Splitting vector bundles
and $\mathbb{A}^1$-fundamental groups of
higher dimensional varieties

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Abstract

We study the $\mathbb{A}^1$-homotopy groups of some $\mathbb{A}^1$-connected smooth varieties of dimension $\geq 3$ in the context of the $\mathbb{A}^1$-homotopy classification problem. We construct pairs of $\mathbb{A}^1$-connected smooth proper varieties over any field all of whose $\mathbb{A}^1$-homotopy groups are abstractly isomorphic, yet which are not $\mathbb{A}^1$-weakly equivalent. The examples come from pairs of Zariski locally trivial projective space bundles over projective spaces.

Projectivizations of vector bundles give rise to $\mathbb{A}^1$-fibration sequences, and when the base of the fibration is a smooth $\mathbb{A}^1$-connected variety, the associated long exact sequence of $\mathbb{A}^1$-homotopy groups can be analyzed in detail. In the case of the projectivization of a rank 2 vector bundle, the structure of the $\mathbb{A}^1$-fundamental group depends on the splitting behavior of the vector bundle via a certain obstruction class. For projective bundles of vector bundles of rank $\geq 3$, the $\mathbb{A}^1$-fundamental group is insensitive to the splitting behavior of the vector bundle, but the structure of higher $\mathbb{A}^1$-homotopy groups is influenced by an appropriately defined higher obstruction class.

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1 Introduction

The purpose of this note is to study some aspects of the $\mathbb{A}^1$-homotopy classification problem introduced in [AM10]. We focus here on varieties having dimension $\geq 3$ and, in particular, projective bundles over a smooth $\mathbb{A}^1$-connected base. For $\mathbb{A}^1$-connected smooth varieties of dimension $\leq 2$ the homotopy classification is governed by the $\mathbb{A}^1$-fundamental sheaf of groups (see Section 2 for a definition; henceforth we abbreviate $\mathbb{A}^1$-fundamental sheaf of groups to $\mathbb{A}^1$-fundamental group). In [AM10, Corollary 5.5.1] it is observed that, at least over algebraically closed fields having characteristic 0, two $\mathbb{A}^1$-connected smooth proper surfaces are $\mathbb{A}^1$-weakly equivalent if and only if they have isomorphic $\mathbb{A}^1$-fundamental groups.

For smooth varieties of dimension $\geq 3$ the $\mathbb{A}^1$-fundamental group is still expected to play a significant role in $\mathbb{A}^1$-homotopy classification, but as we will see many new subtleties arise. Foremost, while in dimension 2 one can simply enumerate all possible $\mathbb{A}^1$-homotopy types of $\mathbb{A}^1$-connected smooth proper varieties and compute all the corresponding $\mathbb{A}^1$-fundamental groups (this computation is performed in [AM10, §5]), the corresponding problem in dimension 3 looks significantly more complicated and necessitates a new approach. The first result, whose justification is the essential content of Section 5, states informally that the $\mathbb{A}^1$-fundamental group is not a sufficiently refined invariant to distinguish $\mathbb{A}^1$-homotopy classes of smooth proper varieties beginning in dimension 3.

**Theorem 1** (See Theorem 5.5). Suppose $d$ is an integer $\geq 3$. There exist pointed $\mathbb{A}^1$-connected smooth proper $d$-folds $(X, x)$ and $(X', x')$ that fail to be $\mathbb{A}^1$-weakly equivalent, yet for which $\pi^\mathbb{A}^1_i(X, x)$ is isomorphic to $\pi^\mathbb{A}^1_i(X', x')$ for all $i > 0$.

In a sense, this result is “expected” in analogy with the situation for closed 3-manifolds where lens spaces provide such examples (see Remark 5.1). Moreover, the $\mathbb{A}^1$-Whitehead theorem of [MV99, §3 Proposition 2.14] behaves in the fashion that is expected from topological intuition. The examples arise from arguably the next simplest class of varieties beyond projective spaces: projective bundles of direct sums of line bundles over projective spaces (we say such a vector bundle is split), and this motivates us to study the $\mathbb{A}^1$-homotopy classification problem for projective bundles of not-necessarily split vector bundles.

