

Mathematical Appendix

This section contains some elements of calculus that are **necessary** to the understanding of the course and the solutions of the exercises. It is important to **become familiar with ALL the concepts** developed below since general functional forms will be used in the course. Note that the list of results below is not exhaustive and should not be seen as a math course for economists. In other words, using a **math book is strongly recommended**.

- A function $f(x)$ maps elements x of a set X with elements y of a set Y . We will generally work with continuous functions where $X = [\underline{x}, \bar{x}]$ (an interval) where \underline{x} is a number, possibly $-\infty$ and \bar{x} is also a number, possibly $+\infty$.
- For any element x_1 and $x_2 > x_1$ of X , we can find $f(x_1)$ and $f(x_2)$ and we can compute the ratio $\frac{f(x_2)-f(x_1)}{x_2-x_1}$. We can also always rewrite x_2 as $x_2 = x_1 + \Delta$ and the ratio becomes $\frac{f(x_1+\Delta)-f(x_1)}{\Delta}$. The first derivative of $f(\cdot)$ computed in x_1 is simply $\lim_{\Delta \rightarrow 0} \frac{f(x_1+\Delta)-f(x_1)}{\Delta}$. It is called $f'(x)$. The first derivative $f'(x)$ is a function itself, and we can apply the same methodology as before to determine how it varies with x , i.e. compute the second derivative $f''(x)$. Similarly, $f''(x)$ is a function and we can compute the third derivative, and so on. When one of the derivative is equal to a constant for all x , then the next derivatives are equal to 0 for all x . A function is linear if $f(x) = ax + b$ (a and b are constant); concave if $f''(x) \leq 0$ and convex if $f''(x) \geq 0$.
- Integration. Consider a function $f(x)$. It can be represented in a two-axes graph with a x -axis (taking the values of x , say $[\underline{x}, \bar{x}]$), and a y -axis (taking the values $f(x)$). Consider x_a and $x_b > x_a$ both in $[\underline{x}, \bar{x}]$ and suppose we want to measure area A . A rough approximation of this is $(x_b - x_a)[f(x_b) - f(x_a)]$ (the surface of a rectangle). We can decompose the interval $[x_a, x_b]$ in k intervals each of length Δ and get a much better approximation: $(x_a + \Delta - x_a)[f(x_a + \Delta) - f(x_a)] + f(x_a + 2\Delta - x_a - \Delta)[f(x_a + 2\Delta) - f(x_a + \Delta)] + \dots + f(x_b - x_a + (k - 1)\Delta)[f(x_b) - f(x_a + (k - 1)\Delta)]$. Now, by decomposing the interval in infinitesimal intervals $\Delta \rightarrow 0$, we get the precise measure we are looking for. It is simply $\int_{x_a}^{x_b} f(x)dx$. The sign \int is the equivalent of a sum, dx represents the infinitesimal length of sub-intervals.
- Maximization. Suppose $f(x)$ is concave, first increasing, then decreasing. The maximum of this function is the point x^* such that $f'(x^*) = 0$. In that case x^* is called an interior solution. If $f(x)$ is concave but increasing everywhere, then the maximum is obtained at the upper bound of X , i.e. in our simple case in \bar{x} . Here the solution is called a corner solution. We will generally have interior solutions in this course. So $\max f(x)$ will simply means look for the point x^* such that $f'(x^*) = 0$.