Admin: Homework 11 problem 2 — don’t use Matlab!

Let $A$ be a Hermitian matrix.

Ways to show that $A \succeq 0$
- compute all $e$-values, check $\geq 0$
  (lame)
- write $A = B + C$ with $B \succeq 0$
  (or $B + C + D$, etc)
- show that $x^T A x \geq 0$ for all $x$

Ways to show $A \not\succeq 0$
- find an $e$-value $< 0$
- find $x$ so $x^T A x < 0$
- show that a submatrix of $A$ is not $\succeq 0$
- check $\det(A) < 0$

Maybe also consider $\text{rank}(A)$, ...

Today: Quantum mechanics

Morals: Matrix exponentiation & eigenvectors to solve differential equations

Relationship between symmetric/Hermitian and orthogonal/unitary matrices

Commuting matrices and time-dependent differential equations

Tensor products to build bigger vector spaces

What is Quantum Mechanics?
Quantum mechanics generalizes probability theory:

**Probability theory**
\[
\begin{pmatrix}
p_1 \\
p_2
\end{pmatrix} \in \mathbb{R}^2
\]
\[p_j \geq 0, \quad p_1 + p_2 = 1\]

**Quantum mechanics**
\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix} \in \mathbb{C}^2
\]
\[|\alpha_1|^2 + |\alpha_2|^2 = 1\]

**Definition:** **Quantum Mechanics**

- **System** ↔ **Vector space (over \( \mathbb{C} \))**
- **State of the system** ↔ **Vector \( \psi \) with \( \| \psi \| = 1 \)**
State of the system $\rightarrow$ Vector $\Psi$ with $\|\Psi\| = 1$

Energy/"Hamiltonian" $\rightarrow$ Hermitian matrix $H$

- its real eigenvalues are
- discrete energy levels

$\text{energy}(\Psi) = \Psi^\dagger H \Psi$

Observe: The "energy" is real:

$(\Psi^\dagger H \Psi)^\dagger = \Psi^\dagger H^\dagger \Psi$

$= \Psi^\dagger H \Psi$ since $H^\dagger = H$.

Dynamics $\rightarrow$ "Schrödinger's equation"

$\frac{d}{dt} \Psi(t) = -i H \Psi(t)$

Solution: $\Psi(t) = e^{-i Ht} \Psi(0)$

Observe: $H$ can be diagonalized as

$H = U \Lambda U^\dagger$

where $U$ is unitary and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are real eigenvalues.

$\Rightarrow e^{-i Ht} = U \left( e^{-i\lambda_1 t} \ 0 \ \ 0 \ \ e^{-i\lambda_n t} \right) U^\dagger$

1. If $\Psi$ is an eigenvector, $H \Psi = \lambda \Psi$, $e^{-i Ht} \Psi = e^{-i\lambda t} \Psi$

$\Rightarrow$ energy eigenvectors are static, except for accumulating a phase.

2. $(e^{-i Ht})(e^{-i Ht})^\dagger = e^{-i Ht} e^{i Ht}$
\[
\begin{align*}
&= U \begin{pmatrix} e^{i\lambda t} & \vdots & e^{i\lambda nt} \\ \vdots & \ddots & \vdots \\ e^{-i\lambda nt} & \vdots & e^{-i\lambda t} \end{pmatrix} U^+ \\
&= UU^+ \\
&= I
\end{align*}
\]

\[\Rightarrow \text{time evolution is unitary} \]
\[\text{(it preserves vector lengths)}\]

**Theorem:**

Matrix \( A \) is unitary \( \iff A = e^{iH} \) for some Hermitian matrix \( H \).

**Proof:**

We just showed that \( e^{iH} \) is unitary for any Hermitian matrix \( H \).

For the converse statement: Assume \( A \) is unitary.

Diagonalize it

\[ A = U \begin{pmatrix} e^{i\alpha_1} & 0 & \cdots & 0 \\ 0 & e^{i\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\alpha_n} \end{pmatrix} U^+ \]

for real angles \( \alpha_1, \ldots, \alpha_n \)

\[= \exp \left( i \cdot U \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} U^+ \right) \]

\[= \exp \left( i \cdot \text{a Hermitian matrix} \right) \checkmark \]

**Example:** Vector space \( \mathbb{C}^2 \)

\[ H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{real symmetric} \quad \Rightarrow \text{Hermitian} \]

Q: If \( \varphi(t=0) = (0) \), what is \( \varphi(t=\pi/2) \)?

