Fault-Tolerant Universality from Fault-Tolerant Stabilizer Operations and Noisy Ancillas

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Q: Do \( \{ \text{stabilizer operations, prepare } \rho \} \) form a universal set?

**Motivation:** [Knill ‘04] Estimated threshold of 5-10%.
Def: Stabilizer operations = CNOT, Hadamard, Phase gates, + Prepare, measure $|0\rangle / |1\rangle$.

Gottesman-Knill Theorem: Stabilizer operations are efficiently classically simulable.
Q: Do \(\{\text{stabilizer operations, prepare } \rho\}\) form a universal set?

\[(x, y, z) \leftrightarrow \frac{1}{2}(I + xX + yY + zZ)\]
Q: Do \( \left\{ \text{stabilizer operations,} \atop \text{prepare } \rho \right\} \) form a universal set?

Fact: Any mixture of Pauli eigenstates (points in octahedron) is classically simulable.
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\[ \frac{1}{2}(1 - \sqrt{\frac{3}{4}}) \]

Theorem: [R '04] Yes for \( |H\rangle \) w/ <14.6\% error

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Theorem: [R ‘04] Yes for \ket{H} w/ <14.6\% error

\[ \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) \]
Improved distillation procedure

1. With equal probabilities \( \frac{1}{2} \), apply \( H \) to \( \rho \).

Assume \( \rho \) lies along \( H \) axis:

\[
\rho = \frac{1}{2} \left( I + x (X + Z) \right)
= \frac{1}{2} \begin{pmatrix} 1 + x & 1 + x \\ 1 + x & 1 - x \end{pmatrix}
\]
Improved distillation procedure

1. Symmetrize $\rho$ into
   $$\rho = \frac{1}{2} \left( I + x (X + Z) \right) = \frac{1}{2} \left( \frac{1+x}{1+x} \frac{1+x}{1-x} \right).$$

2. Take 7 copies of $\rho$. Decode according to the [[7,1,3]] Steane/Hamming quantum code, rejecting if errors detected.

3. Conditioned on acceptance, the output state $\rho'$ is
   $$\rho' = \frac{1}{2} \left( I + \frac{x^3 (7+8x^4)}{1+14x^4} \right) (X + Z).$$
Proof of improved distillation procedure

\[
\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \quad \rho' = \frac{\left( \langle 0_L | \rho^\otimes n | 0_L \rangle \langle 0_L | \rho^\otimes n | 1_L \rangle \langle 1_L | \rho^\otimes n | 0_L \rangle \langle 1_L | \rho^\otimes n | 1_L \rangle \right)}{\text{tr}}
\]

For a CSS code in which \( X_L = X^{-n} \), \( Z_L = Z^{-n} \),

\[
|0_L\rangle = \frac{1}{\sqrt{|C|}} \sum_{a \in C} |a\rangle \quad |1_L\rangle = X_L |0_L\rangle
\]

where \( C \) is the set of codewords for a classical code.

Thus \( \langle 0_L | \rho^\otimes n | 0_L \rangle \propto \sum_{a,b \in C} \langle a | \rho^\otimes n | b \rangle \).

E.g. \( \langle 0001111 | \rho^\otimes 7 | 0110011 \rangle = (\rho_{00})^1 (\rho_{01})^2 (\rho_{10})^2 (\rho_{11})^2 \).

Generally,

\[
\langle a | \rho^\otimes n | b \rangle = \begin{pmatrix} n-\frac{1}{2}(|a|+|b|+|a\oplus b|) & \frac{1}{2}(-|a|+|b|+|a\oplus b|) \\ \frac{1}{2}(|a|-|b|+|a\oplus b|) & \frac{1}{2}(|a|+|b|-|a\oplus b|) \end{pmatrix}
\]

\[
\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}
\]
Universality via **Magic states distillation**

**Theorem:** [R, ‘04] Stabilizer operations + Prepare $|H\rangle$ w/ $\leq \frac{1}{2}(1 - \frac{1}{\sqrt{2}})$ error $\Rightarrow$ Universality.

**Appl. 1:** Stabilizer op. fault-tolerance

$\Rightarrow$ Universal fault-tolerance.

