Countervailing incentives in allocation mechanisms with type-dependent externalities*

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Abstract

I study an allocation mechanism of a single item in the presence of type-dependent externalities between bidders. The type-dependency introduces countervailing incentives and the allocation sometimes requires that types in an interior subset obtain their reservation utility. Furthermore, truth-telling requires the ex-ante allocation to satisfy a non-trivial monotonicity condition. I show that this problem is technically different from the one analyzed in related single agent settings. I provide a procedure to characterize the main properties of the ex-post allocation. Typically, the solution does not entail a single reserve price. More specifically, each agent faces an allocation rule contingent on whether he and his rival's types fall below, in or above the (endogenously determined) subset of types that obtain their reservation utility.

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1 Introduction

Consider two firms on a market competing for the acquisition of a license. There are three possible outcomes for each firm. The license may not be allocated at all, in which case both firms get the status quo payoff; the firm wins the license and its payoff increases; or the rival wins and the payoff of the firm decreases. Overall, the winner induces a *negative externality* on the loser and the designer of the allocation mechanism should reflect this in her pricing strategy. The effect of negative externalities on prices has been studied in the literature starting with Katz and Shapiro [15] and Kamien, Oren and Tauman [14]. As shown in these early studies, the asymmetric ex-post interaction between firms allows the seller to extract some payments even from the firms that do not obtain the license. The analysis has been generalized by Jehiel et al. [10],[11] with the characterization of the optimal allocation mechanism when firms have private information about their valuation for the good, and extended by various authors. In most of this literature, externalities are taken to be unrelated to the agent's valuation of the good.[1] The optimal mechanism exhibits the same qualitative properties as the standard optimal auction without externalities.[2] The only difference is that the seller can extract payments from non acquirers (which can be implemented via an entry fee), and this extra payment as well as the reserve price increase in the externality.

Yet, constructing underlying games and markets in which valuations and externalities are unrelated can be quite heroic. To see this, consider again the licensing example. The valuation for the license and the externality suffered when a rival obtains it depends on the intrinsic ability of the firm to both exploit the innovation and sustain competition from a rival licensee. This suggests that valuation for the good, suffered externality and imposed externality are linked through underlying variables, and that the specific structure of the industry will determine the sign and amount of the correlation. Carrillo [5] was the first to analyze that possibility and to characterize the optimal contract with multiple agents and valuation-dependent externalities.[3] In that case, the reservation utility of the agent becomes valuation-dependent (the agent suffers the externality if he does not show up to bid and the rival is allocated the good). This extra feature gives rise to countervailing incentives. At equilibrium, the utility of a participating firm may bind at the top, at the bottom or for interior valuations on the reservation utility. Carrillo [5] restricts the attention to functional forms that allow only for the two first situations. Formally, the reservation utility of the

[1] In Jehiel et al. [10], the size of the externality suffered by an agent is unknown to him but depends only on the identity of the winner and not on his valuation. In Jehiel et al. [11], the externality suffered by an agent is private information and depends on the identity of the winner but it is not correlated to his valuation. Other mechanism design problems consider identity-dependent externalities (e.g. Aseff and Chade [1] for the case of multi-unit auctions). Most of these papers will be discussed in further detail.

[2] See Myerson [21] for the seminal paper on optimal auctions and Engelbrecht-Wiggans [7], McAfee and McMillan [20] and Klemperer [16] for surveys.

[3] Valuation-dependent externalities have also been studied by Jehiel and Moldovanu [12]. However, the authors restrict to particular procedures and the analysis is not relevant to the optimal allocation problem.

agent is flat up to a given cutoff valuation and steeply increasing in the valuation after the cutoff. The author analyzes three cases. In case 1, the equilibrium utility binds at the bottom (on the flat part of the reservation utility). In case 2, the equilibrium utility binds at the top (on the steep part of the reservation utility); and in case 3, the equilibrium utility binds at the bottom and at the top. Closely related, Figueroa and Skreta [8] studies the role of optimal threats when several outside options coexist. Formally, the authors consider a model where the outside option can be either flat or steep and address three cases. In case A, the reservation utility is flat and the equilibrium utility binds at the bottom. This case is the same as case 1 in Carrillo [5]. In case B, the reservation utility is steep, and the equilibrium utility binds at the top. This case is the same as case 2 in Carrillo [5]. The novelty arises in case C, when the reservation utility can be either flat or steep. The authors show that the seller must randomize between the two possible reservation utilities, and the equilibrium utility may bind for interior types. They provide the solution in a numerical example in which only one bidder has private information, reducing the problem to a single agent mechanism design problem.

The objective of this paper is to study the problem faced by the seller when agents have a single but type-dependent outside option and *both* remain privately informed. Furthermore, we want to concentrate on the case where the equilibrium utility binds for interior types. We therefore consider the case where the reservation utility is weakly increasing (a case in-between flat and steep) in the valuation.[4]

This problem turns out to be challenging to solve. I first characterize two limit cases, one in which the equilibrium utility binds at the top and one in which it binds at the bottom. The first corresponds to the extreme case where the reservation utility becomes steep, while the second corresponds to the extreme case where it becomes flat. In both cases, the seller specifies a type-dependent reserve price and allocates the good to the agent with the highest valuation provided the reserve price is met (Proposition 1). This result is useful as it establishes there is no discontinuity at the limit.

Second, I characterize the main properties of the optimal ex-post allocation mechanism in the general case. The mechanism must be such that the agent receives no rent for valuations at which the equilibrium utility binds, hereafter called the 'set of binding types'. Also, the mechanism requires some form of monotonicity of the ex-ante allocation to make sure truth-telling is a global maximum. These two requirements act as constraints on the optimization program of the seller.

[4]We shall mention valuation-dependent positive externalities have also been studied in Brocas [4] and Chen and Potipiti [6]. In such studies, the outside option is not type-dependent but countervailing incentives may arise through a tension in the incentives to report. Similar countervailing incentives may also arise under negative valuation-dependent externalities as shown in Brocas [3]. For the case of positive externalities, such countervailing incentives can be quite problematic. Chen and Potipiti [6] study this case.

I show in Lemma 2, that the solution of the unconstrained program violates both constraints. This is the case because the allocation to the right of each bound of the set of binding types differs from the allocation to the left. There are two reasons for this. First, there is a tension between the rents that must be given below the set of binding types and above it. Indeed, giving one extra unit of rent to a given type requires to also increase the rents of other types, which are in different proportions if types lie below or above the set of binding types. Second, there is also a tension between the constraints and the interest of the seller on the set of binding types. Indeed, those types must receive no rent but the efficient solution requires to leave rents under asymmetric information.

I develop a procedure to restore both constraints. The procedure consists in designing a mechanism as close as possible to the optimal unconstrained mechanism but such that both constraints are satisfied at the same time. This procedure is used to characterize the main properties of the optimal ex-post allocation (Proposition 2). The ex-post allocation lies in between the two limit cases. The solution obtained in the limit case where the equilibrium utility binds at the top prevails when both types lie below the set of binding types. Similarly, the solution obtained in the limit case where the equilibrium utility binds at the bottom prevails when both types lie above the set of binding types. Everywhere else, the seller gives priority to the allocations associated with the highest possible surplus as long as the constraints require to allocate with positive probability. The seller sometimes does not allocate the good while it would be profitable, to make sure the constraints are satisfied.

Contrary to standard allocation problems, interestingly, the solution does not entail a single reserve price. Rather, each agent faces an allocation rule contingent on whether he and his rival's types fall below, in or above the set of binding types. Moreover, at equilibrium, the agent with the highest type does not necessarily obtain the good. Last, the seller will resort to a stochastic mechanism for some pairs of types. The equilibrium ex-post allocation has therefore novel properties.

The paper is organized as follows. Section 2 presents the model. In section 3, I characterize the main properties of the optimal mechanism. Section 4 concludes. All proofs are relegated to the Appendix.

2 The model

An indivisible good is offered for sale among two risk-neutral potential buyers 1 and 2, indexed by i and j. Buyer i (he) derives utility v_i when he gets the good. We will call v_i, his "willingness to pay", "type" or "valuation" and $v = (v_i, v_j)$ the vector of valuations of both agents. Each v_i is drawn independently from a known distribution with c.d.f. $F(v_i)$ and density $f(v_i)$. $F(\cdot)$ is strictly increasing and continuously differentiable on the interval $[\underline{v}, \overline{v}]$, with $0 < \underline{v} < \overline{v}$. Also, $F(\underline{v}) = 0$ and $F(\overline{v}) = 1$. The valuation for the good of the seller (she) is zero. As usual in mechanism design problems, we restrict our analysis to distributions that satisfy the monotone hazard rate property.

Assumption 1 *$\frac{F(v_i)}{f(v_i)}$ is increasing in v_i and $\frac{1-F(v_i)}{f(v_i)}$ is decreasing in v_i.*

Bidder i suffers an externality $-\alpha_i(v)$ when bidder $j \neq i$ gets the good. This externality is always negative and is a function of both the valuation of the agent who gets the good (v_j) and that of the agent who suffers from not getting it (v_i). In order to keep the analysis as tractable as possible, we shall restrict to the following linear form:

Assumption 2 *$\alpha_i(v) = \alpha_a v_i + \alpha_b v_j + \gamma$ where α_a, α_b and γ are such that $\alpha_i(v) > 0 \quad \forall v_i, v_j$*

Under asymmetric information, the reservation utility of each agent is given by the outcome of the auction if he does not show up, so it is mechanism dependent. In the presence of negative externalities, each agent wants not only to acquire the good, but also to avoid the externality that results when the rival gets it. Then, he is prone to pay and enter the auction if participating buys him a chance to prevent the other agent from acquiring the good. This generates rents that can be captured by the seller. Assuming that the seller can commit to any mechanism proposed to the buyers, in the optimal mechanism, the seller commits to give the good for free to one agent if the other does not participate.[5] In order to achieve entry of both bidders, the seller threatens them with their worst outcome if they do not participate, which is simply to suffer the externality with probability one. This reduces the lower bound of their payoff and therefore increases the rents that can be extracted from them. Moreover, the threat is costless, since it occurs only out-of-equilibrium. Naturally, this heavily relies on the commitment assumption.[6] In the rest of the analysis, we only need to make sure that participating guarantees bidders at least as much utility as their worst outside option.

An auction mechanism A consists of message spaces $\{M_1, M_2\}$ for the two buyers and a pair of winning probabilities and payments $\{x_i(\cdot), T_i(\cdot)\}$. Agents bid simultaneously by announcing their willingness to pay $(\tilde{m}_1, \tilde{m}_2) = \tilde{m}$. The revenue of the seller is $\sum_{i=1}^{2} T_i(\tilde{m})$ and the utility of agent i is $v_i x_i(\tilde{m}) - \alpha_i(v) x_j(\tilde{m}) - T_i(\tilde{m})$. The revelation principle implies that any Bayesian equilibrium $(\tilde{m}_1^\star(\cdot), \tilde{m}_2^\star(\cdot))$ for an auction consisting of $\{M_1, M_2, x_i(\cdot), T_i(\cdot)\}$ can be obtained as a Bayesian equilibrium for a direct mechanism that induces truth-telling.

[5]Our analysis departs from case C analyzed in Figueroa and Skreta [8] here. In my paper, each buyer faces an unambiguous outside option and the optimal threat is trivial. This feature would disappear if the analysis were extended to more than 2 buyers suffering suitably chosen asymmetric externalities. In that case, the optimal threat would be to allocate the good to each rival with some probability.

[6]Although standard in the literature on auctions with externalities (see e.g. Carrillo [5], Jehiel et al. [10],[11] etc.) and sometimes not even discussed, this assumption is strong. If an agent does not show up, the seller will have ex-post incentives to conduct the auction with only one bidder rather than give him the good for free. In Brocas [2], we show that when this assumption is relaxed, there is a coordination problem in the behavior of agents that gives rise to multiple equilibria.