**Answer:**

\[
\begin{array}{c|c}
\text{E-values} & \text{E-vectors} \\
\hline 
\alpha_1 = 1 & \tilde{v}_1 = \frac{1}{\sqrt{2}} (1) \\
\alpha_2 = -1 & \tilde{v}_2 = \frac{1}{\sqrt{2}} (1) \\
\end{array}
\]
\[ H = C_1 v_1 v_1^† + C_2 v_2 v_2^† \]

\[ e^{-iHt} = e^{-iC_1 t} v_1 v_1^† + e^{-iC_2 t} v_2 v_2^† \]

\[ = \frac{e^{-it}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \frac{e^{it}}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} \cos t & -i \sin t \\ -i \sin t & \cos t \end{pmatrix} \]

\[ \quad \Rightarrow \quad \Phi(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Phi(t) = \begin{pmatrix} \cos t \\ -i \sin t \end{pmatrix} \]

\[ \quad \Rightarrow \quad \Phi(\pi/2) = \begin{pmatrix} 0 \\ i \end{pmatrix} \]

**Remark:** Many physical systems are infinite dimensional. To track an electron through space, you need a dimension for every point in \( \mathbb{R}^3 \)!

**Exact diagonalization of Hamiltonians** is often impossible, and we need to rely on approximations and perturbation theory.

**Perturbation theory:** If we understand the spectrum of \( A \), and \( \|B\| \ll \|A\| \), what can we say about the spectrum of \( A+B \)?

**Simultaneous Diagonalizability of Matrices**

"Normal" matrices are nice because they can be unitarily diagonalized.

\[ A \text{ normal} \Rightarrow \text{in some orthonormal basis,} \]

\[ A = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \]

This makes matrix operations easy.

\[ A^{\text{diag}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \text{ in this basis} \]
\[ A^{00} = \begin{pmatrix} \lambda_1^{00} & 0 \\ 0 & \lambda_2^{00} \end{pmatrix} \text{ in this basis} \]

\[ e^{A^t} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} \Rightarrow \text{trivial to solve differential equations} \quad \frac{d}{dt} \vec{v} = A \vec{v} \]

\[ A \vec{x} = \vec{b} \text{ is solved by} \]
\[ \vec{x} = \sum_{s=1}^{n} \frac{1}{\lambda_s} (\vec{v}_s \cdot \vec{b}) \vec{v}_s \]  
(since \( \vec{b} = \sum_{s} (\vec{v}_s \cdot \vec{b}) \vec{v}_s \))

\[ \text{Rank}(A) = \# \text{ of nonzero } \lambda_s \]
\[ \text{Det}(A) = \lambda_1 \lambda_2 \cdots \lambda_n \]

etc...

But what if you want to work with two matrices, at the same time?

**Examples:** (from physics)

Total energy = Kinetic energy + Potential energy
(a Hermitian matrix)

Energy of two-particle system = Energy of 1st particle + Energy of 2nd particle + Interaction energy

**Problem:**

Say that matrices \( A \) and \( B \) are both unitarily diagonalizable.

When are they diagonalizable using the same unitary?

Equivalently, when do they share a basis of eigenvectors?

**Definition:** Matrices \( A \) and \( B \) commute if
\[ AB = BA. \]

**Examples:**

- Any two \( n \times n \) diagonal matrices commute:
  \[ A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \]
  \[ \Rightarrow AB = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}, \quad BA = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix} \]

- All matrices commute with the identity:
  \[ IA = AI = A \]

- Not all matrices commute:
  \[ A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
  \[ AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

**Theorem:** For normal matrices \( A \) and \( B \), there is an orthonormal basis of eigenvectors for \( A \) and \( B \):

\[ AB = BA. \]

**Proof:**

**Easy direction \( \Rightarrow \):**

Assume that \( A \) and \( B \) are diagonalized by the same unitary \( U \):

\[ A = UD_A U', \]
\[ B = UD_B U'. \]

with \( D_A \) and \( D_B \) diagonal.

\[ \Rightarrow AB = (UD_A U')(UD_B U') \]
\[ = UD_A D_B U' \]
\[ BA = UD_B D_A U'. \]

Since \( D_A \) and \( D_B \) are diagonal, \( D_A D_B = D_B D_A \), so \( AB = BA. \)
Hard direction $\Leftarrow$:
Assume that $A$ and $B$ commute: $AB = BA$.