**Remark 2.** For varieties having dimension $\geq 3$, other aspects of geometry introduce complexity in the structure of the $\mathbb{A}^1$-fundamental group. For example, there exist smooth proper non-projective $\mathbb{A}^1$-connected varieties [AD09, §6], and there exist smooth proper $\mathbb{A}^1$-connected varieties that are not rational [AM10, Example 2.3.4] and both of these phenomena are reflected somewhat in the structure of the $\mathbb{A}^1$-fundamental group.

Voevodsky showed that integral Chow cohomology rings are unstable $\mathbb{A}^1$-homotopy invariants, and we distinguish $\mathbb{A}^1$-homotopy classes of varieties with isomorphic $\mathbb{A}^1$-homotopy groups by direct computation of Chow cohomology rings. However, with the exception of $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, we do not obtain a full $\mathbb{A}^1$-homotopy classification of projective space bundles over projective spaces with total space of dimension 3.

Section 4 studies the $\mathbb{A}^1$-fundamental groups of projective bundles over an $\mathbb{A}^1$-connected base variety; by Proposition 3.5 the restriction on the base actually implies that all such bundles are Zariski locally trivial. Besides our stated interest in the $\mathbb{A}^1$-homotopy classification problem,
another motivation for this investigation is an old question of Schwarzenberger regarding the existence of non-split small rank vector bundles on projective spaces (see, e.g., [MFK94, p. 227] or [OSS80, §4.4] for discussion of this question). A precise form of this question is the conjecture of Hartshorne asserting that a rank 2 vector bundle on $\mathbb{P}^n$ splits so long as $n \geq 7$; see [Har74, Conjecture 6.3], though no counterexamples are known even for $n = 6$. Of course, if this conjecture is true, then the $\mathbb{A}^1$-homotopy type of the projectivization of any rank 2 vector bundle on projective space is indistinguishable from the $\mathbb{A}^1$-homotopy type of the projectivization of a split bundle.

By means of the $\mathbb{A}^1$-Postnikov tower (see Definition 2.11), the $\mathbb{A}^1$-homotopy type of a space is determined by its $\mathbb{A}^1$-homotopy sheaves and certain $k$-invariants, and we begin an investigation of these invariants for projective space bundles. Just as in topology, we study the $\mathbb{A}^1$-homotopy groups of projective space bundles using the theory of $\mathbb{A}^1$-fibrations, via techniques pioneered by F. Morel and M. Wendt [Mor06b, Wen10b], building on results of F. Morel and L.-F. Moser [Mor07, Mos11]. Projective bundles fit into $\mathbb{A}^1$-fibration sequences which give rise to long exact sequences relating the $\mathbb{A}^1$-homotopy groups of the base, fiber and total space (see Corollary 3.22).

Proposition 4.2 and Theorem 4.13 demonstrate that if a vector bundle splits, then the $\mathbb{A}^1$-homotopy groups of the associated projective bundle are very restricted. In particular, the $\mathbb{A}^1$-fundamental group of the projectivization of a split vector bundle is a split extension of the $\mathbb{A}^1$-fundamental group of the base by the $\mathbb{A}^1$-fundamental group of the fiber (i.e., projective space), and we completely describe the group structure on this extension. Furthermore, the higher $\mathbb{A}^1$-homotopy groups of a split bundle are simply a product of the $\mathbb{A}^1$-homotopy groups of the base and fiber.

In contrast to the split case, a certain obstruction class, which we refer to as the Euler class (see Definition 4.8), intercedes in the structure of the $\mathbb{A}^1$-fundamental group of a potentially non-split bundle: the $\mathbb{A}^1$-fundamental group of the total space is an extension of the $\mathbb{A}^1$-fundamental group of the base by a quotient of the $\mathbb{A}^1$-fundamental group of the fiber. Moreover, this extension of sheaves of groups is no longer obviously split. We show that, if the Euler class we construct is trivial, which happens, e.g., if $X = \mathbb{P}^n$, $n \geq 2$, then the $\mathbb{A}^1$-fundamental group of the total space is again a split extension of the $\mathbb{A}^1$-fundamental group of the base by the $\mathbb{A}^1$-fundamental group of the fiber. We summarize our calculations in the situation of projective space.