A direct mechanism is characterized by the interim probability that agent i gets the good, $X_i(v_1, v_2) = x_i(\tilde{m}_1^\star(\cdot), \tilde{m}_2^\star(\cdot))$ and the associated transfers $t_i(v_1, v_2) = T_i(\tilde{m}_1^\star(\cdot), \tilde{m}_2^\star(\cdot))$.

Let $u_i(v_i, v_i')$ be the expected *utility* of bidder i when he participates in the auction, his valuation is v_i, he announces v_i', and the other bidder discloses his true valuation v_j. We denote by $u_i(v_i) \equiv u_i(v_i, v_i)$ his expected utility under truthful revelation. We have:

$$u_i(v_i, v_i') = E_{v_j}\Big[v_i X_i(v_i', v_j) - \alpha_i(v_i, v_j) X_j(v_i', v_j) - t_i(v_i', v_j)\Big] \tag{1}$$

Also, let $w_i(v_i)$ be the *reservation utility* of agent i, that is, his expected payoff when he does not participate in the auction, in which case the rival gets the good for sure. It is:

$$w_i(v_i) = -\alpha_a v_i - \alpha_b \int_{\underline{v}}^{\overline{v}} v_j\, f(v_j) dv_j - \gamma \tag{2}$$

Last, we call *informational rents* the difference between the utility of an agent who participates in the auction and his reservation utility, that is $u_i(v_i) - w_i(v_i)$. The mechanism must satisfy the following three kinds of constraints.

First, individual-rationality. Agents must prefer to participate in the auction rather than not:

$$u_i(v_i) \geqslant w_i(v_i) \quad \forall\, i, v_i$$

Second, incentive-compatibility. Bidders must be better-off by disclosing their true valuation:

$$u_i(v_i) \geqslant u_i(v_i, v_i') \quad \forall\, i, v_i, v_i'$$

Third, feasibility of the selection rule:

$$X_1(v) \geqslant 0, \quad X_2(v) \geqslant 0, \quad X_1(v) + X_2(v) \leqslant 1 \quad \forall\, v$$

The problem of the seller is to maximize her expected revenue under the individual-rationality, incentive-compatibility and feasibility constraints.

Lemma 1 *In the optimal mechanism, the seller solves the following program $\mathcal{P}$:*

$$\mathcal{P}:\ \max \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{\overline{v}} \Big[t_1(v) + t_2(v)\Big] f(v_1) f(v_2) dv_1 dv_2$$

$$\text{s.t.}\quad u_i(v_i) - u_i(v_i') = \int_{v_i'}^{v_i} E_{v_j}\Big[X_i(s, v_j) - \alpha_a X_j(s, v_j)\Big] ds \quad \forall\ i,\ v_i' \leqslant v_i \tag{IC$_1$}$$

$$E_{v_j}\Big[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)\Big] \leqslant E_{v_j}\Big[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)\Big] \quad \forall\ i,\ v_i' \leqslant v_i \tag{IC$_2$}$$

$$u_i(v_i) \geqslant w_i(v_i) \quad \forall\ i,\ v_i \tag{IR}$$

$$X_i(v_i, v_j) \geqslant 0 \quad \forall\ i,\ v_i,\ v_j \tag{F$_0$}$$

$$X_1(v_i, v_j) + X_2(v_i, v_j) \leqslant 1 \quad \forall\ v_i,\ v_j \tag{F$_1$}$$

Proof. See Appendix 1.

As usual, (IC_1) is the (first-order) local optimality condition which ensures that stating the true valuation $v_i' = v_i$ is a local optimum. (IC_2) is the (second-order) monotonicity condition. It ensures the convexity of the equilibrium utility, and therefore that the local optimum is a global maximum. Using (IC_1) and (2), we have:

$$\frac{d}{dv_i} u_i(v_i) = E_{v_j}[X_i(v)] - \alpha_a E_{v_j}[X_j(v)] \quad \text{and} \quad \frac{d}{dv_i} w_i(v_i) = -\alpha_a \tag{3}$$

Given that informational rents are costly, the seller wants to minimize $u_i(v_i) - w_i(v_i)$. As a result, there will be at least one agent who receives no informational rents at equilibrium, called a "binding type". Formally, it is a type $\hat{v}$ for which the (IR) constraint binds: $u_i(\hat{v}) = w_i(\hat{v})$.[7] From (3), we have:

$$\frac{d}{dv_i} (u_i(v_i) - w_i(v_i)) = \alpha_a \Big(1 - E_{v_j}[X_j(v)]\Big) + E_{v_j}[X_i(v)] \tag{4}$$

Combining (3) and (4) , we have four qualitatively different cases:[8] (i) when $\alpha_a = 0$, $w_i(v_i)$ is constant, $u_i(v_i)$ is increasing in v_i and $\hat{v} = \underline{v}$; This corresponds to the model analyzed by Jehiel et al. [10], and shares the same technical aspects as in the first case analyzed in Carrillo [5] as well as the the first case studied in Figueroa and Skreta [8]; (ii) when $\alpha_a > 0$, $w_i(v_i)$ is decreasing in v_i, $u_i(v_i)$ is not always monotonic in v_i and $\hat{v} = \underline{v}$. This case turns out to be technically similar to case (i);[9] (iii) when $\alpha_a \leqslant -1$, $w_i(v_i)$ is increasing in v_i, $u_i(v_i)$ is increasing in v_i and $\hat{v} = \overline{v}$; This case shares common features with the second case analyzed in Carrillo [5] as well as the the second case studied in Figueroa and Skreta [8];[10] Last, (iv) when $\alpha_a \in (-1, 0)$, $w_i(v_i)$ is increasing in v_i, $u_i(v_i)$ is increasing in v_i and $\hat{v} \in [\underline{v}, \overline{v}]$.

In this paper, we are interested in analyzing the mechanism design problem in case (iv). It is illustrated in Figure 1.

[INSERT FIGURE 1 HERE]

The literature in incentive theory has fully analyzed optimal contracting under type-dependent reservation utilities in the single agent case (see e.g. Lewis and Sappington [17], Maggi and Rodriguez [18] and Jullien [13]). To our knowledge the optimal mechanism when the binding type is interior has not been analyzed in a multi-agent setting except in simple

[7] At this stage, we cannot establish whether the binding type is unique or not.

[8] Recall that, given (F_1), we have $1 - E_{v_j}[X_j(v)] \geqslant E_{v_j}[X_i(v)]$.

[9] However, it delivers different and novel economic implications due to the fact that the equilibrium utility may not be monotonic (in Lemma 1, the r.h.s. of (IC1) is not necessarily positive). At equilibrium, intermediate types will receive less utility compared to low and high types. See Brocas [3] for the complete solution of this case.

[10] However, it also delivers new economic implications in our setting. In particular, in the optimal mechanism, the bidder with the lowest valuation gets the good and the auction must be implemented with price ceilings rather than reserve prices. See Brocas [3] for details.

limit cases. The authors either restricted attention to corner binding types (as in the first two cases in Carrillo [5] and Figueroa and Skreta [8]) or treated cases boiling down to single agent problems (as in the third case in Figueroa and Skreta [8]). Our objective is to offer a complementary analysis. To economize on notations, let $H(v_i) = E_{v_j}[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)]$ from now on.

3 Optimal mechanism

The equilibrium utility $u_i(v_i)$ must be weakly increasing (by (IC_1)) and weakly convex (by (IC_2)) in v_i. Furthermore, the slope of the reservation utility $w_i(v_i)$ is constant and equal to $-\alpha_a$. Therefore, for any mechanism A with vector of interim probabilities $X = (X_1(v), X_2(v))$ satisfying (IC_1)-(IC_2)-(F_0)-(F_1) there is at most one interval $\hat{V}_i(A) = [\hat{v}_i^a(A), \hat{v}_i^b(A)]$ such that for all $v_i \in \hat{V}_i(A)$:[11]

$$\frac{d}{dv_i} u_i(v) = \frac{d}{dv_i} w_i(v) \tag{5}$$

Furthermore, the convexity of $u_i(v_i)$ combined with the linearity of $w_i(v_i)$ implies that:

$$\frac{d}{dv_i}(u_i(v_i) - w_i(v_i)) < 0 \;\; \forall\, v_i < \hat{v}_i^a(A) \quad \text{and} \quad \frac{d}{dv_i}(u_i(v_i) - w_i(v_i)) > 0 \;\; \forall\, v_i > \hat{v}_i^b(A) \tag{6}$$

Since informational rents are costly, (IR) will bind on $\hat{V}_i(A)$, that is:

$$u_i(v_i) = w_i(v_i) \qquad \forall\, v_i \in \hat{V}_i(A)$$

Definition 1 *The set of binding types is the compact* $\hat{V}_i(A) = [\hat{v}_i^a(A), \hat{v}_i^b(A)]$.

Naturally, $u_i(v_i) > w_i(v_i)$ for all $v_i \notin \hat{V}_i(A)$. In the absence of adequate incentives, an agent with a valuation $v_i < \hat{v}_i^a(A)$ will over-state his type and an agent with a valuation $v_i > \hat{v}_i^b(A)$ will under-state his type. To induce truthful revelation, informational rents have to be decreasing up to $\hat{v}_i^a(A)$ and increasing after $\hat{v}_i^b(A)$, as depicted in Figure 1 (which is drawn for the limit case where $\hat{v}_i^a(A) = \hat{v}_i^b(A)$). Using ($IC_1$), the equilibrium rent is then

$$u_i(v_i) = \begin{cases} w_i(v_i^a) - \int_{v_i}^{\hat{v}_i^a} H(s)ds & v_i < \hat{v}_i^a(A) \\ w_i(v_i) & v_i \in \hat{V}_i(A) \\ w_i(v_i^b) + \int_{\hat{v}_i^b}^{v_i} H(s)ds & v_i > \hat{v}_i^b(A) \end{cases} \tag{7}$$

and the necessary condition for (IR) to bind on $\hat{V}_i(A)$ is

$$H(v_i) = -\alpha_a \qquad \forall\, v_i \in \hat{V}_i(A), \forall\, i \tag{$\widehat{\text{IR}}$}$$

Definition 2 *The lower subset is* $\underline{V}_i(A) = [\underline{v}, \hat{v}_i^a(A))$.

[11] Naturally, it may be that $du_i(v_i)/dv_i > dw_i(v_i)/dv_i$ for all v_i or $du_i(v_i)/dv_i < dw_i(v_i)/dv_i$ for all v_i.

Definition 3 *The upper subset is* $\overline{V}_i(A) = (\hat{v}_i^b(A), \overline{v}]$.

From now on and to economize on notations, we will not reflect the relationship between those sets and the mechanism A, although the reader should keep in mind their mechanism-dependency.

