Example: 9-qubit Shor code

Composition of

\[
\begin{array}{c|c|c|c|c}
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
\end{array}
\]

with

\[
\begin{array}{c|c|c|c|c}
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
\end{array}
\]

leads to

\[
\begin{array}{c|c|c|c|c}
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
\end{array}
\]

[9,1,3] CSS, not self dual
corrects \( \leq 1 \) X error in every block of 3
and \( \leq 1 \) Z error total

Note: \( \lvert T \rangle \) stabi\( \ell \) by \( \frac{1}{\sqrt{2}} \)

\[
\begin{array}{c|c|c|c|c}
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
\end{array}
\]
on each block

\[\Rightarrow \lvert T \rangle = (1000\rangle + 1111\rangle)^{\otimes 3}\]

\( \lvert T \rangle \) prepares a block

automatically fault tolerant
(every error equivalent to a weight-one fault)

\[\Rightarrow\text{Steane-style X EC easy}\]

\[\text{Steane-style X EC even easier}\]

\[\begin{array}{c|c|c|c|c}
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\end{array}\]

\[\Rightarrow \text{don't need to verify } \lvert T \rangle\]
\[\text{don't need to keep it in memory}\]

\[\text{Steane-style X EC even easier}\]

\[\begin{array}{c|c|c|c|c}
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\end{array}\]

\[\begin{array}{c|c|c|c|c}
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ & \_ \\
\end{array}\]

\[\Rightarrow\text{one qubit ancilla only}\]

\( \text{but Z error correction still difficult}\)

\( \text{? : Can we simplify X, Z EC & make X, Z errors symmetric?}\)

- note \( \text{H} \) doesn't work to swap \( X, Z \)

Subsystem stabilizer codes

- since correcting more X errors than needed, can throw away

some Z stabilizers, get down to

\[\begin{array}{c|c|c|c|c}
X & X & X & X & X \\
X & X & X & X & X \\
X & X & X & X & X \\
\end{array}\]

"Baron-Shor code"
also Knill & Laflamme
Remark 1: $X \leftrightarrow \varepsilon$ symmetrical

$A = H^0$ followed by a transposition of the qubits

Remark 2:

$A = \sigma_1$ but first qubit is protected to distance $3$.

i.e. any nontrivial operator acting nontrivially on qubit 1 has weight $\geq 3$.

Proof: Eq. can detect the block a $\varepsilon$ error falls into and correct it, only causing a logical error on some other logical qubit.

QECC: $C \subseteq H_2^n$

subsystem QECC: $(C \otimes C') \subseteq H_2^n$

$\begin{array}{c}
\text{do n't care about these qubits} \\
|T\rangle \otimes |\bar{T}\rangle \text{ typical codeword}
\end{array}$

$\Rightarrow$ for $X \in C$, use $1 \otimes \bar{0} \otimes \bar{0} \otimes \bar{0} = (|000\rangle + |111\rangle) \otimes |\bar{0}\rangle$

$\Rightarrow$ single-qubit syndrome extraction

$\Rightarrow$ for $Z \in C$, use $|\bar{0}\rangle \otimes \bar{0} \otimes \bar{0} \otimes \bar{0} = (|\bar{0}\rangle + |\bar{1}\rangle) \otimes |\bar{0}\rangle$

$\Rightarrow$ single-qubit syndrome extraction.

Remark. Bacon's construction trivially generalizes to work for any two classical linear codes. (e.g. $25$-qubit $E8$ code, ...)

$q$-qubit $O$-S implicit...

Remark 3. According to numerical diagonalization, ground space of

$H = -X_1 X_4 - X_1 X_5 - X_6 X_9 - X_7 X_9 X_{10} X_6 - X_6 X_{10} X_4$

is spanned by $15, 1347$, $17$, $47$.

$\Rightarrow$ codespace is ground space of a local Hamiltonian.

In 3D, the "compass model" with $V$ terms in the third direction, is conjectured to be self-stabilizing.