Theorem: Threshold erasure noise is $\frac{1}{2}$.

Proof sketch:
1. Large random codes give good protection against random errors.
2. For any $\epsilon > 0$, $\exists n_0$ large enough so that
   \[ P(n) = \frac{1}{n} \text{ fails} \leq \epsilon \]
   for all $n > n_0$.

Remarks:
- Generalizes to Pauli noise.

Theorem: Consider a projective Pauli channel $P$ that applies $X$, $Y$, or $Z$ with
   respective rates $p_x$, $p_y$, $p_z$. Let $p_z = 1 - p_x - p_y$.
   Let $H = \frac{1}{2} \sum (1, i, -1, -i) = -2p_x \log p_x - 2p_y \log p_y - 2p_z \log p_z$.
   Then for any $R < 1 - H$, $\exists$ a QECC that encodes
   $R$ qubits, such that the rate of erasure decays a rate $p$.

Best distance of any QECC $d < \frac{1}{2}$.

Haah's bound:
- Encoding into a rank stabilizer code
takes $O(n^2)$.

Chuang-Gottesman: $O(n^2)$.

- Computation + Teleportation

- $U$ is a Clifford operator
  \[ U = H \]
  \[ HXH = Z \]
  \[ HZH = X \]

- $U$ is not a Clifford.

Theorem: The Clifford group together with
the $\frac{3}{8}^{th}$ gate
\[ P \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \]
\[ P \cdot \frac{1}{2} = e^{-i\pi / 4} \cdot \frac{1}{2} \]
\[ P \cdot \frac{1}{2} = e^{-i\pi / 4} (1, i, -1, -i) \times \]
\[ \text{note } (1, i, -1, -i) \in \text{Clifford} \]
\[ (1, i, -1, -i) \times \frac{1}{2} = \frac{1}{2} \]
\[ (1, i, -1, -i) \times (1, i, -1, -i) = Z \]
\[ (1, i, -1, -i) \times (1, i, -1, -i) \times \frac{1}{2} = \frac{1}{2} \]

- Uzialo

- $U = P$
  \[ P \cdot P^+ = P^+ \cdot P = \frac{1}{2} \]
  \[ P \cdot P^+ = e^{-i\pi / 4} (1, i, -1, -i) \times \]
  \[ \text{note } (1, i, -1, -i) \in \text{Clifford} \]
  \[ (1, i, -1, -i) \times (1, i, -1, -i) = Z \]
  \[ (1, i, -1, -i) \times (1, i, -1, -i) \times \frac{1}{2} = \frac{1}{2} \]
**Corollary:** We get universal computation from ability to apply single-qubit Pauli operators in the Bell basis, prepare the states

- ability to apply single-qubit Pauli operators
- measure in the Bell basis
- prepare the states

\[ |00\rangle, |11\rangle, (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \]

\[ |00\rangle, |11\rangle, \text{CNOT}_x(|00\rangle + |11\rangle) \]

assuming adaptive classical control.

**Exercise:** Replace \( |00\rangle \) with \( |0\rangle \) and \( |1\rangle \) by \( |1\rangle \).

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**C** is a Clifford group.

\[ \mathcal{C} = \text{Clifford} \]

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**Ancilla factorizes**

- Very weakly: create many ancillas for computing
- Test them for errors, discard the bad ones
- Tolerates more noise at cost of overhead.
Distillation: take many noisy copies of a state
output one less noisy copy.

100\rightarrow 110
XX \rightarrow Z

\textbf{The overhead}

\textbf{Goal:} Simulate an ideal QC with T gates.

\begin{itemize}
  \item Need an effective error rate \( \frac{1}{T} \).
  \item \( (\log T)^2 \) logical errors
  \item Encoding error increases \( \log \log T \) gates.
  \item Overhead is \( \Omega(\log T) \).
\end{itemize}

\textbf{Logical teleportation}

Claim: Transversal \textbf{Bell} measurement on any stabilizer.

\begin{itemize}
  \item \textbf{Bell} measurement for any stabilizer.
  \item \textbf{Error} rate < \( \frac{1}{\log T} \).
  \item \( m = \log T \) logical errors.
  \item \( O(\log T) \) overhead = \( O((\log T)^2) \).
\end{itemize}