Using (1) and (7), the seller's optimization program $\mathcal{P}$ is now equivalent to program $\hat{\mathcal{P}}$:

$$\hat{\mathcal{P}} : \max \sum_I \sum_J \int_I \int_J \Big[X_i(v)\pi_i^{IJ}(v_i, v_j) + X_j(v)\pi_j^{IJ}(v_i, v_j) \Big] dF(v)$$

$$- \sum_i \Big(\int_{\underline{v}}^{\hat{v}_i^a} w_i(\hat{v}_i^a) dF(v) + \int_{\hat{v}_i^a}^{\hat{v}_i^b} w_i(v) dF(v) + \int_{\hat{v}_i^b}^{\overline{v}} w_i(\hat{v}_i^b) dF(v) \Big)$$

$$\text{s.t.} \quad (\text{IC}_2) - (\text{F}_0) - (\text{F}_1) - (\widehat{\text{IR}})$$

where $\pi_i^{IJ}(v_i, v_j)$ is the virtual surplus of selling to agent i when $v_i \in I$ and $v_j \in J$,

$$\pi_i^{IJ}(v_i, v_j) = v_i - \alpha_j(v) - \frac{1 - F(v_i)}{f(v_i)} 1_{I=\overline{V}_i} + \frac{F(v_i)}{f(v_i)} 1_{I=\underline{V}_i} + \alpha_a \frac{1 - F(v_j)}{f(v_j)} 1_{J=\overline{V}_j} - \alpha_a \frac{F(v_j)}{f(v_j)} 1_{J=\underline{V}_j},$$

$I \in \{\underline{V}_i, \hat{V}_i, \overline{V}_i\}$ and $J \in \{\underline{V}_j, \hat{V}_j, \overline{V}_j\}$. There are 9 possible combinations of I and J, yielding 9 possible virtual surplus. Each of them represents the net surplus that the auctioneer can extract by selling the good to agent i rather than keeping it, adjusted for the informational rents that she is obliged to grant due to the asymmetry of information vis-a-vis bidders. When externalities are present, agents are willing to pay to prevent the allocation to their rival. Therefore, under complete information, the seller can extract v_i from agent i by selling the good to i or she can extract $\alpha_j(v)$ from agent j if she keeps the good. The net surplus of the sale to i is therefore $v_i - \alpha_j(v)$. Under asymmetric information, the seller leaves extra rents reflected in the extra terms. Note that the distortion due to informational rents act differently depending on whether a type lies in the lower subset, the upper subset of the set of binding types. By increasing the probability of allocating the good to agent i at a point v_i in the lower subset, the seller must grant extra rents to types all below v_i (in proportion $F(v_i)$). Conversely, by increasing the probability of allocating the good to agent i at a point v_i in the upper subset, the seller must grant extra rents to all types above v_i (in proportion $1 - F(v_i)$).[12] Now, given the interdependency between types and externalities, increasing the probability of allocating the good to type v_i also affects the rents to be granted to agent j with type v_j.

[12] Note that in any principal-agent problem in which the incentive compatibility constraint and the individual rationality constraint bind at the top (respectively bottom), distortions involve a term $\frac{F(v)}{f(v)}$ (resp. $\frac{1-F(v)}{f(v)}$). When the binding type is interior, both features are present.

3.1 Optimal mechanism in two limit cases

We first analyze limit cases in which the equilibrium utility is equal to the reservation utility at the bottom or at the top of the distribution of valuations. We determine values of α_a for which it occurs and we characterize the solution to $\hat{\mathcal{P}}$. We also need to impose a further condition to restrict to the regular case, that is, to ensure that the virtual surplus is increasing in v_i.

Assumption 3 $\alpha_b \leqslant 0$.

Proposition 1 *The optimal mechanism $\hat{A}$ has the following limit properties.*

(i) *When $\alpha_a \to 0^-$, γ is small enough and the condition $-\alpha_b(\overline{v} - \underline{v}) < \underline{v} - 1/f(\underline{v})$ is satisfied, the optimal mechanism is symmetric, $\hat{v}_i^a = \hat{v}_i^b = \underline{v}$ for all i and the allocation rule is*[13]

$$X_i^*(v_i, v_j) = \begin{cases} 1 & \text{if} \quad v_i > v_j \quad \text{and} \quad v_i > r_i^*(v_j) \\ 0 & \text{otherwise} \end{cases}$$

where $r_i^*(v_j) = \arg\min\{v_i | \pi_i^{\overline{V}_i \overline{V}_j}(v_i, v_j) \geqslant 0; \overline{v}\}$ is decreasing in v_j.

(ii) *When $\alpha_a \to -1^+$ and γ is large enough, the optimal mechanism is symmetric, $\hat{v}_i^a = \hat{v}_i^b = \overline{v}$ for all i and the allocation rule is*[14]

$$X_i^{**}(v_i, v_j) = \begin{cases} 1 & \text{if} \quad v_i > v_j \quad \text{and} \quad v_i > r_i^{**}(v_j) \\ 0 & \text{otherwise} \end{cases}$$

where $r_i^{**}(v_j) = \arg\min\{v_i | \pi_i^{\underline{V}_i \underline{V}_j}(v_i, v_j) \geqslant 0; \overline{v}\}$ is decreasing in v_j.

Proof. See Appendix 2.

When $\alpha_a \to -1^+$, the outside option becomes very steep and the binding type is $\overline{v}$. By contrast, when $\alpha_a \to 0^-$, the outside option becomes flat and the binding type is now $\underline{v}$. The optimal mechanism specifies a reserve price above which the agent with the highest valuation obtains the good. Indeed, if $v_i > v_j$, then the difference between the valuations is greater than the difference between i's willingness to pay to prevent the sale to j and j's willingness to pay to prevent the sale to i: Formally $v_i - v_j > \alpha_j - \alpha_i = (v_i - v_j)(\alpha_b - \alpha_a)$. Therefore, in equilibrium, it is relatively more profitable to sell the good to i. Interestingly, the reserve price faced by agent i depends on the valuation of agent j. This is due to the dependency between valuations and externalities. As the valuation of the rival increases, the externality the rival suffers if the good is sold to i decreases, and so is his willingness to pay to avoid

[13] γ must be small enough to make sure that $\underline{v}$ is the unique binding type. The extra condition makes sure that such a low value of γ exists while $\alpha_i(v) \geqslant 0$ is preserved.

[14] γ must be large enough to make sure that $\overline{v}$ is the unique binding type.

the sale to i. Therefore, it becomes more beneficial to sell to i. Note that technically, the problem is similar to the cases analyzed in Carrillo [5] and followers, and it requires a simple adaptation of the procedure introduced by Myerson [21]. Note also that regularity results from the combination of assumptions 1 and 3.

3.2 The effect of countervailing incentives on the multi-agent problem

We now turn to the case in which the set of binding types is interior. In what follows, we fix $\hat{V}_i$. We first need to show how countervailing incentives impact the design of the optimal mechanism. To do so, we introduce the relaxed optimization program $\mathcal{P}^{UNC}$:

$$\mathcal{P}^{UNC} : \max \sum_I \sum_J \int_I \int_J \Big[X_i(v)\pi_i^{IJ}(v_i, v_j) + X_j(v)\pi_j^{IJ}(v_i, v_j) \Big] dF(v)$$

$$- \sum_i \Big(\int_{\underline{v}}^{\hat{v}_i^a} w_i(\hat{v}_i^a) dF(v) + \int_{\hat{v}_i^a}^{\hat{v}_i^b} w_i(v) dF(v) + \int_{\hat{v}_i^b}^{\overline{v}} w_i(\hat{v}_i^b) dF(v) \Big)$$

$$\text{s.t.} \quad (\mathrm{F}_0) - (\mathrm{F}_1)$$

Lemma 2 *The optimal unconstrained mechanism A^{UNC} has the following properties.*
(i) *The allocation rule is such that in each IJ*

$$X_i^{UNC}(v_i, v_j) = \begin{cases} 1 & \text{if } \ v_i > v_j \ \text{ and } \ v_i > r_i^{IJ}(v_j) \\ 0 & \text{otherwise} \end{cases}$$

where $r_i^{IJ}(v_j) = \arg\min\{v_i | \pi^{IJ}(v_i, v_j) \geqslant 0; \overline{v}\}$ is decreasing in v_j.
(ii) *A^{UNC} does not satisfy (IC_2) at $\hat{v}_i^a$ and $\hat{v}_i^b$;*
(iii) *A^{UNC} does not satisfy $(\widehat{IR})$.*

Proof. See Appendix 3.

Lemma 2 shows two main features of our problem. First, assumptions 1 and 3 guarantee only 'weak' regularity: the problem is regular almost everywhere. Namely (IC_2) is satisfied for all $v_i < \hat{v}_i^a$, $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$ and $v_i > \hat{v}_i^b$ but the function $H(v_i)$ admits downwards jumps at $\hat{v}_i^a$ and $\hat{v}_i^b$. This violates (IC_2) at those points. This is reminiscent of the previous literature on countervailing incentives (see for instance Maggi and Rodriguez [18]), and it relates to a tension between the way the seller wants to solve the trade-off between rents and efficiency when types lie in the upper and the lower sets. To see this, note that the rent $u_i(v_i) - w_i(v_i)$ is decreasing in the lower subset and increasing in the higher subset. In the lower subset, the seller would like to decrease the rent by making the slope of the rent less negative (and make the expected utility come closer to the outside option). This pushes her to increase the probability of allocating the good compared to the full information setting. In the upper subset however, the seller would like to decrease the rent by making the slope of the rent less positive. This pushes her to decrease the probability of allocating the good compared to the full information setting. Overall, the seller would like to allocate the good relatively more often in the lower subset compared to the interior set of binding types, and also relatively more often in the set of binding types compared to the upper subset. Therefore, there are discontinuities at $\hat{v}_i^a$ and $\hat{v}_i^b$. These contribute to violations of (IC_2).

Second, $(\widehat{IR})$ is not satisfied for free. The set of points for which the derivative of the utility profile coincides with that of the outside option is not $\hat{V}_i$. Besides, it may not be a subset either. More generally, there does not exist any $\hat{V}_i$ such that $V_i^{UNC} = \hat{V}_i$. Figure 2 depicts the allocation in the unconstrained mechanism (drawn for the case $\hat{v}_i^a = \hat{v}_i^b = \hat{v}$ for all i) and Figure 3 graphs the corresponding $H(v_i)$.

[Insert Figures 2 & 3 here]

Given Lemma 2, we need to deal with two distinct violations. On one hand, we need to restore (IC_2), which requires to distort the unconstrained allocation to have $H(v_i)$ monotonic everywhere. On the other hand, we need to restore $(\widehat{IR})$, which requires to select allocations such that $H(v_i) = -\alpha_a$ exactly on the set of binding types.

We shall note that the violations of (IC_2) are of a different nature compared to those occurring in the earlier literature in either single agent or multi-agent settings (with or without countervailing incentives).

In multi-agent contracting problems, violations have been studied in Myerson [21] for the non regular case. The procedure to restore (IC_2) consists in replacing the virtual surplus with an *ironed virtual surplus* that coincides with the virtual surplus except on some intervals on which both surpluses are equal in expectation. This procedure is not particularly useful here because Assumptions 1 and 3 deliver regularity. However, regularity is not enough in our case: the monotonicity of the virtual surplus does not guarantee the monotonicity of $H(v_i)$.

In single-agent contracting problems, the seminal work of Guesnerie and Laffont [9] examines violations of (IC_2). In such situations, (IC_2) requires the allocation to be monotonic. When the unconstrained allocation violates (IC_2), the solution is simply to construct an allocation such that bunching occurs on intervals containing the points at which (IC_2) is violated, and that coincides with the unconstrained allocation everywhere else. This is not useful either in our setting.

Firstly, (IC_2) requires some form of monotonicity of the *ex-ante* allocation: $H(v_i)$ is an expectation over the ex-post probabilities X_i and X_j. However, there is not a unique or obvious way of distorting the *ex-post* allocation (X_i, X_j) to restore (IC_2). It could be done by distorting the reserve price of agent i, that of agent j or the decision rule to allocate to i

versus j. This difficulty is not present in the single-agent setting because it is as if the ex-ante and ex-post allocations coincide (as a direct consequence of having only one agent).

Secondly, suppose that we modify the ex-post allocation of agent i in the neighborhood of $\hat{v}_i^a$. That is, we distort the probability of allocating the good to agent i for some values v_j of agent j. This in turn affects the probability of allocating the good to j for those v_j. Therefore, attempting to restore (IC_2) vis-à-vis agent i modifies the shape of the function $H(v_j)$, which may now not be monotonic where it used to be. In other words, restoring (IC_2) where it is violated may affect the mechanism at values at which (IC_2) is satisfied. This results from the interdependency of the allocation rules.

These difficulties do not arise in a single agent setting. To clarify that point, we consider in Appendix B a related single agent allocation problem in which (IC_2) is violated, and we show how the standard Guesnerie and Laffont [9] procedure can be applied.

