Lecture 13: Kraus representation theorem, POVMs, general state distinguishing

**Def:** A general quantum operation \( \mathcal{E} \) is a
- linear map, that
- preserves Hermiticity,
- preserves trace, and
- \( \mathcal{E} \) completely positive, i.e.
\[ \mathcal{E} \otimes I \text{ preserves positivity for all extensions.} \]

**Example:** The transpose map \( T \) is positive (leaves eigenvalues unchanged) but not completely positive.

on \( \sum_x (|x\rangle \otimes |x\rangle) \) maximally entangled state

\[
\left( T \otimes I \right) \left[ \sum_{xy} |x\rangle \langle y| \otimes |x\rangle \langle y| \right]
\]

\[
= \sum_{xy} |y\rangle \langle x| \otimes |x\rangle \langle y| \]

EG. in 2D, \(
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
100 X 001 & 101 X 101 \\
110 X 011 & 111 X 111
\end{pmatrix}
\)

not positive semi-definite

**Theorem (Kraus representation theorem):** Any superoperator satisfying the above four conditions has a Kraus representation (and therefore can be implemented via Stinespring dilation) with at most \( (\dim \mathcal{H})^2 \) Kraus operators.

Thus we have three fully equivalent definitions of general quantum operations:
1. completely positive superoperator
2. \( \text{Tr}_B \mathcal{U}(\rho \otimes 1 \otimes \mathcal{X}) \mathcal{U}^* \)
3. Kraus representation

More examples: depolarizing noise, partial trace, spin relaxation, composition of two superoperators, general quantum measurements ...

note: superoperators are generally not invertible (unless unitary)
Kraus Representation Theorem
for a proof see [Preskill lecture notes, Ch.3.2].
Main idea: the "Jamiołkowski isomorphism" (or Choi-J.)
Let \( |1\rangle = \sum_{i=1}^{d} i |i\rangle \).
Then a superoperator \( \rho \) is characterized by

\[
(\rho \otimes I)(14 \times 14)
= (\rho \otimes I)(\sum_{i,j} i X_{i,j} \otimes i X_{i,j})
= \sum_{i,j} \rho (i i X_{i,j} \otimes i X_{i,j})
\]
\[\geq 0 \quad \text{since } \rho \text{ is completely positive.}\]

General (open-system) measurements (POVMs)

\[
\rho \quad \begin{array}{c}
\text{measure in } \{ \ket{1}, \ket{2} \} \text{ basis}
\end{array}
\]

\[
\mathcal{P}_{L\mathcal{E}L} = \text{Tr} \left[ (\rho \otimes I \otimes I_{k}) U (\rho \otimes I \otimes I_{k}) U^\dagger \right]
= \sum_{i,j} \langle i| \langle k| \rho M_{i}^\dagger M_{j} \ket{i} \ket{k} \rangle
= \sum_{i,j} \langle i| M_{i} \rho M_{j}^\dagger \ket{i} \ket{j}
= \text{Tr} (\rho M_{k}^\dagger M_{k})
\]

and conditioned on outcome \( k \), the state is

\[
\rho' = \frac{\rho \mathcal{P}_{L\mathcal{E}L} \otimes I \otimes I_{k} \otimes I \otimes I_{k} \otimes I_{k} \otimes I_{k}}{\text{Tr}[\mathcal{P}_{L\mathcal{E}L} \otimes I \otimes I_{k}]} M_{k} \rho M_{k}^\dagger
\]

Def: A measurement is specified by operators \( M_{k} \) with \( \sum_{k} M_{k}^\dagger M_{k} = I \).
On a state \( \rho \), the probability of measuring \( k \) is

\[
\mathcal{P}_{L\mathcal{E}L} = \text{Tr} (\rho M_{k}^\dagger M_{k})
\]
and the new state is then given by

\[
\rho' = M_{k} \rho M_{k}^\dagger \mathcal{P}_{L\mathcal{E}L}
\]

Def: POVM (positive operator-valued measurement):

\[
E_{k} \geq 0, \quad \sum_{k} E_{k} = I \quad \mathcal{P}_{E_{k}} = \text{Tr} (E_{k})
\]
\( E_{k} M_{k}^\dagger M_{k} \) works and conversely \( M_{k} = \mathcal{P}_{E_{k}} \) implements the POVM.
Examples...
Notice: POVM elements need not be orthogonal.
Keep in mind: Church of the Larger Hilbert Space
Classic example. Optimal distinguishing measurements for symmetrical set of single-qubit pure states.

Assume we are given one of the states $|1\rangle, \ldots, |L\rangle$, with equal probabilities $1/L$, and our goal is to guess which one.

In general, there is no closed-form expression for the optimal distinguishing POVM. But if the states satisfy a symmetry, then the optimal POVM should satisfy the same symmetry (proof?), so we can find it.