**Fact:** Stabilizer operations + Any other single-qubit unitary $\Rightarrow$ Universality.

**Corollary:** Stabilizer operations + (ability to prepare repeatedly any pure state which is not a stabilizer state) gives universality.
Universality from single-qubit pure states

**Theorem:** Stabilizer operations + (ability to prepare any single-qubit pure state which is not a Pauli eigenstate) is universal.

**Proof:**
Universality from single-qubit pure states

**Theorem:** Stabilizer operations + (ability to prepare any single-qubit
pure state which is not a Pauli eigenstate) is universal.

**Proof:**
Universality from multi-qubit pure states

**Theorem:** Stabilizer operations + (ability to prepare any pure state which is not a stabilizer state) is universal.

**Proof:** 9 sequence of Clifford unitaries and postselected Pauli measurements which reduces $|\psi\rangle$ down to a single-qubit pure state which is not a Pauli eigenstate.

By induction, true for $n=1$.

\[ |\psi\rangle = \alpha|0\rangle|\psi_0\rangle + \beta|1\rangle|\psi_1\rangle \]

with $\alpha, \beta \neq 0$, $|\psi_0\rangle$ and $|\psi_1\rangle$ stabilizer states (else apply induction).

By applying Clifford unitaries, w.l.o.g. $|\psi_0\rangle = |0^{n-1}\rangle$.

\[ \cdots \cdots |\psi\rangle = \alpha|0\rangle|0^{n-1}\rangle + \beta|1\rangle|+^{n-1}\rangle \]

But $\alpha|0\rangle + \frac{\beta}{2^{(n-1)/2}}|1\rangle$, $\frac{\alpha}{2^{(n-1)/2}}|0\rangle + \beta|1\rangle$
can’t both be stabilizer states!
Application to fault-tolerant computing

[Knill, quant-ph/0404104]

Given scheme for fault-tolerantly applying stabilizer circuits, extend it to a universal fault-tolerant scheme.

Universal fault-tolerance \quad \overset{\sim}{\leftrightarrow} \quad \text{Stabilizer op. fault-tolerance}

E.g., Knill’s scheme has threshold of 5-10% for fault-tolerant stabilizer operations, and the same threshold for fault-tolerant universal operations.
Open questions

**Fact:** Any mixture of Pauli eigenstates (points in octahedron) is classically simulable. \( \Rightarrow \) Universality from \(|H\rangle\) w/ \(\frac{1}{2}(1 - \frac{1}{\sqrt{2}})\) error is tight.

**Open:** Is stabilizer operations + (ability to prepare repeatedly single-qubit mixed state \(\rho\)) universal for all \(\rho\) outside the octahedron?
Open questions

**Open**: Is stabilizer operations + (ability to prepare repeatedly single-qubit mixed state $\rho$) universal for all $\rho$ outside the octahedron?

**Open**: What about perturbations to the states $\rho$? What about asymmetries? What if we only have fidelity lower bound? Can we characterize stable fixed points for stabilizer codes?

**Open**: Can we give a provable reduction of fault-tolerance to problem of preparing stabilizer states with independent errors?
Stabilizer operation fault-tolerance Universal fault-tolerance

\[ |00\rangle_L + |11\rangle_L \]

|0\rangle |0\rangle_L \quad + |1\rangle |1\rangle_L \quad |T\rangle_L

\[ |T\rangle \]

**Magic states distillation:**

Stabilizer operations

\[ \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) \quad \frac{1}{2}(1 - \sqrt{\frac{3}{7}}) \]

+ Prepare \(|H\rangle\) or \(|T\rangle\) with <14.6\% or <17.3\% error resp.

\[ \Rightarrow \text{Universality.} \]

\[ \text{[Knill '04]} \quad \text{[Bravyi Kitaev '04]} \]

E.g., Knill’s scheme has same threshold for fault-tolerant stabilizer
Universality

Stabilizer operations (Clifford unitaries + prepare/measure Paulis) are not quantum universal.

**Q:** What additional operations are needed to get universality?

**Fact:** [Y. Shi, 2002] Stabilizer operations + any single-qubit unitary not in $C$ is universal.

**Theorem:** Stabilizer operations + (ability to prepare repeatedly any pure state which is not a stabilizer state) gives universality.

**Q:** Is stabilizer operations + (ability to prepare single-qubit mixed state $\rho$) universal?