Given these considerations, we need to develop a new procedure to restore (IC_2) and obtain an allocation also consistent with $(\widehat{IR})$. We start with a simple observation: for any mechanism that satisfies (IC_2) everywhere, there exists at most one subset $\hat{V}_i$ such that $H(v_i) = -\alpha_a$ for all $v_i \in \hat{V}_i$. Conversely, for each subset $\hat{V}_i$, it is possible to construct mechanisms that satisfy (IC_2) and such that $H(v_i) = -\alpha_a$ for all $v_i \in \hat{V}_i$. Obviously, among those mechanisms, the seller prefers the one that is 'as close as possible' to the optimal unconstrained allocation obtained by fixing $\hat{V}_i$.

3.3 Optimal mechanism with countervailing incentives

Recall that violations of (IC_2) arise because the unconstrained allocation to the right and to the left of points $\hat{v}_i^a$ and $\hat{v}_i^b$ differ (see Figure 2). In particular, for any $\hat{V}_i$, there exists a set of points (v_i, v_j) such that it is optimal to allocate to one agent to the right of $\hat{v}_i^a$ and to the other to the left of that point. This is the case because it is optimal to allocate the item to the agent with the highest valuation on one side and to the agent with the lowest valuation on the other. The same holds for $\hat{v}_i^b$ and this also applies to agent j. We can construct a set containing all such points. Consider the convex hull of this set, $\hat{C}$. By construction, there is no tension in the complement of $\hat{C}$. In particular, and other things being equal, it is preferred to allocate the item to agent i when $v_i \geqslant v_j$ and to agent j when $v_i < v_j$.

Let us fix an allocation in $\hat{C}$. We shall first make sure that this allocation does not violate $(\widehat{IR})$, that is we have "enough room" to construct an allocation that satisfies the constraint.

Definition 4 *An allocation is feasible if for all i and $j \neq i$, we have both $E_{v_j \in \hat{C}} X_i(v_i, v_j) - \alpha_a E_{v_j \in \hat{C}} X_j(v_i, v_j) < -\alpha_a$ for all $v_i \leqslant v_i^a$, and $E_{v_j \in \hat{C}} X_i(v_i, v_j) - \alpha_a E_{v_j \in \hat{C}} X_j(v_i, v_j) \leqslant -\alpha_a$ for all $v_i \in (v_i^a, v_i^b)$.*

An allocation that violates at least one of these properties cannot be candidate for optimality. Therefore, we restrict to allocations that are feasible allocations. Note that for every v_i, there exists an interval $[\hat{c}_l(v_i), \hat{c}_h(v_i)]$ of values v_j such that $(v_i, v_j) \in \hat{C}$. In what follows, we assume the distribution is uniform to avoid effects due exclusively to the relative likelihood of types. This assumption will be discussed in Remark 2 at the end of this section.

Assumption 4 *$F(\cdot)$ is uniform on $[\underline{v}, \overline{v}]$.*

The main result of the paper consists in a characterization of the main properties of the optimal allocation.

Proposition 2 *The optimal mechanism $\hat{A}$ has the following properties when $\alpha_a \in (-1, 0)$:*

1. *The optimal allocation in the complement of $\hat{C}$ has the following properties:*
   (i) *$X_j = 0$ for any $v_j < \min(v_i, \hat{c}_l(v_j))$ and $X_i = 0$ for any $v_j > \max(v_i, \hat{c}_h(v_j))$;*
   (ii) *for all $v_i \in \hat{V}_i$, there exists a point $m(v_i) \in \left(\underline{v}, \min(v_i, \hat{c}_l(v_j))\right)$ such that*
   $$X_i(v_i, v_j) = \begin{cases} 1 & \text{if } v_j \in (m(v_i), c(v_i)) \\ 0 & \text{if } v_j < m(v_i) \end{cases}$$
   and set in such a way that $(\widehat{IR})$ is satisfied.
   (iii) *$X_i(v_i, v_j) = 0$ for all $v_j < \min(v_i, \hat{c}_l(v_j))$ and $v_i < r_i^{**}(v_j)$ unless (IC_2) is violated.*
   (iv) *$X_i(v_i, v_j) = 1$ for all $v_j < \min(v_i, \hat{c}_l(v_j))$ and $v_i > r_i^{*}(v_j)$.*
   (v) *$X_i(v_i, v_j) \in \{0, 1\}$ for all $v_j < \min(v_i, \hat{c}_l(v_j))$ and $v_i \in (r_i^{**}(v_j), r_i^{*}(v_j))$.*
2. *The optimal allocation in $\hat{C}$ is such that for all $v \in IJ$, and for all $IJ \in \hat{C}$*
   (i) *If $v_i > v_j$ and $\pi_i^{IJ}(v) > \pi_j^{IJ}(v)$, then $X_j(v_i, v_j) = 0$ and $X_i(v_i, v_j) \in \{0, 1\}$*
   (ii) *If $v_i > v_j$ and $\pi_i^{IJ}(v) < \pi_j^{IJ}(v)$, then either the good is not allocated or $X_i(v_i, v_j) = b(v_i, v_j)$ and $X_j(v_i, v_j) = 1 - b(v_i, v_j)$ where $b(v_i, v_j)$ is a probability.*
3. *The sets of binding types $\hat{V}_i \subset (\underline{v}, \overline{v})$ and the complement of $\hat{C}$ is never empty.*

Proof. See Appendix 4.

Even though Proposition 2 does not offer a full characterization, it summarizes the main interesting qualitative properties of the allocation. We will explain the mechanism in detail before discussing its implications.

Let us first discuss the properties of the ex-post allocation in the complement of $\hat{C}$ (item 1.). First, as in the limit cases and for the same reason, it is relatively more profitable to sell the good to the agent with highest valuation anywhere outside $\hat{C}$ (part (i)).[15] Second, it is always possible to find an allocation outside $\hat{C}$ such that $(\widehat{IR})$ is satisfied. For any $v_i \in \hat{V}_i$ and $v_i > v_j$, the surplus extracted from selling to i is increasing in v_j. The seller allocates the good to i up to the point $m(v_i)$ where $(\widehat{IR})$ is satisfied (part (ii)). Third, for any v_i below $\hat{V}_i$ and $v_i > v_j$, it is never optimal in the unconstrained mechanism to allocate the good to i when $v_i < r_i^{**}(v_j)$. The optimal mechanism in that region will be as close as possible to that allocation, provided (IC_2) is satisfied and $H(v_i) < -\alpha_a$ (part (iii)).[16] Similarly, for any v_i above $\hat{V}_i$ and $v_i > v_j$, it is always optimal in the unconstrained mechanism to allocate the good to i when $v_i > r_i^{**}(v_j)$. In that region, increasing the probability of allocating the good to i does not conflict with (IC_2). However, it may be necessary to give the good to i more often than optimal to satisfy $H(v_i) > -\alpha_a$ (part (iv)). Last, when $v_i > v_j$ and for intermediate values of v_i (namely, $v_i \in (r_i^{**}(v_j), r_i^{*}(v_j)))$, the good is either allocated to i or not allocated at all (part (v)). Intuitively, for each value v_i, the seller allocates the good to i according to the unconstrained allocation when it is possible. If it is not possible because the good must be allocated more often ($H(v_i)$ is too small and must be increased), then the seller favors pairs (v_i, v_j) that generates the smallest loss. If it is not possible because the good must be allocated less often ($H(v_i)$ is too large and must be reduced), then the seller discriminates against pairs (v_i, v_j) associated with the smallest surplus. Also, consider two pairs (v_i, v_j) and (v_i, v_j') for which it is best to allocate the good to i rather than j and such that the surplus derived from that sale is higher when valuations are (v_i, v_j). If the seller cannot allocate the item in both cases, then she should grant the entire probability weight to the pair (v_i, v_j) and none to (v_i, v_j'). This is a consequence of assumption 4.[17] These equilibrium properties are illustrated in Figure 4.

The allocation in $\hat{C}$ follows the same general principles (item 2.). However, depending on the region IJ we consider, it is sometimes optimal in the unconstrained mechanism to allocate the good to j when $v_i > v_j$ (and conversely). Other things being equal, the seller wants to allocate the good to either agent as long as it is profitable and does not conflict with the constraints (that is adopt the rule in A^{UNC}). Whenever it is necessary to allocate the good more often, the seller allocates the good to i or j at points associated with the smallest 'loss'. Whenever it is necessary to allocate the good less often, the seller discriminates against pairs of valuations associated with the smallest surplus. In other words, the decision to allocate the good is made such that priority is always given to pairs (v_i, v_j) yielding the highest possible surplus (from either agent). However, allocating the good to agent j when $v_i > v_j$ is the source of the discontinuities in the unconstrained mechanism. Therefore, it is necessary to reverse the allocation sometimes at some points. At equilibrium, agent i may receive the

[15]Note that given $1 + \alpha_a \geqslant 0$ and Assumption 3, we have $1 + \alpha_a \geqslant \alpha_b$ and it is optimal to allocate the good to the agent with the highest valuation under complete information.

[16]Technically, we may give the good too often to i in order to satisfy IC_2 if $m(v_i)$ cross several times $r_i^{**}(v_j)$.

[17]If the distribution is not uniform, the seller must also account for the likelihood of those events.

good with some probability when it is optimal to allocate the good to j. Overall, whenever $v_i > v_j$ and the surplus obtained by selling to i is greater, i obtains the good with probability 1 or the good is not allocated (part (i)). If instead, the surplus obtained by selling to j is greater, there are three scenarii (part (ii)). In the first one, the good is not allocated. In the second j is allocated the good. In the last one, the seller randomizes between the agents, allocating to the highest type with probability $b(v)$.

Last, the set of binding types is never the full support $[\underline{v}, \overline{v}]$ when $\alpha_a \in (-1, 0)$ (item 3.). If it were, the optimal unconstrained allocation would be to allocate the good to i whenever $v_i > v_j$ and provided $v_i > r_i^{\hat{V}\hat{V}}(v_j)$. In such allocation, it is rarely optimal to allocate to i for small values of v_i while it is often optimal to allocate to i for large values of v_i (and conversely for j). At the same time, the optimal allocation requires $H(v_i) = -\alpha_a$ for all v_i. When $\alpha \in (0, 1)$, this latter constraint requires to allocate the good more often than optimal for low values of v_i and less often than optimal for high values of v_i. As this is costly to the seller, it creates a motive for reducing the set of binding types. Note that this last property of the optimal allocation also implies that the complement of $\hat{C}$ is never empty.

[Insert Figure 4 here]

3.4 Properties of the optimal mechanism and discussion

The optimal mechanism has three novel and interesting properties. All relate to the *competition* between two agents facing countervailing incentives.

First, the ex-post allocation of the good is contingent on whether each of the two types lies in the lower set, the set of binding types or the upper set. In other words, a bidder does not face a single reserve price, but rather a family of reserve prices. This is the case because the trade-off between efficiency and rents is solved differently if types lie in either of these sets. The ability to extract rents varies as a function of the relative "strength" of the bidders, and the allocation rule must be tailored to it.

Second, the good is not always allocated to bidder i when $v_i > v_j$. This occurs in particular when v_i lies in the upper set while v_j lies in the lower set. For some values, it is relatively less costly to allocate the good to v_j in terms of informational rents. Said differently, the virtual surplus extracted from j is higher.[18] This property is obviously related to the first property mentionned above.

[18]This occurs also when $\alpha_a \geqslant -1$ but for different reasons. When $\alpha_a \geqslant -1$ and when $v_i > v_j$, agent j is willing to pay to prevent the sale to i more than what i is willing to pay to obtain the good. It is therefore optimal to allocate the good to j under complete and asymmetric information. When $\alpha_a \in (-1, 0)$, other things being equal, agent j is willing to pay to prevent the sale to i less than what i is willing to pay to obtain the good. Under complete information, agent j never obtains the good. Incentives to allocate to j are entirely due to countervailing incentives under asymmetric information.

