Lecture 21: Notes on finance

Admin: Homework 6 due next Thursday.

Textbook:

Outline:

Discounted value
Typical vs. expected returns
Pricing stocks: β and diversification
CAPM (capital asset pricing model)
Factor models
Pricing derivatives
Black-Scholes formula
Binomial lattice model for stock prices
Exotic options

How to Price Assets

Efficient market hypothesis:
(If there's a market) Look up the price, and
that price will be correct.

"The efficient market hypothesis is the most remarkable
error in the history of economic theory" —Robert Shiller

Present, discounted value

Forget bonds, stocks, options — what is the value of cash?

Principle: Money today > money tomorrow.

Problem: What is the value today of $100 one year from now?
\[ \frac{100}{1 + r} \], where \( r \) is the "risk-free interest rate"

**What is \( r \)?**

It depends on who you are (individual, company, bank, government) on the duration on the application, e.g., we sometimes assume that you can lend or borrow at the same rate \( r \).

**Examples:** (From Vanguard, but you can get similar and more detailed) information from any finance site.

<table>
<thead>
<tr>
<th>CDs</th>
<th>1 month</th>
<th>3 month</th>
<th>6 month</th>
<th>9 month</th>
<th>1 year</th>
<th>18 month</th>
<th>2 year</th>
<th>3 year</th>
<th>5 year</th>
<th>7 year</th>
<th>10 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>New issue CDs</td>
<td>0.20%</td>
<td>0.40%</td>
<td>0.50%</td>
<td>0.50%</td>
<td>0.60%</td>
<td>0.75%</td>
<td>1.05%</td>
<td>1.45%</td>
<td>2.15%</td>
<td>2.70%</td>
<td>3.20%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Other securities</th>
<th>1 year</th>
<th>2 year</th>
<th>3 year</th>
<th>5 year</th>
<th>7 year</th>
<th>10 year</th>
<th>20 year</th>
<th>30 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treasuries</td>
<td>0.15%</td>
<td>0.55%</td>
<td>1.03%</td>
<td>1.70%</td>
<td>2.07%</td>
<td>2.36%</td>
<td>2.84%</td>
<td>3.08%</td>
</tr>
<tr>
<td>Agencies</td>
<td>0.35%</td>
<td>0.89%</td>
<td>1.39%</td>
<td>1.94%</td>
<td>2.07%</td>
<td>3.30%</td>
<td>3.91%</td>
<td>4.04%</td>
</tr>
<tr>
<td>Municipals (AAA)</td>
<td>0.50%</td>
<td>0.93%</td>
<td>1.06%</td>
<td>1.61%</td>
<td>2.30%</td>
<td>3.03%</td>
<td>3.72%</td>
<td>3.90%</td>
</tr>
<tr>
<td>Municipals (AA)</td>
<td>0.53%</td>
<td>0.99%</td>
<td>1.25%</td>
<td>1.92%</td>
<td>2.71%</td>
<td>3.56%</td>
<td>4.51%</td>
<td>4.10%</td>
</tr>
<tr>
<td>Municipals (A)</td>
<td>0.65%</td>
<td>1.08%</td>
<td>1.60%</td>
<td>2.39%</td>
<td>3.15%</td>
<td>3.72%</td>
<td>4.51%</td>
<td>4.75%</td>
</tr>
<tr>
<td>Municipals (BBB)</td>
<td>1.22%</td>
<td>2.73%</td>
<td>1.70%</td>
<td>4.00%</td>
<td>4.30%</td>
<td>4.60%</td>
<td>4.92%</td>
<td>4.62%</td>
</tr>
<tr>
<td>Corporates (AAA)</td>
<td>0.33%</td>
<td>0.61%</td>
<td>1.18%</td>
<td>1.88%</td>
<td>2.36%</td>
<td>3.19%</td>
<td>3.77%</td>
<td>4.04%</td>
</tr>
<tr>
<td>Corporates (AA)</td>
<td>0.68%</td>
<td>1.18%</td>
<td>1.75%</td>
<td>2.50%</td>
<td>3.35%</td>
<td>3.70%</td>
<td>4.60%</td>
<td>4.39%</td>
</tr>
<tr>
<td>Corporates (A)</td>
<td>0.92%</td>
<td>1.44%</td>
<td>1.90%</td>
<td>2.85%</td>
<td>3.55%</td>
<td>4.09%</td>
<td>5.55%</td>
<td>4.79%</td>
</tr>
<tr>
<td>Corporates (BBB)</td>
<td>1.06%</td>
<td>1.54%</td>
<td>2.39%</td>
<td>3.87%</td>
<td>4.63%</td>
<td>5.02%</td>
<td>6.76%</td>
<td>6.38%</td>
</tr>
</tbody>
</table>