**Theorem 3** (See Theorems 4.6 and 4.13). Suppose $\mathcal{E}$ is a rank $m$ vector bundle on $\mathbb{P}^n$, $n \geq 2$ and $m \geq 2$, and fix an identification $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. If $m \geq 3$, then we have isomorphisms

$$\pi_{1}^{\mathbb{A}^1}(\mathbb{P}(\mathcal{E})) \xrightarrow{\sim} G_m \times \pi_{1}^{\mathbb{A}^1}(X).$$

If $m = 2$, and furthermore $e(\mathcal{E})$ is trivial (see Definition 4.8), then there is a split short exact sequence of the form

$$1 \longrightarrow \pi_{1}^{\mathbb{A}^1}(\mathbb{P}^1) \longrightarrow \pi_{1}^{\mathbb{A}^1}(\mathbb{P}(\mathcal{E})) \longrightarrow G_m \longrightarrow 1.$$

There are two possible group structures on $\pi_{1}^{\mathbb{A}^1}(\mathbb{P}(\mathcal{E}))$ depending on the class of $c_1(\mathcal{E}) \mod 2$.

**Remark 4.** In [Mor07, Theorem 9], Morel introduces an Euler class, defined via obstruction theory, that provides the obstruction to splitting a rank $n$ vector bundle ($n \geq 4$) on a smooth affine scheme of dimension $\leq n$. The Euler class we define is conceived in the same spirit: it provides
an obstruction to splitting a vector bundle up to the $\mathbb{A}^1$-2-(co)skeleton of the base (see Definition 2.11) in a manner independent of the dimension of $X$ (or whether it is affine), though at the expense of introducing some $\mathbb{A}^1$-connectivity hypotheses. We expect our Euler class coincides with Morel’s Euler class, but we have not checked this. In a sense, our discussion is classical (cf. [MS74, Chapter 12]), with some additional contortions necessitated in order to avoid imposing orientability hypotheses.

Remark 5. In the case where $\mathcal{E}$ is a rank 2 vector bundle and $e(\mathcal{E})$ is non-trivial the situation is definitely different; we explain this in more detail in Example 4.17. The situation for higher $\mathbb{A}^1$-homotopy groups is more complicated: see Corollary 4.3 for some statements in the case of split vector bundles and Lemma 4.23 for a statement regarding possibly non-split vector bundles in the presence of strong $\mathbb{A}^1$-connectivity hypotheses on the base variety (satisfied, for example, if the base variety is a high-dimensional projective space).

Overview of sections

Section 2 recalls a number of general facts from $\mathbb{A}^1$-homotopy theory and $\mathbb{A}^1$-algebraic topology, especially aspects related to the theory of the $\mathbb{A}^1$-fundamental sheaf of groups. Since a number of references in the subject are still not in final form, we have chosen to repeat statements of results to simplify the exposition. Section 3 collects a number of facts about specific fiber sequences related to classifying spaces for $\text{SL}_n, \text{GL}_n$ and $\text{PGL}_n$, which will be necessary for analyzing the various obstructions that arise. Finally, the main results are proven in Sections 4 and 5. We refer the reader to the introduction to each section for a more detailed list of contents.

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2 Review of some $\mathbb{A}^1$-algebraic topology

In this section, we review some basic facts from $\mathbb{A}^1$-homotopy theory, together with some facts about the geometry of projective bundles. The material in this section will be used throughout the paper and we have tried to include enough to make the remainder of the paper reasonably self-contained. Many statements here are contained in Morel’s papers [Mor06b] and [Mor07], because the numbering and content in those papers is likely to change, we have chosen to insert precise theorem statements and some comments about the ingredients involved in proofs to streamline the exposition.
\( \mathbb{A}^1 \)-homotopic preliminaries

Fix a field \( k \); in the body of the text \( k \) is assumed to have characteristic 0 if no alternative hypotheses are given. Write \( Sm_k \) for the category of schemes separated, smooth and finite type over \( k \). When we use the word sheaf we will always mean Nisnevich sheaf on \( Sm_k \). Write \( Spc_k \) for the category of simplicial Nisnevich sheaves on \( Sm_k \); objects of the category \( Spc_k \) will be referred to as \( k \)-spaces, or simply as spaces if \( k \) is clear from context. Sending a smooth scheme \( X \) to the representable functor \( \text{Hom}_{Sm_k}(\cdot, X) \) and taking the associated constant simplicial sheaf (i.e., all face and degeneracy maps are the identity map) determines a fully faithful functor \( Sm_k \to Spc_k \). Spaces will generally be denoted by calligraphic letters, while schemes will be denoted by capital roman letters; we use the composite functor just mentioned to identify \( Sm_k \) with its essential image in \( Spc_k \) and we use the same notation for both a scheme and the associated space. Write \( Spc_k \), for the category of pointed \( k \)-spaces, i.e., pairs \((\mathcal{X}, x)\) consisting of a \( k \)-space \( \mathcal{X} \) and a morphism \( x: \text{Spec } k \to \mathcal{X} \).

