Any AND-OR formula of size $N$ can be evaluated in time $N^{1/2+o(1)}$ on a quantum computer.

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Def: \{\text{AND, OR, NOT}\} Formula = Tree of nested gates
Def: \{AND, OR, NOT\} Formula = Tree of nested gates

\[ \varphi(x) \]

- Input variables may appear more than once...
- But gates cannot have fan-out!
- (Unless formula is read-once)

(Only in a circuit can subexpressions be reused)
Def: \{AND, OR, NOT\} Formula = Tree of nested gates

\[ \phi(x) \]

Problem: Evaluate \( \phi(x) \). (Formula/Game tree evaluation problem)
• Problem: Evaluate the formula, with minimal queries to the inputs bits $x_i$.

• Classical history

  • Some formulas, e.g., $\text{OR}(x_1, x_2, \ldots, x_N)$, require $\Omega(N)$ time

  • Randomized algorithm in E-time $O(N^{0.754})$ for balanced binary AND-OR formulas [Snir ‘85, Saks & Wigderson ‘86]
    • Flip coins to decide which subtree to evaluate next, short-circuit

  • Optimal [SW ‘86, Santha ‘95]

  • General formulas, ??
Problem history: Quantum computation

- Classical history
  - Randomized algorithm in E-time $\Theta(N^{0.754})$ for balanced binary formulas
  - Other formulas may require $\Omega(N)$ time

- Quantum history
  - $\Omega(\sqrt{N})$ queries required for read-once [Barnum, Saks ‘04]
  - Grover search: Evaluates $\text{OR}(x_1, x_2, \ldots, x_N) = \begin{cases} 1 & \text{if } \exists \text{ an } i : x_i = 1 \\ 0 & \text{otherwise} \end{cases}$ using $O(\sqrt{N})$ queries ($O(\sqrt{N \log \log N})$-time)
  - Can be applied recursively to evaluate shallow trees:
    - Evaluates regular depth-$d$ AND-OR formula in $\sqrt{N} \ O(\log N)^{d-1}$ queries [Buhrman, Cleve, Wigderson ‘98]
    - Search on faulty oracles [Høyer, Mosca, de Wolf ‘03] ⇒ $O(\sqrt{N \ c^d})$ queries
• Classical history
  • Randomized algorithm in E-time $\Theta(N^{0.754})$ for balanced binary formulas
  • Other formulas may require $\Omega(N)$ time

• Quantum history
  • $\Omega(\sqrt{N})$ queries required for read-once [Barnum, Saks ‘04]
  • Grover search: Evaluates $\text{OR}(x_1, x_2, \ldots, x_N) = \begin{cases} 1 & \text{if } \exists \text{ an } i : x_i = 1 \\ 0 & \text{otherwise} \end{cases}$ using $O(\sqrt{N})$ queries ($O(\sqrt{N \log \log N})$-time)
  • Can be applied recursively to evaluate shallow trees

Quantum Leap!

• Farhi, Goldstone, Gutmann 2007: Breakthrough quantum algorithm for evaluating balanced binary AND-OR formula in $N^{1/2+o(1)}$ time
**Farhi, Goldstone, Gutmann ‘07 algorithm**

- **Theorem** ([FGG ‘07, CCJY ‘07]): A balanced binary NAND formula can be evaluated in time $N^{1/2+o(1)}$.
- Convert formula to a tree:
- Attach an infinite line to the root
Farhi, Goldstone, Gutmann ‘07 algorithm

- **Theorem** ([FGG ‘07, CCJY ‘07]): A balanced binary NAND formula can be evaluated in time $N^{1/2+o(1)}$.
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Theorem ([FGG ‘07, CCJY ‘07]): A balanced binary NAND formula can be evaluated in time $N^{1/2+o(1)}$.

- Convert formula to a tree:
  - Attach an infinite line to the root
  - Add edges above leaf nodes evaluating to one…

Diagram:

- $\circ = 0$
- $\bullet = 1$
Continuous-time quantum walk [FGG '07]

$x_{11} = 1$

$x_{11} = 0$
FGG quantum walk $|\psi_t\rangle = e^{iAGt}|\psi_0\rangle$
FGG quantum walk $|\psi_t\rangle = e^{iAGt} |\psi_0\rangle$
FGG quantum walk \( |\psi_t\rangle = e^{iA_G t} |\psi_0\rangle \)

\[ \varphi(x) = 0 \]

Wave reflects!

\[ \varphi(x) = 1 \]

Wave transmits!
[FGG ‘07] algorithm

- **Theorem** ([FGG ‘07, CCJY ‘07]): A balanced binary AND-OR formula can be evaluated in time $N^{1/2+o(1)}$.
  