**Remark:** Higher annual yields for lower-rated securities reflect higher risks of default. Higher annual yields over longer time periods, even for US government treasuries, reflects risks of inflation...

**TYPICAL VS. EXPECTED RETURNS**

Investing is not a one-shot game.

**Example:** Double-or-quarter gambling investing

\[
Z_n = \overbrace{X_1, X_2, \ldots, X_n}^{i.i.d.} \sim \begin{cases} 
\frac{2}{4} & \text{with prob } \frac{3}{2} \\
\frac{4}{1} & \text{with prob } \frac{1}{2}
\end{cases} \quad \mathbb{E}[X_i] = \frac{9}{8} = \mu
\]

\[ \Rightarrow \mathbb{E}[Z_n] = \left(\frac{9}{8}\right)^n \quad \text{— a great investment!} \]
But typically, \( \frac{n}{2} = O(n) \) \( X_i \)'s will be 2, and
\[ \frac{n}{2} = O(n) \] will be \( \frac{1}{4} \).

\[ \Rightarrow \text{typically, } Z_n \approx 2^{\frac{n}{4}} \left( \frac{1}{4} \right)^{\frac{n}{4}} = \frac{1}{2^{\frac{n}{4}}} \]

-- a terrible investment!

What's going on?

If your goal is to maximize your expected wealth, this is a great investment.

But most investors have a degree of risk aversion, and care more about typical outcomes.

A way of modeling this is to assume a logarithmic utility function, so the goal is to maximize \( E[\log Z_n] \) instead of \( E[Z_n] \). Also, even if you don't want to think in terms of utility, considering \( E[\log Z_n] \) has the technical advantage that it allows us to apply the Law of Large Numbers to study typical results.

Question: If you have $1 to invest, how much should you gamble?

If you gamble \( \lambda \) fraction, \( 0 \leq \lambda \leq 1 \), always reallocating,
\[ Z_n(\lambda) = \frac{1}{n} \sum_{i=1}^{n} X_i(\lambda) \]
\[ X_i(\lambda) = \begin{cases} (1-\lambda) + 2\lambda & \text{w/ prob } \frac{1}{2} \\ (1-\lambda) + \frac{1}{2} & \text{w/ prob } \frac{1}{2} \end{cases} \]

\[ \Rightarrow \log Z_n(\lambda) = \sum_{i=1}^{n} \log X_i(\lambda) \]

\[ \frac{1}{n} \log Z_n(\lambda) \xrightarrow{n \to \infty} E[\log X_i(\lambda)] \]

by Law of Large Numbers

\( E[\log X_i] = \frac{1}{2} \log (1+\lambda) + \frac{1}{2} \log (1-\frac{3}{4}\lambda) \)

\[ 0 = A \cdot E[\log X_i] \approx \frac{1}{1+\lambda} - \frac{3}{1-\frac{3}{4}\lambda} \]

\[ (1-\frac{3}{4}\lambda) - \frac{3}{4}(1+\lambda) \]

\[ = \frac{1}{4} - \frac{3}{4}(1+\lambda) \]

\[ \Rightarrow \lambda = \frac{1}{6} \]
\[ \Rightarrow E[\log X] = \frac{1}{2} \log \frac{49}{48} \approx 0.01 \]

\[ \log Z_n = \frac{n}{2} \log \frac{49}{48} \text{ typically} \]

\[ \Rightarrow Z_n \approx e^{\frac{n}{2} \log \frac{49}{48}} \approx e^{0.01n} \approx 1.01^n \]