Last, the seller must sometimes randomize between the two agents. This occurs for values where discontinuities arise in the unconstrained mechanism. Tailoring the rules to solve optimally the trade-off between rent extraction and efficiency creates a misalignment of incentives. Intuitively, in this competitive setting, solving the trade-off vis-à-vis one agent conflicts with the truth-telling requirements related to the other agent. Said differently, the seller would like to allocate more or less often to one given agent than what is necessary to make the other reveal. It is therefore necessary to further distort the probability of allocating the good compared to the unconstrained mechanism. An agent who should receive the item with probability one (respectively zero) may now obtain it less (respectively more) often.

We want to emphasize that characterizing the optimal ex-post allocation in this class of problem turns out to be challenging. The reader might argue we could limit ourselves to the characterization of the ex-ante allocation instead, and provide a complete solution. Such an approach has been used for example in Maskin and Riley [19] in a different setting. The authors exploit the symmetry of the problem to rewrite it as a single agent problem.[19] In their case, this allows to transform the problem into a standard optimal control problem and they can characterize properties of the optimal ex-ante allocation. In our case, however, the correlations between externalities and valuations prevent us from rewriting the problem in a standard form. Technically, we need to obtain an objective function that depends on the probability of allocating the good to the agent, but this always fails: the objective function depends also on the probability of allocating the good to the rival.[20] Therefore, this procedure is not helpful in our setting.

Remark 1. It is important to note that the problem is substantially different from the problem where the 'type' is the vector formed by a valuation and an externality. This multidimensional approach has been introduced by Jehiel et al. [11]. In that paper, the types in the various dimensions are not independently distributed. However, the authors assume full support on the Cartesian product of the support of each dimension. Therefore, there always exists an exogenously given "lowest type". Then, similarly to standard one-dimensional mechanism problems, it is sufficient to check that the individual rationality constraint is satisfied for that type, and at equilibrium it binds at that type. In our case, however, there is no "lowest type" with such properties. This generates other complications.

Remark 2. We have restricted ourselves to the case of a uniform distribution. In the more general case, the seller needs to trade-off the virtual surplus obtained from a given allocation

[19] It is easy to see that we can restrict to symmetric mechanisms: if an optimal mechanism A is asymmetric, then given the symmetry of the problem its counterpart A' is also optimal. Therefore, the symmetric mechanism that consists in implementing A with probability μ and A' with probability $1-\mu$ is also optimal.

[20] Assume $\alpha_b = 0$ to simplify. Let $P_i(v_i) = E_{v_j} X_i(v_i, v_j)$ and $Q_j(v_i) = E_{v_j} X_j(v_i, v_j)$, the expected transfer paid by agent i is $T(v_i) = E_{v_j} t_i(v_i, v_j) = P_i(v_i) - (\alpha_a v_i + \gamma) Q_j(v_i) - u_i(v_i)$. Noting that probabilities must satisfy $\int_{\underline{v}}^{\overline{v}} Q_j(v_i) dv_i = \int_{\underline{v}}^{\overline{v}} P_j(v_j) dv_j$ is not enough to obtain the expected transfer as a function of $P_i(v_i)$ only when $\alpha_a \neq 0$. Imposing symmetry is not useful either.

and the likelihood of that allocation. As such, it is not always optimal to favor the values associated with the highest surplus. Extending to a general distribution requires to take this trade-off into account. This would slightly affect the allocation in points 1(ii) and 2. However, the qualitative properties of the mechanism would remain. The results are simply more intuitive and easily explained if we abstract from this trade-off.[21]

4 Concluding remarks

This article studies the allocation mechanism of a single item in the presence of type-dependent externalities between two bidders. The type-dependency introduces countervailing incentives. Therefore, the optimal allocation must be sometimes such that the equilibrium utility is equal to the reservation utility on an interior subset of types. I have shown that this problem is technically different from the one analyzed in related single agent settings because the seller must manage two constraints that conflict with her objective. First, truth-telling requires the ex-ante allocation to satisfy a non-trivial monotonicity condition. Second, the allocation must be such that some types receive no rent. I have provided a procedure to characterize the main properties of the ex-post allocation. This procedure consists in constructing a mechanism as close as possible to the optimal unconstrained allocation, and such that the constraints are restored.

Compared to analyses in the earlier literature, the equilibrium ex-post allocation has novel properties. Indeed, the solution does not entail a single reserve price. More specifically, each agent faces an allocation rule contingent on whether he and his rival's types fall below, in or above the set of binding types. Namely, the solution that would obtain in the limit case where the equilibrium utility binds at the top prevails when both types lie below the set of binding types. Similarly, the solution that would obtain in the limit case where the equilibrium utility binds at the bottom prevails when both types lie above the set of binding types. Everywhere else, the seller gives priority to allocations associated with the highest possible surplus as long as the constraints require to allocate with positive probability. The seller sometimes refrains from allocating the good while it would be profitable, to make sure the constraints are satisfied. At equilibrium, the seller may randomize between the two agents and the agent with the lowest valuation may obtain the good.

The results emphasized here complement the existing literature on auctions with externalities. In particular, it shows that the typical environment affects crucially the analysis and complicates heavily the characterization of the mechanism. Previous results suggested that the standard independent value auction framework could be extended relatively easily to the presence of externalities. Our analysis suggests that building appropriate models of externalities may lead to different predictions from what has been obtained elsewhere.

[21] Details about the more general case are available upon request.

Appendix A

Appendix 1. Note that $u_i(v_i, v_i') = u_i(v_i', v_i') + (v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)]$. Then the incentive compatibility constraint is equivalent to:

$$u_i(v_i, v_i) \geq u_i(v_i', v_i') + (v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)]. \tag{8}$$

Using this inequality twice, the incentive compatibility constraint is equivalent to

$$\begin{aligned}(v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)] &\leq u_i(v_i, v_i) - u_i(v_i', v_i') \\ &\leq (v_i - v_i')[E_{v_j} X_i(v_i, v_j) - \alpha_a E_{v_j} X_j(v_i, v_j)]. \end{aligned} \tag{9}$$

Then the agent reveals truthfully if:

$$E_{v_j}\left[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)\right] \leq E_{v_j}\left[X_i(v_i, v_j) - \alpha_a X_j(v_i, v_j)\right] \quad \forall\ v_i' \leq v_i \tag{IC$_2$}$$

(9) must hold for all v_i' and all $v_i = v_i' + \delta$ with $\delta > 0$. Since $E_{v_j} X_i(v_i, v_j) - \alpha_a E_{v_j} X_j(v_i, v_j)$ is increasing in v_i, we can take the Riemann integral. Then, the agent reveals truthfully if we also have:

$$u_i(v_i) - u_i(v_i') = \int_{v_i'}^{v_i} E_{v_j}[X_i(s, v_j) - \alpha_a X_j(s, v_j)]ds \quad \forall\ v_i' \leq v_i \tag{IC$_1$}$$

To complete the proof, we need to verify that (IC$_1$) and (IC$_2$) imply (8). Suppose $v_i' \leq v_i$, then given (IC$_1$) and (IC$_2$), we have:

$$\begin{aligned} u_i(v_i, v_i) &= u_i(v_i', v_i') + \int_{v_i'}^{v_i} E_{v_j}[X_i(s, v_j) - \alpha_a X_j(s, v_j)]ds \\ &\geq u_i(v_i', v_i') + \int_{v_i'}^{v_i} E_{v_j}[X_i(v_i', v_j) - \alpha_a X_j(v_i', v_j)]ds \\ &= u_i(v_i', v_i') + (v_i - v_i')[E_{v_j} X_i(v_i', v_j) - \alpha_a E_{v_j} X_j(v_i', v_j)]. \end{aligned}$$

The seller maximizes her expected revenue (the sum of transfers) under constraints (IC$_1$) and (IC$_2$) (to induce truthtelling) and the remaining constraints (IR), (F$_0$) and (F$_1$).[22] □

Appendix 2. When $\alpha_a \to -1+$, $H(v_i) \to E_{v_j}[X_i(v) + X_j(v)] \leqslant 1$, and $-\alpha_a \to 1$. At equilibrium, $\hat{V}_i = [\hat{v}_i^a, \overline{v}]$ for all i. We now construct a case where $\hat{v}_i^a = \overline{v}$.

The mechanism such that the seller keeps the good if $\max_i\{\pi_i^{\underline{V}_i \underline{V}_j}(v)\} < 0$ and allocates it to the bidder with the highest $\pi_i^{\underline{V}_i \underline{V}_j}(v)$ maximizes $\hat{\mathcal{P}}$ under (F$_0$) and (F$_1$). We have $\pi_i^{\underline{V}_i \underline{V}_j}(v) > \pi_j^{\underline{V}_i \underline{V}_j}(v)$ if $v_i > v_j$. When the cutoff is interior,

$$\left[\frac{d}{dv_i}[v_i + \frac{F(v_i)}{f(v_i)}]|_{r_i^{**}} - \alpha_b\right] \frac{d}{dv_j} r_i^{**}(v_j) - \alpha_a \frac{d}{dv_j}[v_j + \frac{F(v_j)}{f(v_j)}] = 0$$

[22] Note that the proof is similar to Myerson [21] except that we do not provide a sufficient condition for (IR) to hold at this stage.

showing that $r_i^{**}(v_j)$ is decreasing in v_j. We need to check now that (IC$_2$) is satisfied. Curves $r_i^{**}(v_j)$ and $r_j^{**^{-1}}(v_j)$ cross at $\breve{v}$. We have $\frac{d}{dv_i}[v_i + \frac{F(v_i)}{f(v_i)}]|_{\breve{v}} = h$, then $\frac{\partial}{\partial v_j} r_i^{**}(v_j)|_{\breve{v}} = \frac{\alpha_a h}{h - \alpha_b} < -1$, which ensures that $\breve{v}$ is unique. For all $v_i \lesseqgtr \breve{v}$, $r_j^{**}(v_i) \gtreqless r_i^{**^{-1}}(v_i)$. When $v_i < \breve{v}$, $E_{v_j}\hat{X}_j = 0$ and $E_{v_j}\hat{X}_i = 1 - F(r_i^{**^{-1}}(v_i))$. When $v_i > \breve{v}$, $E_{v_j}\hat{X}_i = 1 - F(v_i)$ and $E_{v_j}\hat{X}_j = F(v_i) - F(r_j^{**}(v_i))$. Therefore, (IC$_2$) is satisfied everywhere. Last, we have $E_{v_j}[\hat{X}_i(v) + \hat{X}_j(v)] = 1$ for all $v_i \geqslant r_i^{**}(\underline{v})$ and therefore $H(v_i) \to 1$ for all $v_i \geqslant r_i^{**}(\underline{v})$. To ensure $\overline{v}$ is the unique binding point, we must have $r_i^{**}(\underline{v}) \geqslant \overline{v}$. It is easy to see that $r_i^{**}(v_j)$ increases in γ. A sufficient condition is to choose γ large enough so that $r_i^{**}(\underline{v}) \to \overline{v}$. Naturally, if γ is overly big, $r_i^{**}(v_j) \to \overline{v}$ for all v_j and the solution is degenerate.