  Analysis by scattering theory.

[ACRŠZ ‘07] algorithm

- **Theorem:**
  
  - An “approximately balanced” AND-OR formula can be evaluated with $O(\sqrt{N})$ queries (optimal for read-once!).
  
  - A general AND-OR formula can be evaluated with $N^{1/2+o(1)}$ queries.

Running time is $N^{1/2+o(1)}$ in each case, after efficient preprocessing.
Remarks on formula evaluation algorithms:

**Classical vs. Quantum**

- Classical complexity of evaluating balanced k-ary alternating AND-OR tree is \((k/2)^{\text{depth}} = N^{-(1-1/\log_2 k)}\) — approaches \(N\) as \(k\) increases.

- Classical complexity of evaluating general AND-OR formulas is not known?

- Classical complexity of evaluating iterative MAJ_3 formula is unknown: between \(\Omega\left(\left(\frac{7}{3}\right)^d\right)\) and \(o\left(\left(\frac{8}{3}\right)^d\right)\)
  
  - (the generalization of the optimal AND-OR algorithm is not optimal when applied to MAJ_3 trees)

- Quantumly, complexity is \(\sqrt{N}\) queries always, all the way up to \(k=N\) (i.e., evaluating OR\((x_1,\ldots,x_N)\), Grover search).

- General AND-OR formulas can be evaluated with \(N^{1/2+o(1)}\) queries.

- Expanding MAJ_3 into AND-OR gates gives \(O(\sqrt{5^d})\) quantumly.

- Also, the algorithm generalizes to give optimal algorithm for evaluating iterated \(f\), where \(f\) is any 3-bit function.
**Formula evaluation algorithm**

1. Convert formula $\varphi$ into a graph $G(\varphi)$
2. Define classical random walk on $G(\varphi)$
3. Quantize that walk
Convert formula $\varphi$ into a graph $G(\varphi)$

Define classical random walk on $G(\varphi)$

Quantize that walk

Substitution rules:

- AND
- OR
- NOT

$\varphi(x)$
Convert formula $\varphi$ into a graph $G(\varphi)$

Define classical random walk on $G(\varphi)$

Quantize that walk

Substitution rules:

- AND
- OR
- NOT

$\varphi(x)$
Substitution rules:

- **AND**
- **OR**
- **NOT**

1. Convert formula $\varphi$ into a graph $G(\varphi)$
2. Define classical random walk on $G(\varphi)$
3. Quantize that walk
Start with classical random walk on the tree... and...

- \( P(\text{stepping to subtree}) \propto \sqrt{\text{size of that subtree}} \)
- (For a balanced tree, walk is uniform)

Convert formula \( \varphi \) into a graph \( G(\varphi) \)

Define classical random walk on \( G(\varphi) \)

Quantize that walk
Convert formula $\varphi$ into a graph $G(\varphi)$

- Define classical random walk on $G(\varphi)$
- Quantize that walk

- $P(\text{stepping to subtree}) \propto \sqrt{\text{(size of that subtree)}}$
- (For a balanced tree, walk is uniform)
- Make leaves (inputs) evaluating to 0 probability sinks

If $x_9 = 0$, STOP!
If $x_i = 0$, STOP!

- Classically, roll a dice to determine next step
- Quantumly, the dice is part of the quantum state. Instead of randomizing the dice between steps, apply a unitary operator to it.

Transition probabilities

$$\{ p_1, p_2, \cdots, p_6 \}$$

$$U = \text{reflection about the state}$$

$$\sqrt{p_1} \left| \bullet \right\rangle + \sqrt{p_2} \left| \bullet \bullet \right\rangle$$

$$+ \sqrt{p_3} \left| \bullet \bullet \bullet \right\rangle + \sqrt{p_4} \left| \bullet \bullet \bullet \bullet \right\rangle$$

$$+ \sqrt{p_5} \left| \bullet \bullet \bullet \bullet \bullet \right\rangle + \sqrt{p_6} \left| \bullet \bullet \bullet \bullet \bullet \bullet \right\rangle$$
Classically, roll a dice to determine next step

Quantumly, the dice is part of the quantum state. Instead of randomizing the dice between steps, apply a unitary operator to it.