This is much worse than the achievable expectation, \((\frac{3}{4})^n = 1.125^n\), but it is much safer!

You can (almost) maximize your expected wealth and (almost) maximize your typical wealth, at the same time, by splitting your money in half between the two investment strategies, at the very beginning, and not reallocating this split.

Then
\[ E[Z_n] = \frac{1}{2} \left(\frac{3}{4}\right)^n \] from the \(\frac{1}{2}\) that's all in

\[ E[\log Z_n] = \frac{n}{2} \log \frac{49}{48} \] from the \(\frac{1}{2}\) that's dynamically reallocated.

**Moral**: If you care more about typical outcomes than the extremes, or if you are otherwise risk-averse, then you should choose your investments at each time step to maximize the expected rate of return

i.e., \(E[\log X]\)

and not try to maximize the expected wealth \(E[X]\).

Of course often, especially when we are looking over small time increments, we might argue \(P[X-1 > E] = 0\). Since \(\log(x) \approx x - 1\) for \(x \approx 1\), then these two perspectives are roughly equivalent.

**Capital Asset Pricing Model (CAPM)**

**Basic model** for portfolio diversification

\[ \text{n assets} \]

\[ \text{random rates of return} \quad R_i = \frac{X_i}{X_{i,0}} - 1 \]

\[ \text{Let} \quad \gamma_i = E[R_i] \quad \text{and} \quad \sigma_i^2 = \text{Cov}(R_i, R_j). \]

\[ \Rightarrow \text{In a portfolio with asset allocations} \ (\alpha_1, \ldots, \alpha_n), \ \sum \alpha_i = 1. \]
\( X_{0,t} = \alpha_t \),
\[
X_t = \frac{X_t}{X_0} = \sum_{\delta} X_{\delta,t} = \sum_{\delta} \frac{X_{\delta,t}}{X_0} \cdot \alpha_{\delta}
\]
\[
= \sum_{\delta} (R_{\delta} + 1) \alpha_{\delta} = 1 + \sum_{\delta} \alpha_{\delta} R_{\delta}
\]
\[
\Rightarrow R = \frac{X_t}{X_0} - 1 = \sum_{\delta} \alpha_{\delta} R_{\delta}
\]
has expectation and variance
\[
\mathbb{E}[R] = \sum_{\delta} \alpha_{\delta} \mu_{\delta}
\]
\[
\text{Var}(R) = \sum_{\delta} \alpha_{\delta} \sigma_{\delta}^2
\]

**Problem:** For a given return \( r_t \), find the minimum-variance portfolio.

minimize \( \sum_{\delta} \alpha_{\delta} \sigma_{\delta}^2 \)

subject to \( \sum_{\delta} \alpha_{\delta} = 1 \)
\[
\sum_{\delta} \alpha_{\delta} \geq r.
\]
\[
L = \sum_{\delta} \alpha_{\delta} \mu_{\delta} - \lambda (\sum_{\delta} \alpha_{\delta} - 1) - \mu (\sum_{\delta} \alpha_{\delta} - 1)
\]
\[
\Rightarrow \text{Need to solve the system of linear equations}
\]
\[
\begin{cases}
\sum_{\delta} \alpha_{\delta} \mu_{\delta} = r \\
\sum_{\delta} \alpha_{\delta} = 1 \\
-\lambda (\sum_{\delta} \alpha_{\delta} - 1) = 0
\end{cases}
\]