When $\alpha_a \to 0-$, $H(v_i) \to E_{v_j}[X_i(v)] \geqslant 0$, and $-\alpha_a \to 0$. At equilibrium, $\hat{V}_i = [\underline{v}, \hat{v}_i^b]$ for all i. We now construct a case where $\hat{v}_i^b = \underline{v}$. The mechanism such that the seller keeps the good if $\max_i\{\pi_i^{\underline{V}_i\underline{V}_j}(v)\} < 0$ and allocates it to the bidder with the highest $\pi_i^{\underline{V}_i\underline{V}_j}(v)$ otherwise maximizes $\hat{\mathcal{P}}$ under (F$_0$) and (F$_1$). We have $\pi_i^{\underline{V}_i\underline{V}_j}(v) > \pi_j^{\underline{V}_i\underline{V}_j}(v)$ if $v_i > v_j$. Moreover, when the cutoff $r_i^*(v_j)$ is interior, we have

$$\left[\frac{d}{dv_i}[v_i - \frac{1-F(v_i)}{f(v_i)}]|_{r_i^*} - \alpha_b\right] \frac{d}{dv_j} r_i^*(v_j) - \alpha_a \frac{d}{dv_j}[v_j - \frac{1-F(v_j)}{f(v_j)}] = 0$$

proving that $r_i^*(v_j)$ is decreasing in v_j. Curves $r_i^*(v_j)$ and $r_j^{*^{-1}}(v_j)$ cross at $\breve{v}$. $\frac{\partial}{\partial v_j} r_i^*(v_j)|_{v^o} \leq 1$ and $\breve{v}$ is unique. For all $v_i \lesseqgtr \breve{v}$, $r_j^*(v_i) \gtreqless r_i^{*^{-1}}(v_i)$. When $v_i < \breve{v}$, $E_{v_j}\hat{X}_j = 0$ and $E_{v_j}\hat{X}_i = 1 - F(r_i^{*^{-1}}(v_i))$. When $v_i > \breve{v}$, $E_{v_j}\hat{X}_i = 1 - F(v_i)$ and $E_{v_j}\hat{X}_j = F(v_i) - F(r_j^*(v_i))$. Therefore, (IC$_2$) is satisfied everywhere. To ensure $\underline{v}$ is the unique binding point, we must have $\breve{v} \to \underline{v}$. It is easy to see that $r_i^*(v_j)$ increases in γ. A sufficient condition is to choose γ small enough so that $r_i^*(\underline{v}) \to \underline{v}$, which requires $\gamma \leqslant \underline{v} - 1/f(\underline{v}) - \alpha_b\underline{v}$. Given γ is bounded below by $-\alpha_b\overline{v}$ to guarantee $\alpha_i(v) \geqslant 0$, it is sufficient to have a distribution and a parameter $\alpha_b < 0$ such that $\underline{v} + \alpha_b(\overline{v} - \underline{v}) - 1/f(\underline{v}) < 0$. □

Appendix 3. Fix $\hat{V}_1$ and $\hat{V}_2$. $\pi_i^{IJ}(v_i, v_j)$ is increasing in v_i and v_j and $\pi_j^{IJ}(v_j, v_i) - \pi_i^{IJ}(v_i, v_j)$ is decreasing in v_i and increasing in v_j for all IJ. Let $r_i^{IJ}(v_j) = \min\left\{v_i \,|\, \pi_i^{IJ}(v_i, v_j) \geqslant 0\right\}$ and $h^{IJ}(v_j) = \min\left\{v_i \,|\, \pi_i^{IJ}(v_i, v_j) \geqslant \pi_j^{IJ}(v_j, v_i)\right\}$.

For all IJ, $r_i^{IJ}(v_j)$ is decreasing in v_j and $h^{IJ}(v_j)$ is increasing in v_j. Given the symmetry of the virtual surplus functions, we have:

(i) $r_i^{\underline{V}_i,\underline{V}_j}(v_j)$ and $r_j^{\underline{V}_i,\underline{V}_j}(v_i)$; $r_i^{\hat{V}_i,\hat{V}_j}(v_j)$ and $r_j^{\hat{V}_i,\hat{V}_j}(v_i)$; $r_i^{\overline{V}_i,\overline{V}_j}(v_j)$ and $r_j^{\overline{V}_i,\overline{V}_j}(v_i)$; $r_i^{\underline{V}_i,\hat{V}_j}(v_j)$ and $r_j^{\hat{V}_i,\underline{V}_j}(v_i)$; $r_i^{\underline{V}_i,\overline{V}_j}(v_j)$ and $r_j^{\overline{V}_i,\underline{V}_j}(v_i)$; $r_i^{\hat{V}_i,\overline{V}_j}(v_j)$ and $r_j^{\overline{V}_i,\hat{V}_j}(v_i)$ are symmetric.

(ii) $h^{\underline{V}_i,\hat{V}_j}(v_j)$ and $h^{\hat{V}_i,\underline{V}_j}(v_i)$; $h^{\underline{V}_i,\overline{V}_j}(v_j)$ and $h^{\overline{V}_i,\underline{V}_j}(v_i)$; $h^{\hat{V}_i,\overline{V}_j}(v_j)$ and $h^{\overline{V}_i,\hat{V}_j}(v_i)$ are symmetric. Moreover $h^{IJ}(v_j) = v_j$ for $IJ = \underline{V}_i\underline{V}_j, \hat{V}_i\hat{V}_j, \overline{V}_i\overline{V}_j$.

The mechanism A^{UNC} such that the seller allocates the good to i if $v_i \geqslant \max\{r_i^{IJ}(v_j), h^{IJ}(v_j)\}$ when $v_i \in I$ and $v_j \in J$ and keeps it otherwise maximizes $\hat{\mathcal{P}}$ under (F$_0$) and (F$_1$). Let $\hat{V}_i'$ be the set of types such that $H(v_i) = -\alpha_a$ and let $\Psi_i(\cdot)$ be the mapping between elements $\hat{V}_i$ into elements $\hat{V}_i'$. The mechanism A^{UNC} is candidate for optimality if (IC_2) is satisfied and $\Psi_i(\hat{V}_i) = \hat{V}_i$ for all i.

• We first show that the mechanism does not satisfy (IC_2) at $\hat{v}_i^a$ and $\hat{v}_i^b$: by inspection of $\pi_i^{IJ}(v_i, v_j)$, $\pi_j^{IJ}(v_i, v_j)$ and $\pi_j^{IJ}(v_j, v_i) - \pi_i^{IJ}(v_i, v_j)$, we have

$$h^{\underline{V}_i\cdot}(v_j) < h^{\hat{V}_i\cdot}(v_j) < h^{\overline{V}_i\cdot}(v_j), \quad r_i^{\underline{V}_i\cdot}(v_j) < r_i^{\hat{V}_i\cdot}(v_j) < r_i^{\overline{V}_i\cdot}, \quad r_j^{\underline{V}_i\cdot}(v_i) < r_j^{\hat{V}_i\cdot}(v_i) < r_j^{\overline{V}_i\cdot}(v_i)$$

implying that $H(\hat{v}_i^{a^-}) > H(\hat{v}_i^{a^+})$. The same applies at $\hat{v}_i^b$, therefore (IC_2) is not satisfied at $\hat{v}_i^a$ and $\hat{v}_i^b$.

• We now show that (IC_2) is satisfied everywhere else: $r_i^{IJ}(v_j)$ and $r_j^{IJ^{-1}}(v_j)$ cross at $\breve{v}^{IJ}$. This point is unique[23] because $\frac{d}{dv_j} r_j^{IJ^{-1}}(v_j) \geqslant -1$ and $h^{IJ}(\breve{v}^{IJ}) = r_i^{IJ}(\breve{v}^{IJ})$. When $v_i < \breve{v}^{IJ}$, i never obtains the good and j gets it if $v_j > r_j^{IJ}(v_i)$. Given the reserve price decreases in v_i, then $X_j^{A^{UNC}}(v_i', v_j) > X_j^{A^{UNC}}(v_i, v_j)$ when $v_i' > v_i$. When $v_i \geqslant \breve{v}^{IJ}$, i is allocated the good when $v_j \in (r_j^{IJ^{-1}}(v_i), h^{IJ^{-1}}(v_i))$ and j gets it if $v_j > h^{IJ^{-1}}(v_i)$. Given the properties of r_j^{IJ} and h^{IJ}, we have necessarily $X_i^{A^{UNC}}(v_i', v_j) - X_i^{A^{UNC}}(v_i, v_j) > -\left(X_j^{A^{UNC}}(v_i, v_j) - X_j^{A^{UNC}}(v_i', v_j)\right)$. Let us denote by E^{IJ} the expectation in region IJ, then $E_{v_j}^{IJ} X_i^{A^{UNC}}(v_i, v_j) - \alpha_a E_{v_j}^{IJ} X_j^{A^{UNC}}(v_i, v_j)$ increases in v_i. The same is true for all I and J and overall $H(v_i)$ increases in v_i at any point but at v_i^a and v_i^b.

• Last, as a consequence of the two previous points, $\Phi_i(\hat{V}_i) \neq \hat{V}_i$. □

Appendix 4. The procedure consists in characterizing mechanisms that satisfy (IC_2) and ($\widehat{IR}$) and are as close as possible to A^{UNC} for any pair $(\hat{V}_1, \hat{V}_2)$. We first need to introduce a few useful notations. Let $z_i(v)$ (respectively $z_j(v)$) the virtual surplus derived from selling to i (respectively j) at point v. Define:

$$Y(v) = \begin{cases} i & z_i(v) \geqslant z_j(v) \\ j & z_i(v) < z_j(v) \end{cases}$$

which assigns the identity of the winner in the unconstrained optimal mechanism with no reserve prices (it coincides with A^{UNC} whenever the good is allocated and it allocates to the bidder with highest surplus otherwise). Let

$$\hat{V}_i^k = \{(v_i^{k^-}, v_j), (v_i^{k^+}, v_j) | Y(\hat{v}_i^{k^-}, v_j) \neq Y(\hat{v}_i^{k^+}, v_j)\} \qquad k = \{a, b\}$$

the set of all points in the neighborhood of $\hat{v}_i^k$ such that the allocation to the left does not agree with the allocation to the right. Consider $\hat{V}_{12} = \hat{V}_1^a \bigcup \hat{V}_2^a \bigcup \hat{V}_1^b \bigcup \hat{V}_2^b$ and $\hat{C}$ the convex

[23] Note that $\frac{d}{dv_j} r_j^{IJ^{-1}}(v_j) = \frac{\alpha_a h}{h' - \alpha_b}$ where $h, h' = \frac{d}{dv_i}[v_i - \frac{1-F(v_i)}{f(v_i)}]$ or $\frac{d}{dv_i}[v_i + \frac{F(v_i)}{f(v_i)}]$

hull of $\hat{V}_{12}$. By construction, $\hat{C}$ contains all the points that generate violations of (IC_2) due to inconsistent allocations to agents 1 and 2 in the unconstrained mechanism. Also by construction, at any point below (respectively above) $\hat{C}$ and below (respectively above) the 45^o line, the seller always prefer to allocate the good to i (respectively to j) or to nobody. For all v_i, we define the following sets:

$$\begin{aligned}
&\hat{C}(v_i) = \Big\{v_j \,|\, (v_i, v_j) \in \hat{C}\Big\} = \Big[\hat{c}_l(v_i), \hat{c}_h(v_i)\Big] \\
&\underline{\hat{C}}(v_i) = \{v_j|(v_i, v_j) \in \hat{C}, v_i > v_j\} \qquad \overline{\hat{C}}(v_i) = \{v_j|(v_i, v_j) \in \hat{C}, v_i \leqslant v_j\} \\
&\overline{D}(v_i) = \Big\{v_j \,|\, v_j > v_i\Big\}, \qquad \overline{C}(v_i) = \overline{D}(v_i)\backslash\hat{C}(v_i) \\
&\underline{D}(v_i) = \Big\{v_j \,|\, v_j \leqslant v_i\Big\}, \qquad \underline{C}(v_i) = \underline{D}(v_i)\backslash\hat{C}(v_i)
\end{aligned}$$

$\hat{C}(v_i)$ is the set of all v_j such that v belongs to $\hat{C}$ where $\underline{\hat{C}}(v_i)$ are those below the 45^o line while $\overline{\hat{C}}(v_i)$ are those above the 45^o line, $\underline{C}(v_i)$ is the set of all v_j such that v lies below $\hat{C}$ and the 45^o line; last, $\overline{C}(v_i)$ is the set of all v_j such that v lies above $\hat{C}$ and the 45^o line. Define:

$$\begin{aligned}
&\hat{H}(v_i) = E_{\hat{C}(v_i)} X_i(v_i, v_j) - \alpha_a E_{\hat{C}(v_i)} X_j(v_i, v_j), \\
&\overline{\hat{h}}(v_i) = E_{\overline{\hat{C}}(v_i)} X_i(v_i, v_j) - \alpha_a E_{\overline{\hat{C}}(v_i)} X_j(v_i, v_j) \\
&\underline{\hat{h}}(v_i) \equiv E_{\underline{\hat{C}}(v_i)} X_i(v_i, v_j) - \alpha_a E_{\underline{\hat{C}}(v_i)} X_j(v_i, v_j) \\
&\overline{H}(v_i) = -\alpha_a E_{\overline{C}(v_i)} X_j(v_i, v_j), \qquad \underline{H}(v_i) = E_{\underline{C}(v_i)} X_i(v_i, v_j)
\end{aligned}$$

By construction $H(v_i) = \hat{H}(v_i) + \overline{H}(v_i) + \underline{H}(v_i)$ and $\hat{H}(v_i) = \overline{\hat{h}}(v_i) + \underline{\hat{h}}(v_i)$

Step 1: We construct the optimal mechanism in the complement of $\hat{C}$. We start with the values lying below the 45^o line, that is valuations in $\bigcup_{v_i} \underline{C}(v_i)$. Fix an allocation in $\hat{C}$ and in $\bigcup_{v_i} \overline{C}(v_i)$ such that $\hat{H}(v_i) + \overline{H}(v_i) < -\alpha_a$ for all $v_i \leqslant v_i^a$, and $\hat{H}(v_i) + \overline{H}(v_i) \leqslant -\alpha_a$ for all $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$ (otherwise we cannot design a feasible allocation).