- Probability sinks in the classical r.w. (inputs $x_i=0$) become **phase flips** in the qu. walk $\Rightarrow$ standard phase flip oracle

Transition probabilities $\{p_1, p_2, \ldots, p_6\}$

$$U = \text{reflection about the state}$$

$$\sqrt{p_1}|\cdot\rangle + \sqrt{p_2}|\cdot\cdot\rangle + \sqrt{p_3}|\cdot\cdot\cdot\rangle + \sqrt{p_4}|\cdot\cdot\cdot\cdot\rangle + \sqrt{p_5}|\cdot\cdot\cdot\cdot\cdot\rangle + \sqrt{p_6}|\cdot\cdot\cdot\cdot\cdot\cdot\rangle$$
The Algorithm:

- Start at the root
- Apply **phase estimation** to the quantum walk with precision $1/\sqrt{N}$ (i.e., run the walk for time $\sqrt{N}$)
  - If phase is 0, output “$\varphi(x)=1$”
  - Otherwise output “$\varphi(x)=0$”
Formula evaluation algorithm

- Convert formula $\varphi$ into a graph $G(\varphi)$
- Define classical random walk on $G(\varphi)$
- Quantize that walk

\[ \sqrt{p_1} | \cdot \rangle + \sqrt{p_2} | \cdot \rangle + \sqrt{p_3} | \cdot \rangle + \sqrt{p_4} | \cdot \rangle + \sqrt{p_5} | \cdot \rangle + \sqrt{p_6} | \cdot \rangle \]

$P(\text{stepping to subtree}) \propto \sqrt{\text{size of that subtree}}$

If $x_i = 0$, STOP!
2. Why It Works
The Algorithm:

- Start at the root
- Apply phase estimation to the quantum walk with precision \(1/\sqrt{N}\) (i.e., run the walk for time \(\sqrt{N}\))
  - If eigenvalue is 0, output “\(\phi(x) = 1\)”
  - Otherwise output “\(\phi(x) = 0\)”

Note:
Precision-\(\delta\) phase estimation on a unitary \(U\), starting at an e-state, returns the e-value to precision \(\delta\), except w/ prob. \(1/4\). It uses \(O(1/\delta)\) calls to c-U.

We need to carry out spectral analysis of the quantum walk \(U(x)\).
Szegedy eigenvalue and eigenvector correspondence [FOCS ‘04]

Quantum coined walk $U(x)$:

$$\sqrt{P \circ P^T}$$

Weighted adjacency matrix $A_{G(x)}$ of $G(x)$:

Note: Much like the [FGG] algorithm, edges to input vertices evaluating to 1 are deleted in $G(x)$. 

$\bigcirc = 1$

$\bullet = 0$

$2|E|$ dimensions

$|V|$ dimensions

eigenvalues & eigenvectors
• Start at the root
• Apply phase estimation to the walk with precision $1/\sqrt{N}$
  • If e-value is 0, output “$\varphi(x)=1$”
  • Otherwise output “$\varphi(x)=0$”

Main Theorem:

• $\varphi(x)=1 \Rightarrow A_{G(x)}$ has eigenvalue-0 e.v. with $\Omega(1)$ support on the root.
• $\varphi(x)=0 \Rightarrow A_{G(x)}$ has no eigenvectors overlapping the root with $|\text{eigenvalue}| < 2/\sqrt{N}$.

The Algorithm:

$\therefore$ Algorithm is correct, except w/ error rate $<1/4$ (say)
• **Theorem:** \( \varphi(x) = 1 \iff \exists \lambda = 0 \) eigenstate of \( A_{G(x)} \) supported on root \( r \).

Proof: By induction, we argue that for every \( v \), a vector \( \alpha \) satisfying constraints for vertices above \( v \) must satisfy:

**Induction hypothesis:**
- \( \varphi_v(x) = 0 \Rightarrow \alpha_v = 0 \)
- \( \varphi_v(x) = 1 \Rightarrow \alpha_v \) can be \( \neq 0 \)
**Induction hypothesis:**
- $\varphi_v(x) = 0 \Rightarrow \alpha_v = 0$
- $\varphi_v(x) = 1 \Rightarrow \alpha_v$ can be $\neq 0$

Base case: $v$ an input

- $x_i = 0$:
  - $\lambda = 0$ eigenvector constraint at $c$ is $\alpha_v = 0$. ✓