We write \( \mathcal{H}_s^{Nis}(k) \) for the (Nisnevich) simplicial homotopy category and \( \mathcal{H}_s^{Nis}(k) \) for its pointed analog; each of these categories is the homotopy category of a model structure on \( Spc_k \) or \( Spc_k \). The set of morphisms between two spaces in \( \mathcal{H}_s^{Nis}(k) \) or \( \mathcal{H}_s^{Nis}(k) \) will be denoted \([\cdot, \cdot]_s \), though when considering pointed homotopy classes of maps, the base-point will be explicitly specified unless it is clear from context; see [MV99 §2 Definition 1.2] for a precise definition. We write \( \Sigma_i \) for the simplicial suspension operation \( S^i \wedge - \). As usual, simplicial suspension is left adjoint to (simplicial) looping. We write \( RHS_c^{\mathcal{X}} \) for the space \( \Omega_c^{\mathcal{X}} \) of \( \mathcal{X} \), where \( \mathcal{X} \) is a simplicially fibrant model of \( \mathcal{X} \).

We use the Morel-Voevodsky \( \mathbb{A}^1 \)-homotopy category \( \mathcal{H}(k) \) and its pointed analog \( \mathcal{H}_s(k) \); these categories are constructed as a (left) Bousfield localization of either \( \mathcal{H}_s^{Nis}(k) \) or \( \mathcal{H}_s^{Nis}(k) \) (see [MV99 §3 Definition 3.2]). In particular, the category \( \mathcal{H}(k) \) is equivalent to the subcategory of \( \mathcal{H}_s^{Nis}(k) \) consisting of \( \mathbb{A}^1 \)-local objects, and the inclusion of the subcategory of \( \mathbb{A}^1 \)-local objects admits a left adjoint called the \( \mathbb{A}^1 \)-localization functor; we write \( L_{\mathbb{A}^1} \) for this functor and we assume that it commutes with finite products (one model of the functor \( L_{\mathbb{A}^1} \) is given in [MV99 §3 Lemma 3.20], and one replaces \( Ex \) in this lemma by the Godement resolution functor via [MV99 §2 Theorem 1.66]). We write \([\cdot, \cdot]_{\mathbb{A}^1} \) for morphisms in \( \mathcal{H}(k) \) or \( \mathcal{H}_s(k) \) and explicitly indicate the base-point in the latter case. The most important fact regarding the \( \mathbb{A}^1 \)-local model structure is that it is right proper, i.e., the pullbacks of \( \mathbb{A}^1 \)-weak equivalences along \( \mathbb{A}^1 \)-fibrations are again \( \mathbb{A}^1 \)-weak equivalences (see [MV99 §2 Theorem 3.2]). We will also import various definitions from classical homotopy theory (e.g., regarding connectivity) to the simplicial or \( \mathbb{A}^1 \)-homotopy category by prepending either the word simplicial or \( \mathbb{A}^1 \); when we do not give precise definitions, the terms are defined in analogy with the classical setting.

If \( \mathcal{X} \) is a space, the sheaf of simplicial connected components, denoted \( \pi^s_0(\mathcal{X}) \), is the Nisnevich sheaf on \( Sm_k \) associated with the presheaf \( U \mapsto [U, \mathcal{X}] \). Similarly, the sheaf of \( \mathbb{A}^1 \)-connected components of \( \mathcal{X} \), denoted \( \pi^s_{\mathbb{A}^1}(\mathcal{X}) \) is the Nisnevich sheaf on \( Sm_k \) associated with the presheaf \( U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1} \). A space \( \mathcal{X} \) is simplicially connected if the canonical morphism \( \pi^s_0(\mathcal{X}) \to \text{Spec } k \) is an isomorphism, and \( \mathbb{A}^1 \)-connected if the canonical morphism \( \pi^s_{\mathbb{A}^1}(\mathcal{X}) \to \text{Spec } k \) is an isomorphism. We recall the following result, which provides many examples of \( \mathbb{A}^1 \)-connected varieties, and suffices to establish \( \mathbb{A}^1 \)-connectedness of most of the examples in the remainder of the paper.
Proposition 2.1. If $k$ is a field, and $X$ is a smooth scheme admitting an open cover by open sets isomorphic to affine space (with non-empty intersections), then $X$ is $\mathbb{A}^1$-connected.