Work in a basis so that $A$ is diagonal:

$$A = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_k \end{pmatrix}$$

with $\lambda_1, \ldots, \lambda_k$ distinct eigenvalues

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \cdots \\ B_{k1} & B_{k2} & \cdots & \cdots & B_{kk} \end{pmatrix}$$

written in the same basis

$$AB = \begin{pmatrix} \lambda_1 B_{11} & \lambda_1 B_{12} & \cdots & \lambda_1 B_{1k} \\ \lambda_2 B_{21} & \lambda_2 B_{22} & \cdots & \lambda_2 B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \lambda_k B_{kk} \\ \lambda_k B_{11} & \lambda_k B_{12} & \cdots & \lambda_k B_{1k} \\ \lambda_k B_{21} & \lambda_k B_{22} & \cdots & \lambda_k B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \lambda_k B_{kk} \end{pmatrix}$$

Since $AB = BA$:

$$\lambda_i B_{ij} = \lambda_j B_{ij} \text{ for all } i \neq j$$

$$\Rightarrow \text{If } i \neq j \text{ then } B_{ij} = (0)$$

Thus in this basis,

$$A = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_k \end{pmatrix}$$

$$B = \begin{pmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{kk} \end{pmatrix}$$

Since $B$ is normal, $BB^* = B^*B = (B_{11}B_{11}^* \cdots B_{k1}B_{k1}^*)$ (note that this is true in any basis, as $(UBU^*)(UBU^*)^* = UBB^*U^*$)
each block $B_{ii}$ is normal, too.

Choose a basis for each coordinate block so as to diagonalize the $B_{ii}$'s.

This does not affect $A$! (Since $A$ is proportional to the identity on each block $A_i I$, and the identity $I$ is the same in any basis.)

Now $A$ and $B$ are both diagonal. $\square$

Remark: The same theorem also extends to three or more matrices, with pairwise commutation, e.g., $AB = BA$, $AC = CA$, $BC = CB$.

**Theorem:** If $A$, $B$, and $C$ are normal matrices that pairwise commute, then there is a basis $v_1, ..., v_n$ of simultaneous eigenvectors for all three:

$$A v_j = \lambda_j^A v_j, \quad B v_j = \lambda_j^B v_j, \quad C v_j = \lambda_j^C v_j.$$  

In this basis all three matrices are diagonal:

$$A = \begin{pmatrix} \lambda_1^A & 0 & \cdots & 0 \\ 0 & \lambda_2^A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^A \end{pmatrix}, \quad B = \begin{pmatrix} \lambda_1^B & 0 & \cdots & 0 \\ 0 & \lambda_2^B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^B \end{pmatrix}, \quad C = \begin{pmatrix} \lambda_1^C & 0 & \cdots & 0 \\ 0 & \lambda_2^C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^C \end{pmatrix}.$$  

**Example:** Matrix exponentials

For real or complex numbers $a$ and $b$,$$
e e^{a+b} = e^a e^b$$

(eg. $e^{i t_1} e^{i t_2} = e^{i t_1 + i t_2}$)

What about matrices?

If $A = U A_{\text{diag}} U^H$, $$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = U A_{\text{diag}} e^{D_A} U^H.$$
Similarly, if $B = U_6D_8U_6^*$.

**What is $e^{A+B}$?**

```matlab
>> A = randn(2,2);  A(2,1) = A(1,2);  A
B = randn(2,2);  B(2,1) = B(1,2);  B
A =
   4.8889e-01    7.2689e-01
   7.2689e-01   -3.0344e-01
B =
   2.9387e-01    8.8840e-01
   8.8840e-01  -1.1471e+00

>> expm(A+B)
ans =
   4.0240e+00    2.0574e+00
   2.0574e+00   1.1794e+00
```

```matlab
>> expm(A) * expm(B)
ans =
   4.0214e+00    1.9054e+00
   2.2445e+00   1.1911e+00
>> expm(B) * expm(A)
ans =
   4.0214e+00    2.2445e+00
   1.9054e+00   1.1911e+00
```