Examples:

$$|10\rangle, |1\rangle, |1\rangle \quad |10\rangle, \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$

Pauli eigenstates

$$|10\rangle, |11\rangle, \frac{1}{\sqrt{2}}(|10\rangle + i|11\rangle), \frac{1}{\sqrt{2}}(|10\rangle - i|11\rangle)$$

A measurement using projectors can only have two outcomes, and will not be optimal. Instead, we use a 3-outcome POVM $E_0, E_1, E_2$.

We want to maximize over $2 \times 2$ positive semi-definite matrices satisfying

$$E_0^* E_1 + E_2 = 1, \quad \frac{1}{3} \sum_{j=0}^{2} \text{Tr} \left[ E_j \frac{1}{2} (e^{i \theta_j} 1 + e^{-i \theta_j} 1) \right]$$

Expressed in the Bloch sphere coordinates

$$1 + (1 + X) = \frac{1}{2} (1 + X + i Y), \quad \frac{1}{2} (|10\rangle + e^{i \theta} |11\rangle)(|10\rangle + e^{-i \theta} |11\rangle) = \frac{1}{2} \left( 1 - \frac{1}{2} X + \frac{1}{2} Y \right)$$

The POVM elements can also be expressed in their Pauli coordinates, and by symmetry the $Z$ coordinates should be 0, while the $X$ and $Y$ coordinates should be in the same proportion as above. We want $E_0, E_1, E_2$ to be as close as possible to the above projectors. Set them to be $\frac{1}{3}$ the above projectors.
Definition: The Schumacher- Hausladen-Wootters "pretty good measurement" (PGM) is given by

\[ E_k = \left( \sum \delta \rho \right)^{-\frac{1}{2}} \delta_k \left( \sum \delta \rho \right)^{-\frac{1}{2}} \]

Theorem [Bar-mum + Knill, quant-ph/0004088]

\[ P_{\text{success}}(E_{\text{PGM}}) > P_{\text{success}}(\text{optimal POVM}) \]

Corollary: The failure rates satisfy

\[ P_{\text{fail}}(E_{\text{PGM}}) \leq P_{\text{fail}}(E_{\text{other}}) \leq (1 - P_{\text{success}}(E_{\text{PGM}})) \]

\[ = (1 - P_{\text{success}}(E_{\text{PGM}})) P_{\text{fail}}(E_{\text{PGM}}) \]

\[ \leq 2 P_{\text{fail}}(E_{\text{PGM}}) \]

\[ \Rightarrow \text{For states that are reasonably distinguishable, the pretty good measurement is approximately optimal. (see also [Tjion, 0707.3386])} \]
Distinguishing quantum states:

Recall: For two pure states $|147\rangle$ and $|14\rangle$, the optimal distinguishing measurement, i.e., the measurement that maximizes

$$\min \{ P[\text{say } 147 \mid 147], P[\text{say } 147 \mid 14\rangle] \}$$

achieves

$$= \cos(\pi - \theta)$$

$$= \frac{1}{2} \left( 1 + \cos \left( \frac{\pi}{2} - \theta \right) \right)$$

$$= \frac{1}{2} \left( 1 + \sin \theta \right)$$

$$= \frac{1}{2} \left( 1 + \sin \left( \cos^{-1} (|14\rangle) \right) \right)$$

$$= \frac{1}{2} \left( 1 + \sqrt{1 - |14\rangle^2} \right)$$

What about the general problem, of distinguishing two states $\rho$ and $\sigma$? Important for a notion of distance.

States that are close together should be difficult to distinguish.

Easy cases:

1. $\rho$ and $\sigma$ are pure — see above.

2. $\rho$ and $\sigma$ can be simultaneously diagonalized, i.e., $[\rho, \sigma] = 0$, i.e., in some basis $\rho = (\rho_1, \ldots, \rho_n)$, $\sigma = (\sigma_1, \ldots, \sigma_n)$.

Simplest probability distributions

Optimal measurement samples and outputs the more likely distribution

$$P[\text{correctly identifies } \rho] = P[\text{correctly identifies } \sigma]$$

$$= \frac{1}{2} + \frac{1}{4} \sum_j |q_j - q_j|$$

$$= \frac{1}{2} \left( 1 + TV([q_1, \ldots, q_n], [q_1, \ldots, q_n]) \right)$$

Total variation distance

when each has 50% prior.
Distance measures between quantum states

Euclidean distance $\|1\rangle \langle 1\|$

Fidelity $\langle \psi | \psi \rangle$ usually preferable for pure states
because it does not depend on the global phase

For mixed states $\rho, \sigma$, can also use Euclidean distance
and Fidelity (defined as $\text{Tr} \sqrt{\rho^{\otimes 2} - \rho^{\otimes 2} \sigma^{\otimes 2}}$)

or the trace norm $\|M\|_\infty = \text{Tr} |M| = \text{Tr} \sqrt{M^* M}$

the 1-norm of the eigenvalues

trace distance $\|\rho - \sigma\|_\text{tr}$

Theorem: For any two quantum states $\rho$ and $\sigma$, the
optimal measurement procedure for distinguishing
between them succeeds with probability

$$\frac{1}{2} + \frac{1}{4} ||\rho - \sigma||_\text{tr}.$$