**Example:**

1. Read in financial data, compute estimates for monthly returns and covariance matrix.
```
aapl = FinancialData["AAPL", "Jan. 1, 2008"];
ibm = FinancialData["IBM", "Jan. 1, 2008"];
msft = FinancialData["MSFT", "Jan. 1, 2008"];
td = TemporalData[{aapl, ibm, msft, spx500}];
DateListPlot[td["Paths"], PlotRange -> Automatic,
            PlotStyle -> {Red, Green, Blue, Black}]
td["SliceData", "May 24, 2009"]
Mean /@ td["States"]
```

rateintervaldays = 21;
rates = ConstantArray[0, Dimensions[prices] - {0, rateintervaldays}];
"The rate data is 21 days shorter than the price data, because we compute the rate of return between days."

For[company = 1, company <= Length[prices], company++,
   For[day = 1, day <= Length[prices[[company]]] - rateintervaldays, day++,
      rates[[company, day]] = (prices[[company, day + rateintervaldays]] / prices[[company, day]]) - 1;
   ];
]

averagerates = Map[1/Length[#] Plus @@@ #, rates]
covariances = Covariance[rates // Transpose]
(Diagonal[covariances], averagerates) // Transpose // MatrixForm

{{0.0244186, 0.00981043, 0.0081119, 0.00571022}
 {0.00969944, 0.0027147, 0.0035068, 0.00321498},
 {0.0027147, 0.00342182, 0.00152332, 0.00207787},
 {0.0035068, 0.00152332, 0.00567059, 0.00244974},
 {0.00321498, 0.00207787, 0.00244974, 0.00285552}}

{0.00969944, 0.0244186
  0.00342182 0.00981043
  0.00467059 0.0081119
  0.00285552 0.00571022}
(2) Set up and solve the linear equations:

\[
\text{SetupEquations}[\text{rates}, \text{covariances}] := \text{Module}[\{n, A, b, i, j\},
\]
\[
\text{n = Length}[\text{rates}];
\]
\[
\text{A} \equiv \text{ConstantArray}[0, \{n+2, n+2\}];
\]
\[
\text{"Start with the equations sum}_j \text{ sigma}_{i j} \text{ alpha}_j - r_i \text{ lambda}_i -
\]
\[
\text{mu} = 0, \text{ for all } i;\]
\[
\text{For}[i = 1, i \leq n, i++,
\]
\[
\text{A}[i, j] = \text{covariances}[i, j];
\]
\[
\text{A}[i, n + 1] = -\text{rates}[i];
\]
\[
\text{A}[i, n + 2] = -1;
\]
\[
\text{];}
\]
\[
\text{"This is the equation sum}_j \text{ alpha}_j = 1";
\]
\[
\text{For}[i = 1, i \leq n, i++,
\]
\[
\text{A}[n + 1, i] = 1;
\]
\[
\text{];}
\]
\[
\text{"This is the equation sum}_j r_j \text{ alpha}_j = x";
\]
\[
\text{For}[i = 1, i \leq n, i++,
\]
\[
\text{A}[n + 2, i] = \text{rates}[i];
\]
\[
\text{];}
\]
\[
\text{b} = \text{Table}[0, \{n + 2\}];
\]
\[
\text{b}[n + 1] = 1;
\]
\[
\text{b}[n + 2] = x;
\]
\[
\text{(A, b)}
\]
\[
\text{];}
\]
\[
(A, b) = \text{SetupEquations}[\text{averagerates}, \text{covariances}];
\]
\[
\text{A} // \text{MatrixForm}
\]
\[
\text{b}
\]
\[
\begin{bmatrix}
0.00969944 & 0.0027147 & 0.0035068 & 0.00321498 & -0.0244186 & -1 \\
0.0027147 & 0.00342192 & 0.0052332 & 0.00257887 & -0.00981043 & -1 \\
0.0035068 & 0.00152332 & 0.00467059 & 0.00244974 & -0.0081119 & -1 \\
0.00321498 & 0.00207787 & 0.00244974 & 0.00285652 & -0.00571022 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 \\
0.0244186 & 0.00981043 & 0.0081119 & 0.00571022 & 0 & 0
\end{bmatrix}
\]
\[
\{0, 0, 0, 1, 2\}
\]
Remark (Two-fund theorem): All points on the efficient frontier can be generated by linear combinations of two fixed portfolios.

