3/25/10 Lecture 20: Turaev-Viro codes

Last time:
- for \( |0\rangle, |+\rangle \) preparation, CNOT on toric code
- for encoding a magic state

Goals:
1. Geometrically local codes with easier universality, similar threshold?
2. Realization of nearly arbitrary anyon models from a qubit lattice.

Map:
1. Anyon model (tensor category)
2. Desired microscopic behavior
3. Derive macroscopic anyons
4. Implement micro features w/ local checks
5. Computation
6. Relate to TV & WRT topological invariants

\[1002.2816\]

Stable \( q \) states of matter, loop \( q \) gravity, FT & physics

1. Anyon model: (Unitary modular tensor category (without multiplicities and \( w \)/self-dual particles))
   - Particle types: \( 0, i, j, k, \ldots \)
   - Go dimensions: \( d_i, d_0 = 1 \)
   - Fusion rules: \( s_{i,j,k} \) allowed
   - F-moves: \( s_{i,j,k} = \sum_i F_{i,j,k}^n s_{i,j,k} \)
   - R-moves: \( s_{i,j,k} = R_{i,j,k}^{n} s_{i,j,k} \)

   **Consistency conditions:**
   - Associativity of fusion:
     \( \sum_m f_{j,m} s_{i,k} = \sum_m f_{i,m} s_{j,k} \)
   - Physicality:
     \( F_{i,k}^m = F_{i,k}^m \) for \( i, j, k, m \)
   - Unitarity:
     \( F_{i,m}^n = (F_{i,m}^n)^* \)
   - Normalization:
     \( F_{i,j,k} = \sum_{n} \delta_{i,j} \delta_{j,k}^{m} \)
   - Tetrahedral symmetry:
     \( F_{i,j,k} = F_{i,j,k} \) for \( i, j, k \)
   - Pentagon equation:
     \( \sum_{m} F_{i,j,m} F_{j,k,n} F_{k,i,m} = \sum_{m} F_{i,j,m} F_{j,k,n} F_{k,i,m} \)
   - Hexagon equation:
     \( \sum_{m} f_{j,k,m} \)
2. Microscopic behavior: ribbon graph (string-net) condensate

\[ \Sigma = \text{compact, orientable surface with boundary} \]
\[ \text{e.g., torus } \mathbb{T}^2 \text{ or (n+1)-punctured sphere} \]

\[ H_\Sigma \text{ (code-space)} = \{ \text{formal linear combinations of ribbon graphs} \} \]
\[ \text{modulo isotopy}, \quad \longrightarrow \]
\[ \mathcal{M}_\Sigma = \mathcal{M}_\Sigma \]

**Exercise:** Prove that \( H_\Sigma \) is finite-dimensional.

**Idea:** First show that any

\[ \text{can be written as a linear combination of diagrams} \]

\[ \Rightarrow \dim H_\Sigma = \sum_{j_1, \ldots, j_n} \mathcal{M}_\Sigma \]

Next, remove crossings around.

\[ \left( \frac{\text{Eg in Fib}}{\text{Eg in Fib}} \right) \left( \frac{\text{Eg in Fib}}{\text{Eg in Fib}} \right) \left( \frac{\text{Eg in Fib}}{\text{Eg in Fib}} \right) \left( \frac{\text{Eg in Fib}}{\text{Eg in Fib}} \right) \]

\[ \Rightarrow H = \left( \frac{\text{Eg in Fib}}{\text{Eg in Fib}} \right) \left( \frac{\text{Eg in Fib}}{\text{Eg in Fib}} \right) \]

**Computational basis states for \( H_{\Sigma_{\text{link}}} \)**

all orthonormal

trace inner product for general \( \Sigma \)

3. Macroscopic anyons

\[ n = 2 \text{ punctures: } \quad x \quad \otimes \quad \beta \quad \beta \quad \text{Fib} \]

**vacuum loop**

\[ \langle \beta \rangle = \frac{1}{2} \left( \langle x + \tau \beta \rangle \right) \]

\[ = \frac{1}{2} \sum_{\tau} d_{\beta}(\tau) \]

**Claim:**

\[ \langle \beta \rangle = \langle \beta \rangle \]

**Proof:**

\[ \sum_{\tau} \frac{1}{2} d_{\beta}(\tau) \]

\[ \sum_{\tau} \frac{1}{2} \beta \]

not orthonormal (ND)
3D to 2D reduction

$$j = \sum_k F_{0i}^i k$$

$$\phi = \sum_{s, i, k} R_{s, i}^k$$

to eliminate crossings

$$e^{i\phi} = R_0$$

$$\Rightarrow$$ doubled anyon fusion diagrams

satisfy rules, F moves, braiding

$$\Rightarrow$$ doubled category realized

4. Implement micro features with local checks

begin with a trivalent lattice, a qudit on every edge
let $$\Sigma' = \Sigma$$ with a puncture through every plaquette

then $$\mathcal{H}_{\Sigma'}$$ can be naturally embedded in the qudit Hilbert space

vertex stabilizers enforce fusion constraints

$$S_v = \sum_{i, k} \lambda_{i, k} [X_i, X_k]$$

plaquette stabilizers: move from $$\mathcal{H}_{\Sigma'}$$ to $$\mathcal{H}_{\Sigma}$$ by adding a vacuum loop

5. Computation by F-moves

$$\langle \langle \phi_1 | \phi_2 \rangle \rangle = \langle \langle \phi_1 | \phi_2 \rangle \rangle$$

$$\langle \langle \phi_1 | \phi_2 \rangle \rangle = \langle \langle \phi_1 | \phi_2 \rangle \rangle$$

6. Relationship to topological invariants

tensors and tensor network contraction

Turaev-Viro invariant