- $x_i = 1$:
  - $v$ and $c$ are not connected in $G(x)$, so $\alpha_v$ is not constrained. ✓
• Induction hypothesis:
  • \( \varphi_v(x) = 0 \Rightarrow \alpha_v = 0 \)
  • \( \varphi_v(x) = 1 \Rightarrow \alpha_v \text{ can be } \neq 0 \)

AND gate gadget constraints:
\[
\alpha_{v_1} + \alpha_r = 0 \\
\alpha_{v_2} + \alpha_r = 0 \\
\alpha_{v_3} + \alpha_r = 0
\]
• If any \( \varphi(v_i) = 0 \), \( \alpha_{v_i} = 0 \Rightarrow \alpha_r = 0 \)
• If all \( \varphi(v_i) = 1 \), can scale each \( |\alpha_{T_i}\rangle \) so \( \alpha_{v_1} = \alpha_{v_2} = \alpha_{v_3} \neq 0 \), then set \( \alpha_r = -\alpha_{v_i} \neq 0 \)
• Induction hypothesis:
  - \( \varphi_v(x)=0 \Rightarrow \alpha_v=0 \)
  - \( \varphi_v(x)=1 \Rightarrow \alpha_v \text{ can be } \neq 0 \)

**OR gate gadget constraint:**
\[
\alpha_{v_1} + \alpha_{v_2} + \alpha_{v_3} + \alpha_r = 0 
\]
- \( \alpha_r \text{ can be } \neq 0 \Leftrightarrow \text{at least one } \varphi(v_i)=1 \)

\[\checkmark \text{ OR} \]
Just in case...

\[ \text{AND}(0,0) = 0 \]
\[ \alpha_r = 0 \]

\[ \text{AND}(0,1) = 0 \]
\[ \alpha_r = 0 \]

\[ \text{AND}(1,1) = 1 \]
\[ \alpha_r = -a \]
• **Theorem:** $\varphi(x)=1 \iff \exists \lambda = 0$ eigenstate of $A_{G(x)}$ supported on root $r$.

• **Main Theorem:**
  - $\varphi(x)=1 \implies A_{G(x)}$ has eigenvalue-0 e.v. with $\Omega(1)$ support on the root.
  - $\varphi(x)=0 \implies A_{G(x)}$ has no eigenvectors overlapping the root with $|\text{eigenvalue}|<1/\sqrt{N}$.

• Remains to show support $\alpha_r$ is large $(\Omega(1))$ when $\varphi(r)=0$, and that there is a large spectral gap $(1/\sqrt{N})$ away from $E=0$ when $\varphi(r)=1$.

• Proofs by same induction but quantitative.

\[ \frac{\alpha_p}{\alpha_v} \in (0, s_v \lambda) \quad \text{if true} \]

\[ -\frac{\alpha_v}{\alpha_p} \in (0, s_v \lambda) \quad \text{if false} \]

with $s_v = \sqrt{s_{v_1}^2 + \cdots + s_{v_3}^2} = \sqrt{\text{size}(\varphi_v)}$
[**FGG ‘07**] algorithm

- **Theorem** ([**FGG ‘07, CCJY ‘07**]): A balanced binary AND-OR formula can be evaluated in time $N^{\frac{1}{2}+o(1)}$.

  Analysis by scattering theory.

[**ACRŠZ ‘07**] algorithm

- **Theorem**:
  - An “approximately balanced” AND-OR formula can be evaluated with $O(\sqrt{N})$ queries (optimal for read-once!).
  - A general AND-OR formula can be evaluated with $N^{\frac{1}{2}+o(1)}$ queries.

Running time is $N^{\frac{1}{2}+o(1)}$ in each case, after efficient preprocessing.
**[FGG ‘07] algorithm**

- **Theorem** ([FGG ‘07, CCJY ‘07]): A balanced binary AND-OR formula can be evaluated in time $N^{\frac{1}{2} + o(1)}$.

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- **Theorem ([FGG ‘07, CCJY ‘07]):** A balanced binary AND-OR formula can be evaluated in time $N^{1/2+o(1)}$.
  
  Analysis by scattering theory.

Fixed, by working with coined quantum walks (via Szegedy corr.) instead of continuous-time qu. walks.

- An “approximately balanced” AND-OR formula can be evaluated with $O(\sqrt{N})$ queries (optimal for read-once!).