• Consider $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$. Note that $v_j \in \underline{C}(v_i)$ are such that $v_j < \hat{v}_j^a$. The value of $\underline{H}(v_i)$ is uniquely defined. Given the virtual surplus is increasing in v_j, there exists a value $m(v_i)$ such that it is optimal to set $X_i(v_i, v_j) = 0$ for all $v_j < m(v_i)$ and $X_i(v_i, v_j) = 1$ for all $v_j \in \underline{C}(v_i)$ and $v_j > m(v_i)$. Formally, $\underline{H}(v_i) = F(\min(v_i, \hat{c}_l(v_i))) - F(m(v_i)) = -\alpha_a - \hat{H}(v_i) - \overline{H}(v_i)$.

• Consider $v_i \in [\underline{v}, v_i^a)$. Again, $v_j \in \underline{C}(v_i)$ are such that $v_j < \hat{v}_j^a$. In that region, $H(v_i)$ must be increasing in v_i and such that $H(v_i) < -\alpha_a$. In the unconstrained allocation, the seller gives the good to i when $v_j > r_i^{\underline{V}_i \underline{V}_j -1}(v_i)$ and to nobody otherwise. Depending on the allocation in $\hat{C}(v_i)$ and $\overline{C}(v_i)$, the seller needs to distort her preferred choices if (i) $m(v_i) > r_i^{\underline{V}_i \underline{V}_j -1}(v_i)$, because this implies $H^{\underline{V}_i \underline{V}_j}(v_i) \geqslant -\alpha_a$ and (ii) if $r_i^{\underline{V}_i \underline{V}_j -1}(v_i)$ and $m(v_i)$ cross several times, because $H^{\underline{V}_i \underline{V}_j}(v_i)$ is non monotonic in that case: the seller selects an allocation such that $H(v_i)$ increases without exceeding $-\alpha_a - \epsilon$. Given everything but the reserve price faced by i has been fixed, it is enough to look for a solution as close as possible to the curve $H(v_i)$.

Therefore, we can use the procedure in Guesnerie and Laffont [9] up to our extra constraint $H(v_i) \leqslant -\alpha_a - \epsilon$. The optimal allocation yields a piecewise weakly increasing function $H(v_i)$ that coincides with $H^{\underline{V}_i\underline{V}_j}(v_i)$ (in which case the reserve price is $r_i^{\underline{V}_i\underline{V}_j}(v_j)$) except on N disjoint intervals $[v_o^n, v_1^n]$ increasing in n where $H(v_i) = -\alpha_a - \epsilon_n$ with ϵ_n decreasing in n (in those intervals, the reserve price is slightly above $m(v_i)$).

• Consider $v_i \in (v_i^b, \overline{v}]$. In that case, $v_j \in \underline{C}(v_i)$ are such that $v_j > \hat{v}_j^a$. In that region $H(v_i)$ must be increasing and such that $H(v_i) > -\alpha_a$. Allocating the good to i when $v_j > r_i^{\overline{V}_i J^{-1}}(v_i)$ $J = \underline{V}_j, \hat{V}_j, \overline{V}_j$ does not conflict with (IC_2). The only concern is to give the item sufficiently often to make sure $H(v_i) > -\alpha_a$. Therefore, the seller may decide to give more than optimal. For each J, there exists $\tilde{r}_i^{\overline{V}_i J}(v_i) < r_i^{\overline{V}_i J^{-1}}(v_i)$ such that $X_i(v_i, v_j) = 0$ if $v_j < \tilde{r}_i^{\overline{V}_i J}(v_i)$ and $X_i(v_i, v_j) = 1$ otherwise.

• The argument is symmetric and the optimal allocation in $\bigcup_{v_i} \overline{C}(v_i) \equiv \bigcup_{v_j} \underline{C}(v_j)$ to agent j has the same properties.

Step 2: We now characterize the properties of the optimal allocation in $\hat{C}$. Fix an allocation in the complement of $\hat{C}$ as well as in $\hat{\overline{C}}(v_i)$. Let $k(v_i) = \underline{H}(v_i) + \overline{H}(v_i) + \hat{\overline{h}}(v_i)$. We restrict to allocations that are feasible: if we target a given value of $\underline{\hat{h}}(v_i)$, there exists an allocation that allows to reach that value.

• In $\hat{C}$, it is optimal to sell to i for some values and to j for others. Note first that, other things being equal, it is optimal to set $X_i(v) = 0$ and $X_j(v) \geqslant 0$ when $Y(v) = j$. Moreover, given the virtual surplus are increasing in both v_i and v_j, the benefit of allocating to either agent increases in v_j. Therefore, other things being equal, it is optimal to set $X_i(v_i, v_j) \geqslant X_i(v_i, v_j')$ and $X_j(v_i, v_j) \geqslant X_j(v_i, v_j')$ when $v_j \geqslant v_j'$. Overall, other things being equal, priority should be given to the values associated with higher surplus, which should receive the good with probability 1.

• Suppose we need to distribute a given value of $\underline{\hat{h}}(v_i)$. Given that any allocation to i is weighted by 1, whereas any allocation to j is weighted down by $-\alpha_a < 1$, it is more difficult to reach the targeted $\underline{\hat{h}}(v_i)$ by allocating to j. In particular, reaching $\underline{\hat{h}}(v_i)$ may not be possible by sticking to the optimal unconstrained allocation and the seller may have to allocate to i instead of j with some probability for some values. Note that, assuming it is optimal in the unconstrained allocation to allocate the good to j at a given point, it is best to allocate to j with probability x and i with probability $1 - x$ rather than allocating to i with probability 1. Therefore, the seller may randomize between the bidders.

• When $v_i \in (\hat{v}_i^a, \hat{v}_i^b)$, we must have $\underline{\hat{h}}(v_i) = -\alpha_a - k(v_i)$ (provided this quantity is positive), and the valuations associated with he highest surplus obtain the good up to the point $\underline{\hat{h}}(v_i) = -\alpha_a - k(v_i)$. If this point is not reached, the seller must allocate the item to i with at least some probability when it is best to allocate to j. When $v_i < \hat{v}_i^a$, the valuations associated with the highest surplus obtain the good provided $\underline{\hat{h}}(v_i)$ increases in v_i and lies strictly below $-\alpha_a - k(v_i)$. When $v_i > \hat{v}_i^b$, the valuations associated with the highest surplus obtain the good provided $\underline{\hat{h}}(v_i)$ increases in v_i and lies strictly above $-\alpha_a - k(v_i)$. The seller may also need to allocate to i with positive probability when it would be optimal to allocate to j to guarantee $\underline{\hat{h}}(v_i)$ has the required property.

Step 3: We now show that the complement of $\hat{C}$ is generically non empty, that is at equilibrium we do not have $\hat{v}_a^i = \underline{v}$ and $\hat{v}_b^i = \overline{v}$ for all $i = 1, 2$. Suppose the contrary holds. The unconstrained allocation is $X_i(v) = 1$ if $v_i > v_j$ and $v_i > r_i^{\hat{V}\hat{V}}(v_j)$. In the constrained allocation however, we must have $H(v_i) = -\alpha_a$ for all v_i.

• When $\alpha_a \to 0$, the optimal mechanism entails $\hat{v}_a^i = \underline{v}$ for all i. When α_a decreases however, it becomes costly to have an allocation such that $H(v_i) = -\alpha_a$ for low values of v_i as it requires to give the good to j and or i when this yields negative surplus. If $\hat{v}_a^i > \underline{v}$, inefficient trades when $v_i < \hat{v}_i^a$ can be avoided as we only need to satisfy $H(v_i) < -\alpha_a$. Overall, $\hat{v}_a^i > \underline{v}$ when $\alpha_a < 0$.

• Similarly, when $\alpha_a \to -1$, the optimal mechanism entails $\hat{v}_b^i = \overline{v}$ for all i. When α_a increases, it becomes costly to have an allocation such that $H(v_i) = -\alpha_a$ for high values of v_i as it requires to not allocate the good to i when this yields positive surplus. If $\hat{v}_b^i < \overline{v}$, efficient trades when $v_i > \hat{v}_i^b$ can be undertaken because we must have now $H(v_i) > -\alpha_a$. Overall, $\hat{v}_b^i < \overline{v}$ when $\alpha_a > -1$. □

Appendix B

Consider the following principal-agent problem. A seller can allocate a good to an agent, keep the good or destroy the good. The valuation of the agent is $v \in [\underline{v}, \overline{v}]$ with $\underline{v} < \overline{v}$ and it is drawn from distribution $F(\cdot)$. If the seller keeps the good, she uses it and exerts a negative externality $-\alpha_a v - \gamma < 0$ on the agent. Destroying the good does not generate any value or externality. We assume Assumptions 1 and 2 hold in this setting. Moreover, suppose that $\alpha_a \in (-1, 0)$ (to make sure we are in the case where the binding type may be interior) and $\gamma >> 0$ (to make sure that $-\alpha_a v - \gamma < 0$ for all v).

Denote by $X_1(v)$ the probability of allocating the good to the agent, $X_0(v)$ the probability of keeping the good and $t_1(v)$ the payment from the agent to the seller. The utility of the agent if he reports v' is

$$u(v, v') = vX_1(v') - \alpha_a v X_0(v') - t_1(v')$$

Incentive compatibility requires for all $v \geqslant v'$:

$$u_1(v) - u_1(v') = \textstyle\int_{v'}^{v} [X_1(s) - \alpha_a X_0(s)]ds \quad (\text{IC}_1)$$

$$X_1(v) - \alpha_a X_0(v) \geqslant X_1(v') - \alpha_a X_0(v') \qquad (\text{IC}_2)$$

Given Assumption 2, the worst outside option is obtained when the seller keeps the good and therefore, individual rationality requires $u(v) \geqslant w(v) = -\alpha_a v - \gamma$. Given $\frac{du}{dv} = X_1(v) - \alpha_a X_0(v)$ and $\frac{dw}{dv} = -\alpha_a$, there exists at most a set of types $\hat{V} = [\hat{v}^a, \hat{v}^b]$ such that $\frac{du}{dv} = -\alpha_a$ for all $v \in \hat{V}$. For any set of binding types $\hat{V}$, the expected revenue of the seller is

$$\int_{\underline{v}}^{\hat{v}^a} \Big[X_1(v)[v+\frac{F(v)}{f(v)}]-X_0(v)[\alpha_a(v+\frac{F(v)}{f(v)})+\gamma]\Big]dF(v)+\int_{\hat{v}^a}^{\hat{v}^b} \Big[X_1(v)v_1-X_0(v)[\alpha_a v+\gamma]\Big]dF(v)+$$

$$\int_{\hat{v}^b}^{\overline{v}} \Big[X_1(v)[v-\frac{1-F(v)}{f(v)}]-X_0(v)[\alpha_a(v-\frac{1-F(v)}{f(v)})+\gamma]\Big]dF(v)-F(\hat{v}^a)w(\hat{v}^a)-(1-F(\hat{v}^b))w(\hat{v}^b)-$$

$$\int_{\hat{v}^a}^{\hat{v}^b} w(v)dF(v)$$

The problem of the seller is to maximize the expected revenue under the constraints (IC_2) and $X_1(v) - \alpha_a X_0(v) = -\alpha_a$ for all $v \in \hat{V}$.