Proof:
If you take any two solutions \((x, \lambda, \mu)\) and \((x', \lambda', \mu')\) to the equations

\[
\begin{align*}
\sum_{i} x_i &= 1 \\
-x_i - \lambda + \sum_{j} \sigma_{ij} x_j &= 0 \quad \forall i,
\end{align*}
\]

then a linear combination \(p(x, \lambda, \mu) + (1-p)(x', \lambda', \mu')\) is also a solution. If the solutions achieve different expected returns \(r = x \cdot \omega\) and \(r' = x' \cdot \omega'\), then the linear combination achieves

\[
p \cdot r + (1-p) r',
\]

which varies over all rates when you vary \(p\). \(\square\)
Model with a risk-free asset

4) Now add a risk-free asset (US government Treasuries)

\[
\text{averagerates} = \text{Prepend}[\text{averagerates}, .001];
\]
\[
\text{covariances} = \text{ArrayFlatten}[\{(0, 0), (0, \text{covariances})\}];
\]
\[
\text{covariances} // \text{MatrixForm}
\]
\[
\begin{bmatrix}
0 & 0.00963944 & 0.0027147 & 0.0035068 & 0.00321498 \\
0.0027147 & 0.00342182 & 0.00152332 & 0.00207787 \\
0.0035068 & 0.00152332 & 0.00467059 & 0.00244974 \\
0.00321498 & 0.00207787 & 0.00244974 & 0.00285652
\end{bmatrix}
\]
\[
\{A, b\} = \text{SetUpEquations}[\text{averagerates}, \text{covariances}];
\]
\[
\text{solution} = \text{LinearSolve}[A, b][3];
\]
\[
\text{averagerates}.\text{solution} // \text{Simplify}
\]
\[
\text{solution}(\text{covariances}.\text{solution}) // \text{Simplify}
\]
\[
\text{riskfreeportfolio} = \text{ParametricPlot}\left[\left[\sqrt{\vec{\alpha}}, \vec{\alpha}\right], \{\vec{r}, 0, 1\}, \text{PlotStyle} \rightarrow \text{Red}\right];
\]
\[
\text{Show}[\text{dotplot}, \text{portfolio}];
\]
\[
\text{riskfreeportfolio} // \text{Simplify}
\]
\[
\begin{align*}
-2.1664 \times 10^{-18} + 1. \vec{r}; \\
0.0000136238 - 0.0276476 \vec{r} + 13.8238 \vec{r}^2
\end{align*}
\]

Observe: The new efficient frontier, maximizing the expected return \( r \) for every risk level \( \sigma \), is a straight line, passing through the risk-free asset and tangent to the old efficient frontier.

This has to happen ("One-fund theorem").

Easy proof: A combination \( p (0, r_0) + (1-p)(\overrightarrow{c}, r) \)
achieves return \( p r_0 + (1 - p) \times r \).

and standard deviation \((1 - p) \times \sigma_r\).

This gives a straight line, and the tangent line is the highest possible.

**Corollary:** Every investor's portfolio is a mixture of the riskless asset and the same combination of risky assets!

**Corollary:** There is no need to run these optimizations; just follow the market portfolio:

\[
\text{proportion to invest in asset } i = \frac{\text{total value of asset } i}{\text{total value of all assets}}.
\]

(For example, use the S&P 500 index.)

