν ,

$$-T \leq -\int_A \|\underline{x}^n(a)\| d\nu(a) 1_L \leq \int_A \underline{x}^n(a) d\nu(a) \leq \int_A x^n(a) d\nu(a)$$

Consequently, for ν large enough, the sequence $(\int_A e^n(a, x^n(a)) d\nu(a), \int_A x^n(a) d\nu(a), \bar{y}^n, \int_A q^k(a) d\nu(a))$ belongs to the set

$$M' := \{(u, x, y, z) \in \mathbf{R}^L \times [-T + \mathbf{R}_+^L] \times (Y + B(0, 1)) \times -\mathbf{A}Y \mid -u + cx + (\sqrt{L} + 2c)T \in \mathbf{R}_+^L \text{ and } x = u + y + z\},$$

The set M' is bounded (see Claim 5.3). Consequently, without any loss of generality, we can assume that there exists some element x^* [resp. q^*] such that $x^* = \lim_{n \rightarrow \infty} \int_A x^n(a) d\nu(a)$ [resp. $q^* = \lim_{n \rightarrow \infty} \int_A q^n(a) d\nu(a)$].

The integrability of $\hat{x}(\cdot)$ and $\hat{q}(\cdot)$ are clearly a consequence of Fatou's lemma. \blacksquare

End of the Proof of Lemma 3.1. under the additional assumption that there exists $k > 0$ such that

$$\tilde{M}(a) := \{(x, (y_j), \alpha) \in X(a) \times \prod_j \overline{\text{co}} Y_j \times -\mathbf{A}Y \mid x = e(a, x) + \sum_{j \in J} \theta_j(a) y_j + \alpha \leq k\} \neq \emptyset.$$

(hence in particular under the Survival Assumption \mathbf{S}_2 made by Hildenbrand (1970)).

The existence of the measurable mappings $\hat{x}(\cdot)$, $y_j(j \in J)$ and $q(\cdot)$ is shown as above and we can now impose that the measurable mapping $a \rightarrow \hat{y}(a) := \sum_{j \in J} \theta_j(a) \hat{y}_j(a)$ is bounded by k from above. Hence it is integrable. The fact that the measurable mappings $\hat{x}(\cdot)$ and $q(\cdot)$ are integrable is then a consequence of the integrability of the mappings $\underline{x}(\cdot)$, $e(\cdot)$ (from the Measurability Assumption \mathbf{M} , the integrability of $\hat{y}(\cdot)$ (from above) and the following lemma, taking $D = \mathbf{A}Y$ and $C = \mathbb{R}_+^L$.

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