Note that all virtual surplus are increasing in v. Let $\underline{r} = \min\{v | v + \frac{F(v)}{f(v)} \geqslant 0\}$ (an interior solution satisfies $v + \frac{F(v)}{f(v)} = 0$) and $\overline{r} = \min\{v | v - \frac{1-F(v)}{f(v)} \geqslant 0\}$ (an interior solution satisfies $v - \frac{1-F(v)}{f(v)} = 0$), we have $\underline{r} \leqslant \overline{r}$ (and $\underline{r} = \underline{v}$ if $\underline{v} \geqslant 0$). Note also that $v + \frac{F(v)}{f(v)} + \alpha_a(v + \frac{F(v)}{f(v)}) + \gamma > 0$, $v + \alpha_a v + \gamma > 0$ and $v - \frac{1-F(v)}{f(v)} + \alpha_a(v - \frac{1-F(v)}{f(v)}) + \gamma > 0$. Therefore, and other things being equal, the seller prefers to give the good to the agent rather than keeping it.

Consider a mechanism such that $X_1(v) = X_0(v) = 0$ if $v < \underline{r}$ and $X_1(v) = 1$ if $v > \overline{r}$. This implies that $\hat{V} \subset (\underline{r}, \overline{r})$. Now, for all $v \in (\underline{r}, \hat{v}^a)$, the seller's revenue is maximized if she gives the good to the agent ($X_1(v) = 1$) and when $v \in (\hat{v}^b, \overline{r})$, the seller's revenue is maximized if she destroys the good ($X_1(v) = X_0(v) = 0$). This solution is not incentive compatible.

Given keeping the good is never beneficial, let us restrict the attention to solutions such that $X_0(v) = 0$. In that class, the optimal solution is $X_1(\hat{v}) = -\alpha_a - \epsilon$ for all $v_1 \in (\underline{r}, \hat{v}^a)$, $X_1(\hat{v}) = -\alpha_a$ for all $v \in (\hat{v}^a, \hat{v}^b)$ and $X_1(\hat{v}) = -\alpha_a + \delta$ for all $v \in (\hat{v}^b, \overline{r})$ where $\epsilon \to 0$ and $\delta \to 0$. The system of transfers associated to this mechanism is:

$$t(v) = \begin{cases} \gamma - \epsilon\hat{v}^a + \underline{r}(\alpha_a + \epsilon) & v \in (\underline{v}, \underline{r}) \\ \gamma - \epsilon\hat{v}^a & v \in (\underline{r}, \hat{v}^a) \\ \gamma & v \in (\hat{v}^a, \hat{v}^b) \\ \gamma + \delta\hat{v}^b & v \in (\hat{v}^b, \overline{r}) \\ \gamma + \delta\hat{v}^b + \overline{r}(1 + \alpha_a - \delta) & v \in (\overline{r}, \overline{v}) \end{cases}$$

yielding the expected revenue:[24]

$$\gamma + F(\underline{r})[\underline{r}(\alpha_a + \epsilon) - \epsilon\hat{v}^a] - \epsilon\hat{v}^a[F(\hat{v}^a) - F(\underline{r})] + \delta\hat{v}^b[F(\overline{r}) - F(\hat{v}^b)] + (1 - F(\overline{r}))[\delta\hat{v}^b + \overline{r}(1 + \alpha_a - \delta)]$$

The derivatives with respect to $\hat{v}^a$ and $\hat{v}^b$ respectively are

$$-\epsilon[f(\hat{v}^a)\hat{v}^a + F(\hat{v}^a)] \propto -\epsilon[\hat{v}^a + \frac{F(\hat{v}^a)}{f(\hat{v}^a)}] < 0$$

$$-\delta[\hat{v}^b f(\hat{v}^b) - 1 + F(\hat{v}^b)] \propto -\delta[\hat{v}^b - \frac{1 - F(\hat{v}^b)}{f(\hat{v}^b)}] > 0$$

as long as $\hat{v}^a > \underline{r}$ and $\hat{v}^b < \overline{r}$. Therefore it is optimal to set $\hat{v}^a = \underline{r}$ and $\hat{v}^b = \overline{r}$. Last, it is easy to see that distorting the allocation below $\underline{r}$ or above $\overline{r}$ would decrease the seller's revenue. Moreover, increasing $X_0(v)$ on $(\underline{r}, \overline{r})$ would require decreasing the probability of a better option (allocating the good to the agent).

Overall, in the optimal mechanism, the seller never allocates the good when $v < \underline{r}$, she always allocates it when $v > \overline{r}$ and she allocates it with probability $-\alpha_a$ when $v \in (\underline{r}, \overline{r})$. She destroys the good each time it is not allocated, except when the agent does not show up. In that case, she keeps the item and exerts the negative externality. Equilibrium payments are:

$$t(v) = \begin{cases} \gamma + \underline{r}\alpha_a & v \in (\underline{v}, \underline{r}) \\ \gamma & v \in (\underline{r}, \overline{r}) \\ \gamma + \overline{r}(1 + \alpha_a) & v \in (\overline{r}, \overline{v}) \end{cases}$$

[24]Note that the expected revenue decreases in both δ and ϵ, so these numbers must be as close as possible to 0.

References

[1] Aseff, J. and Chade, H.: An optimal auction with identity-dependent externalities. Unpublished manuscript (2006)

[2] Brocas, I.: Endogenous entry in auctions with negative externalities. Theory and Decision, **54(2)**, 125-149 (2003)

[3] Brocas, I.: Auctions with type-dependent and negative externalities: the optimal mechanism. Unpublished manuscript (2001), last revised (2009)

[4] Brocas, I.: Optimal choice of characteristics for a non-excludable good. RAND Journal of Economics, **39 (1)**, 283-304 (2008)

[5] Carrillo, J.D.: Coordination and externalities. Journal of Economic Theory **78**, 103-129 (1998)

[6] Chen, B. and Potipiti T.: Optimal Selling Mechanisms with Countervailing Positive Externalities and an Application to Tradable Retaliation in the WTO. Journal of Mathematical Economics, **46(5)**, 825-843 (2010)

[7] Engelbrecht-Wiggans, R.: Auctions and bidding models: a survey. Management Science **26**, 119-142 (1980)

[8] Figueroa N., Skreta V.: The role of optimal threats in auction design. Journal of Economic Theory, **144**, 884-897 (2009)

[9] Guesnerie, R. and Laffont, J-J.: "A complete solution to a class of principal-agent problems with an application to the control of a self-managed firm". Journal of Public Economics, **25(3)**, 329-369 (1984)

[10] Jehiel, P., Moldovanu, B., Stacchetti, E.: How (not) to sell nuclear weapons. American Economic Review **86**, 814-829 (1996)

[11] Jehiel, P., Moldovanu, B., Stacchetti, E.: Multidimensional mechanism design for auctions with externalities. Journal of Economic Theory **85**, 258-293 (1999)

[12] Jehiel, P., Moldovanu, B.: Auctions with downstream interaction among buyers. RAND Journal of Economics **31**, 768-791 (2000)

[13] Jullien, B.: Participation constraints in adverse selection models. Journal of Economic Theory **93**, 1-47 (2000)

[14] Kamien, M., Oren, S., Tauman, Y.: Optimal licensing of cost-reducing innovations. Journal of Mathematical Economics **21**, 483-508 (1992)

[15] Katz, M.L., Shapiro, C.: How to license intangible property. Quarterly Journal of Economics **101**, 567-589 (1986)

[16] Klemperer, P.: Auction theory: a guide to the literature. Journal of Economic Surveys **13**, 227-286 (1999)

[17] Lewis, T., Sappington, D.: Countevailing incentives in agency problems. Journal of Economic Theory **49**, 294-313 (1989)

[18] Maggi, G., Rodriguez, A.: On countervailing incentives. Journal of Economic Theory **66**, 238-263 (1995)

[19] Maskin, E., Riley, J.: Optimal auctions with risk averse bidders. Econometrica **52**, 1473-1518 (1984)

[20] McAfee, P., and McMillan, J.: Auctions and bidding. Journal of Economic Literature **25**, 699-738 (1987)

[21] Myerson, R.B.: Optimal auction design. Mathematics of Operation Research **6**, 58-73 (1981)

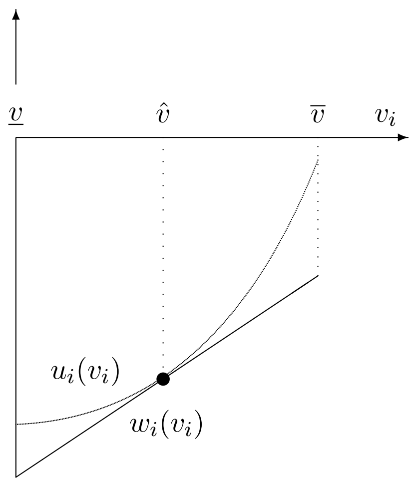

FIGURE 1: Equilibrium and reservation utilities

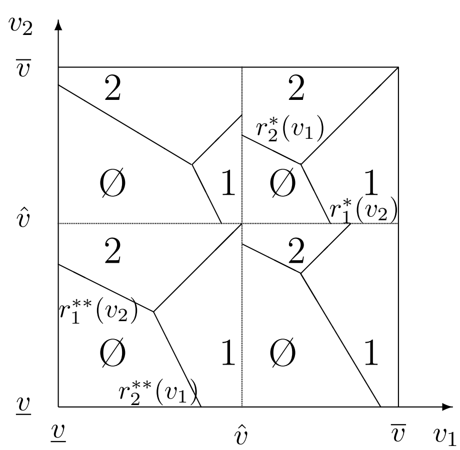

FIGURE 2: optimal unconstrained allocation
when $\alpha_a \in (-1, 0)$

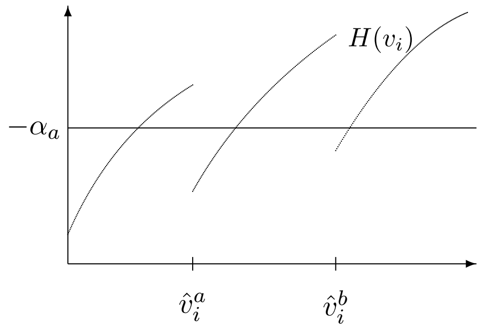

FIGURE 3: Violations of (IC_2) and ($\widehat{IR}$).

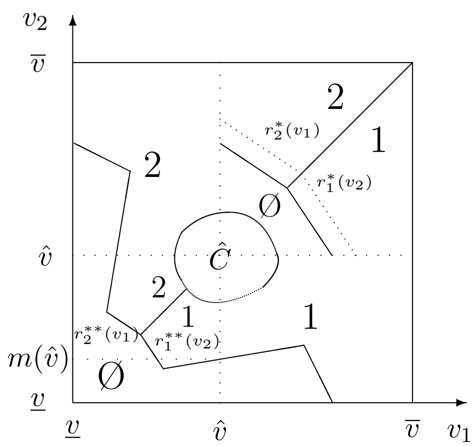

FIGURE 4: optimal allocation in the complement of $\hat{C}$