**Further consequences for asset pricing**

With some more calculations, one can argue that

\[
\frac{r_i - r_0}{\text{market } r_0} = \frac{\sigma_{i,M}}{\sigma_M} \quad \text{called } \beta_i.
\]

A higher \( \beta_i \) implies a higher expected rate of return.

(The risk \( r_i \) does not, because nonsystematic risk can be diversified away.)

In practice, \( \beta_i \) can be estimated, then used to back out a price for an asset:

\[
\text{price } X_0 = \frac{\text{EC}_X}{1 + r_0 + \beta(r_m - r_0)}
\]

\[
= \frac{\text{EC}_X - \text{Cov}(X_1, R_M)(r_m - r_0)}{1 + r_0}.
\]

**Lessons of CAPM:**

- Risk is not reworded, only non-diversifiable risk \( \beta \) (AKA systematic or market risk).
- Diversification is good, and index funds (that track the market) diversify in the correct proportion.

**Problems:**

- Data intensive: Implementing the model on \( n \) assets requires estimating \( n^2 \) parameters.
The optimization can be avoided, but only assuming an equilibrium situation.

One-period, discrete time

Remark: "Factor models" reduce the data required.

Assume

$$R_i = \sum_{j=1}^{k} b_{ij} F_j + \epsilon_i$$

(error uncorrelated with anything)

(correlated factors, such as stock market return, interest rates, inflation, unemployment & other macroeconomic factors)

Example: With a single market return factor,

$$R_i - r_0 = b_i (R_m - r_0) + \epsilon_i$$

Taking $\text{Cov}(\epsilon_i, R_m)$ on both sides,

$$b_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}.$$  

So taking expectations,

$$r_i - r_0 = b_i (r_m - r_0) + \text{E}[\epsilon_i].$$

If $\text{E}[\epsilon_i] = 0$, this is the same as the CAPM pricing formula we asserted above.

**Binomial Lattice Model for Stock Prices**

Binomial lattice model

![Binomial lattice diagram]
\[ P[L^\theta = \alpha \sigma(n-\theta) S_0] = \left( \begin{array}{c} n \\ \hat{\theta} \end{array} \right) \rho^{\hat{\theta}} (1-\rho)^{n-\hat{\theta}} \]

**Remarks:**

1. The three parameters \( \mu, \sigma, \rho \) are chosen to match estimated expected rates of return and variance for the stock.

For example, geometric BM. \( S(t) = e^{\mu t + \sigma B(t)} \)

\[ E[S(t+\delta t)|S(t)] = S(t) e^{(\mu + \frac{\sigma^2}{2}) \delta t} \]

\[ \text{Var}(S(t+\delta t)|S(t)) = (e^{\sigma^2 \delta t} - e^{-\sigma^2 \delta t}) e^{2\mu \delta t} \]

This gives 2 equations for 3 parameters.

2. Lattice with \( ud = 1 \)

3. Lattice with \( p = \frac{1}{2} \)

\( ud = du \quad \text{so the lattice has size only} \quad O\left( \frac{1}{\sqrt{\delta t}} \right), \quad \text{instead of} \quad 2^{T/\delta t}. \)

4. It is important that there are only two places the stock price can go (we’ll see later).

**Stock options**


**European call option:**

At time \( T \), can buy one share at price \( K \),

value at time \( T = \max(S(T)-K, 0) \).

**European put option:**

At time \( T \), can sell one share at price \( K \),

value at time \( T = \max(K-S(T), 0) \).

Other options might use different functions of \( S(T) \)

their value at time \( T \) or even have path-
There is a whole zoo of options that I won’t attempt to summarize here.

**Goal:** What is the value at time $0$?

**Principle:** We know the price of a bond (future cash), so use stock + options to simulate a bond. (Alternatively: use stock + bond to simulate an option.)

**Simpler example:** "Put-call parity"

$$S + P - C = Ke^{-rT}$$

Why? At time $T$, the value of one stock, one put and -1 calls is

$$S(T) + \max(0, K - S(T)) - \max(0, S(T) - K) = K,$$

regardless of whether $S(T) > K$ or $S(T) < K$. => At time $0$, the same portfolio is worth $Ke^{-rT}$.