**Problem:** $AB - BA \neq 0$

```matlab
>> A * B - B * A
ans =
   0   -3.4349e-01
   3.4349e-01    0
```

$\exp(A+B) = I + A + B + \frac{(A+B)^2}{2} + \cdots$

$= \frac{A^2 + AB + BA + B^2}{2}$

$\exp(A) \exp(B) = (I + A + \frac{A^2}{2} + \cdots)(I + B + \frac{B^2}{2} + \cdots)$

$= I + A + B + (AB + \frac{1}{2}A^2 + \frac{1}{2}B^2 + \cdots)$
But if $A$ and $B$ commute, then $BA = AB$, so all the $A$'s can be pulled to the left, and
\[
e^{A+B} = e^A e^B = e^B e^A \quad \text{(if they commute!)}
\]

**Application:** Time-dependent differential equations:
\[
\frac{dy}{dt} = f(t) \ y(t)
\]
\[
\Rightarrow y(t) = y(0) \cdot \exp \left( \int_0^t f(s) \, ds \right)
\]

But this is **false** for matrices
\[
\frac{d\tilde{y}}{dt} = F(t) \ \tilde{y}(t)
\]
\[
\Rightarrow \tilde{y}(t) = \exp \left( \int_0^t F(s) \, ds \right) \tilde{y}(0)
\]

unless all matrices $F(t)$ commute with each other!

If they don't commute, then there isn't a clean solution, and you'll have to solve the differential equation numerically, using a sufficiently fine discretization of time.

\[
\gg \ \text{expm}(A+B)
\]
\[
\text{ans} =
\begin{array}{cc}
4.0240e+00 & 2.0574e+00 \\
2.0574e+00 & 1.1794e+00
\end{array}
\]

\[
\gg \ m = 100;
\]
\[
\ ( \ \text{expm}(A/m) \ * \ \text{expm}(B/m) \ )^m
\]
\[
\text{ans} =
\begin{array}{cc}
4.0240e+00 & 2.0552e+00 \\
2.0596e+00 & 1.1794e+00
\end{array}
\]

\[
\gg \ m = 1000;
\]
\[
\ ( \ \text{expm}(A/m) \ * \ \text{expm}(B/m) \ )^m
\]
\[
\text{ans} =
\begin{array}{cc}
4.0240e+00 & 2.0572e+00 \\
2.0576e+00 & 1.1794e+00
\end{array}
\]

**Trivia:** Is this robust?

What if $\|AB-BA\| < S$?
Are they nearly simultaneously diagonalizable?


First asked in the 1950s


Theorem: There exists a function $\varepsilon = \varepsilon(\delta)$ such that, for any $n \times n$ Hermitian matrices $A$ and $B$, with $\|AB-BA\| < \delta$,

there exist commuting matrices $A'$ and $B'$ with $\|A-A'\| < \varepsilon(\delta)$, $\|B-B'\| < \varepsilon(\delta)$.

Note: $\varepsilon$ does not depend on $n$!

Theorem: [Hastings, 2008]

This holds for $\varepsilon(\delta) \sim \delta^{1/5}$.

Remark: These theorems are all false for almost-commuting triples of Hermitian matrices, and pairs of unitary matrices.

**Tensor Products**

Starting with a vector space $V$, you can get a smaller vector space by taking a subspace.

**How to get a bigger vector space?**

Method 1: ‘Direct sums’

Inputs: vector spaces $V \cong \mathbb{R}^m$, $W \cong \mathbb{R}^n$

Output: $V \oplus W \cong \mathbb{R}^{m+n}$

If $v \in V$, $w \in W$, then $(v, w) \in V \oplus W$

Examples: $\mathbb{R}^2 = \mathbb{R}' \oplus \mathbb{R}'$

$\mathbb{R}^3 = \mathbb{R}' \oplus \mathbb{R}^2$

$\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}'$

$\mathbb{R}^4 = \mathbb{R}' \oplus \mathbb{R}' \oplus \mathbb{R}'$
If $A$ is an $m \times m$ matrix acting on $V$, then
\[
\begin{pmatrix}
\hat{A} & 0 \\
0 & 0
\end{pmatrix}
\]
acts on the $V$ part of $V \oplus W$.