- A general AND-OR formula can be evaluated with $N^{1/2+o(1)}$ queries.

Running time is $N^{1/2+o(1)}$ in each case, after efficient preprocessing.

**[ACR Š Z ‘07] algorithm**

Running time is $N^{1/2+o(1)}$ in each case, after efficient preprocessing.

Where do o(1) terms come from?
Algorithm for very unbalanced trees

- Problem: We lose control of recursion fudge factors in a very deep formula.
- Intuition: Walk from root will not even reach the farthest leaves in time $\sqrt{N}$.

E.g., if depth is $N$, then gap could be only $1/N$.
Algorithm for very unbalanced trees

- **Problem:** Walk might not even reach the bottom of a deep formula in time $\sqrt{N}$

- **Solution:** **Rebalance** the formula tree (in preprocessing)

**Theorem:** ([Bshouty, Cleve, Eberly ‘91, Bonet & Buss ‘94]) For any NAND formula $\varphi$ and $k \geq 2$, can efficiently construct an equivalent NAND formula $\varphi'$ with
  - $\text{depth}(\varphi') = O(k \log N)$
  - $\text{size}(\varphi') \leq N^{1+1/\log k}$

- **Open Classical ?**: Is [BCE‘91] formula rebalancing optimal?
  - Does there exist formula $\varphi$, $k$ such that every equivalent $\varphi'$ of depth at most $k \log N$ has $\text{size}(\varphi') \geq N^{1+1/\log k}$?

- **Open:** What is the effect of general formula rebalancing on the ADV bound?
Remarks on formula evaluation algorithms:

**Classical vs. Quantum**

- Classical complexity of evaluating balanced k-ary alternating AND-OR tree is \((k/2)^{\text{depth}} = N^{-(1-1/\log_2 k)}\) — approaches \(N\) as \(k\) increases

- Classical complexity of evaluating general AND-OR formulas is not known?

- Classical complexity of evaluating iterative \(\text{MAJ}_3\) formula is unknown: between \(\Omega\left((7/3)^d\right)\) and \(o\left((8/3)^d\right)\)
  - (the generalization of the optimal AND-OR algorithm is not optimal when applied to \(\text{MAJ}_3\) trees)
  - [Jayram, Kumar, Sivakumar ‘03]

- Quantumly, complexity is \(\sqrt{N}\) queries always, all the way up to \(k=N\) (i.e., evaluating \(\text{OR}(x_1, \ldots, x_N)\), Grover search)

- General AND-OR formulas can be evaluated with \(N^{\frac{1}{2}+o(1)}\) queries

- Expanding \(\text{MAJ}_3\) into AND-OR gates gives \(O(\sqrt{5^d})\) quantumly.

- Also, the algorithm generalizes to give optimal algorithm for evaluating iterated \(f\), where \(f\) is any 3-bit function
Span-program-based quantum algorithm for formula evaluation

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[quant-ph/0710.2630]

We present a time-efficient and query-optimal quantum algorithm for evaluating adversary-bound-balanced formulas on an extended gate set. The allowed gates include arbitrary two- and three-bit gates, as well as bounded fan-in AND, OR, PARITY and EQUAL gates. The technique behind the formula evaluation algorithm is a new framework for quantum algorithms based on span programs. For example, the classical complexity of evaluating the balanced ternary majority formula is unknown, and the natural generalization of the standard balanced AND-OR formula evaluation algorithm is known to be suboptimal. In contrast, a generalization of the optimal quantum \{AND, OR, NOT\} formula evaluation algorithm is optimal for evaluating the balanced ternary majority formula.

span programs [Karchmer, Wigderson ‘93],...
Classical learning theory:

**Corollary:** AND-OR formulas of size $N$ are (classically) PAC-learnable in time $2^{N^{1/2+o(1)}}$. \cite{O'Donnell & Servedio '03}

Open problems

- Is the phase estimation needed, or can the walk be run directly?
- Is the eigenstate useful as a witness?
- Open Classical ?: Is [BCE'91] formula rebalancing optimal?
  - Does there exist formula $\varphi, k$ such that every equivalent $\varphi'$ of depth at most $k \log N$ has size($\varphi'$) $\geq N^{1+1/\log k}$?
  - Effect of rebalancing on the adversary lower bound
- Optimal algorithm for more formula types, more span-program-based quantum algorithms; see [quant-ph/0710.2630]

(and many more...)