**Assume:** Also that we can borrow or lend a risk-free asset at a rate $r$. (Notice: $d < 1 + r < u$)

**Inductive argument:**

---

**Induction step:**

Assume the value is $Cu$ if the stock goes up

$Cd$ if it goes down.
Consider a portfolio (a stock, a risk-free asset)
Its value in 1 time step is
\[
\begin{cases}
    \alpha u S_0 + \beta (1+r), & \text{if stock goes up} \\
    \alpha d S_0 + \beta (1+r), & \text{if it goes down}
\end{cases}
\]

Now solve for \((\alpha, \beta)\) so
\[
\begin{align*}
    \alpha u S_0 + \beta (1+r) &= C_u \\
    \alpha d S_0 + \beta (1+r) &= C_d.
\end{align*}
\]

Two independent equations, two unknowns \(\Rightarrow\) there is a solution!
\(\Rightarrow\) value now is \(C_0 = \alpha S_0 + \beta\).

**Remark:** The solution is given by
\[
\begin{align*}
    \alpha &= \frac{C_u - C_d}{S_0 (u - d)} \\
    \beta &= \frac{u C_u - d C_d}{(1+r)(u - d)}
\end{align*}
\]
\[\Rightarrow C_0 = \frac{1}{1+r} \left( p^* C_u + (1-p^*) C_d \right) \]
where \(p^* = \frac{1+r-d}{u-d},\) \(1-p^* = \frac{u-(1+r)}{u-d}\)
are probabilities.
\[\Rightarrow C_0 = \frac{1}{1+r} \mathbb{E}^x \left[ C_i \right] \]

The price does not depend on \(p\)!

**Remark:** Observe that
\[
S_0 = \frac{1}{1+r} \mathbb{E}^x \left[ S_i \right] = \frac{1}{1+r} \left( p^* u + (1-p^*) d \right) S_0 \checkmark
\]
Therefore \(p^*\) is called the "risk-neutral probability."

By induction, we conclude
\[C_0 = \frac{1}{(1+r)^T} \mathbb{E}^x \left[ C_T \right] \]
\[= \frac{1}{(1+r)^T} \sum_{T=0}^{T} \left( (1+r)^T \right) (p^*)^{T-j}(1-p^*)^{j} C_T (u^j d^{T-j} S_0) \]

**Continuous-time Black-Scholes formula**

Assume the stock price follows geometric Brownian motion:
\[S(t) = S_0 \exp \left( \mu t + \sigma B(t) \right)\]
Let \( \mu^* \) be such that

\[
S_0 = e^{-rT} \mathbb{E}^*[S(T)]
\]

i.e.,

\[
e^{-rT} = \mathbb{E}^*[e^{\mu^* t + \sigma B(t)}]
= e^{\mu^* t} \mathbb{E}^*[e^{\sigma B(t)}] = e^{(\mu^* + \frac{\sigma^2}{2})t}
\]

\[\Rightarrow \mu^* = r - \frac{\sigma^2}{2}.\]

Then the fair present value of an option that at time \( T \) pays \( f(S(T)) \) is given by

\[
e^{-rT} \mathbb{E}^*[f(S(T))]
= e^{-rT} \int_{-\infty}^{\infty} f(e^{(r - \frac{\sigma^2}{2})T + \sigma B}) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2}} db.
\]

We will prove this formula, and explain more of the assumptions behind it, after we have developed stochastic calculus.

Remark: A problem for using Black-Scholes to determine option prices is that the stock volatility \( \sigma \) is unknown. In fact, people often back out an implied volatility based on the option prices. Also, when quoting option prices a dealer might state prices in terms of the volatility, so that a customer has time to decide what to buy, even though the stock price changes constantly.