Matrices like
\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\]
act separately on $V$ and $W$.

**Method 2:** "Tensor products"

Inputs: $V \cong \mathbb{R}^m$, $W \cong \mathbb{R}^n$

Output: $V \otimes W \cong \mathbb{R}^{m \cdot n}$

**Claim:** You already know tensor products!

Proof by example: Consider two coins...

<table>
<thead>
<tr>
<th>Coin 1</th>
<th>Coin 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$H$</td>
</tr>
<tr>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$p$</td>
<td>$1-p$</td>
</tr>
<tr>
<td>$q$</td>
<td>$1-q$</td>
</tr>
</tbody>
</table>

Coins 1 and 2 together:

\[
\begin{pmatrix}
HH & HT & TH & TT \\
pq & p(1-q) & (1-p)q & (1-p)(1-q)
\end{pmatrix}
\]

Thus $\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{R}^4$

\[
\begin{pmatrix}
p \\
1-p
\end{pmatrix} \otimes \begin{pmatrix}q \\
1-q
\end{pmatrix} = \begin{pmatrix}
pq \\
p(1-q) \\
(1-p)q \\
(1-p)(1-q)
\end{pmatrix}
\]

**Note:** Not all vectors in $\mathbb{R}^m$ are tensor products of vectors in $\mathbb{R}^2$ and $\mathbb{R}^2$.

\[
\begin{pmatrix}
\frac{1}{2} \\
0
\end{pmatrix} \neq \begin{pmatrix}
pq \\
p(1-q)
\end{pmatrix}
\]

for any $p, q$. 
\[
\begin{pmatrix}
0 \\
1/2 \\
\end{pmatrix} \neq \begin{pmatrix}
\nu^p(1-q) \\
(1-\nu)^q \\
\nu(1-\nu)(1-q) \\
\end{pmatrix} \text{ for any } p, q
\]

\[
\text{Independent events} \quad \text{Tensor-product distributions}
\]

\[
\text{Correlated events} \quad \text{Not a tensor product}
\]

**Tensor products of operators**

**How to randomize a coin:**

multiply by \( \begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2 \\
\end{pmatrix} \)

since \( \begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2 \\
\end{pmatrix} \begin{pmatrix}
q \\
1-q \\
\end{pmatrix} = \begin{pmatrix}
1/2 \\
\end{pmatrix} \)

**How to randomize the 2nd of 2 coins:**

multiply by

\[
M = \begin{pmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 1/2 \\
0 & 1/2 & 0 \\
\end{pmatrix} \\
\text{since } M \begin{pmatrix}
a \\
b \\
c \\
d = a-b-c \\
\end{pmatrix}^{HH} = \begin{pmatrix}
(a+b)/2 \\
(a+b)/2 \\
(c+d)/2 \\
(c+d)/2 \\
\end{pmatrix}
\]

And the first coin?

\[
\begin{pmatrix}
1/2 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 1/2 \\
1/2 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 1/2 \\
\end{pmatrix} \begin{pmatrix}
a \\
b \\
c \\
d \\
\end{pmatrix}^{HH} = \begin{pmatrix}
(a+c)/2 \\
(b+d)/2 \\
\end{pmatrix}
\]

**Definition:**

\[
A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B \\
a_{21}B & a_{22}B \\
\end{pmatrix}
\]

\[
\Rightarrow I \otimes \begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2 \\
\end{pmatrix} = \begin{pmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0 \\
0 & 1/2 & 1/2 \\
0 & 1/2 & 1/2 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2 \\
\end{pmatrix} \otimes I = \begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2 \\
\end{pmatrix}
\]
Quantum entanglement

Definition: A quantum state in $\mathbb{C}^m \otimes \mathbb{C}^n$ is entangled if it is not a tensor product state.

Observe:

1 coin $\rightarrow \mathbb{C}^2$
2 coins $\rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$
3 coins $\rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$

\[ \vdots \]
\[ n \text{ coins } \rightarrow (\mathbb{C}^2)^n = \mathbb{C}^{2^n} \]

The dimension grows exponentially!

Having to keep track of an exponentially long vector is why quantum systems are hard to simulate — and also why quantum computers are potentially exponentially faster than standard (classical) computers!