If $(\mathcal{X}, x)$ is a pointed space, then $\pi_i^\text{Nis}(\mathcal{X}, x)$ is the Nisnevich sheaf on $\text{Sm}_k$ associated with the presheaf $U \mapsto [S^i_\text{Nis} \wedge U, \mathcal{X}]_{\text{Nis}}$, and $\pi_i^\mathbb{A}(\mathcal{X}, x)$ is the Nisnevich sheaf on $\text{Sm}_k$ associated with the presheaf $U \mapsto [S^i \wedge U, \mathcal{X}]_{\mathbb{A}^1}$ (here, we take pointed homotopy classes of maps). These are sheaves of groups for $i > 0$, and sheaves of abelian groups for $i > 1$. For notational compactness, we will often suppress base-points when writing $\mathbb{A}^1$-homotopy groups.

A presheaf $\mathcal{F}$ on $\text{Sm}_k$ is $\mathbb{A}^1$-invariant if for every $U \in \text{Sm}_k$, the canonical map $\mathcal{F}(U) \to \mathcal{F}(U \times \mathbb{A}^1)$ is a bijection. A sheaf of groups $G$ (possibly non-abelian) is strongly $\mathbb{A}^1$-invariant if the cohomology presheaves $H^i_{\text{Nis}}(\cdot, G)$ are $\mathbb{A}^1$-invariant for $i = 0, 1$. A sheaf of abelian groups $A$ is strictly $\mathbb{A}^1$-invariant if the cohomology presheaves $H^i_{\text{Nis}}(\cdot, A)$ are $\mathbb{A}^1$-invariant for $i \geq 0$. The main structural properties of the sheaves $\pi_i^\mathbb{A}(\mathcal{X}, x)$ are summarized in the following fundamental results due to Morel.

Theorem 2.2 (Mor06b Theorem 3.1 and Corollary 3.3). If $(\mathcal{X}, x)$ is a pointed space, then for any integer $i > 0$ the sheaves $\pi_i^\mathbb{A}(\mathcal{X}, x)$ are strongly $\mathbb{A}^1$-invariant.

Theorem 2.3 (Mor06b Theorem 3.25). A strongly $\mathbb{A}^1$-invariant sheaf of abelian groups is strictly $\mathbb{A}^1$-invariant.

Notation 2.4. We write $\mathcal{G}r^\mathbb{A}_k$ for the category of strongly $\mathbb{A}^1$-invariant sheaves of groups, and $\mathcal{A}b^\mathbb{A}_k$ for the category of strictly $\mathbb{A}^1$-invariant sheaves of groups.

Simplicial homotopy classification of torsors

Let $G$ be a Nisnevich sheaf of groups. Let $EG$ denote the Čech construction of the epimorphism $G \to \text{Spec} k$, and let $BG$ denote the (Nisnevich) sheaf quotient $EG/G$ for the diagonal (right) action of $G$ on $EG$. The space $BG$, which is the simplicial classifying space for $G$, has a canonical base point, which we denote $*$ in the sequel. We write $H^i_{\text{Nis}}(\mathcal{X}, G)$ for the set of Nisnevich locally trivial $G$-torsors over $\mathcal{X}$, i.e., triples $(P, \pi, G)$ consisting of (right) $G$-space $P$, a morphism $\pi : P \to \mathcal{X}$ equivariant for the trivial right action on $X$, and an isomorphism of Nisnevich sheaves $G \times P \to P \times_{\mathcal{X}} P$. The terminology classifying space is justified by the following result.

Theorem 2.5 (MV99 §4 Propositions 1.15 and 1.16). If $G$ is a Nisnevich sheaf of groups, then for any space $\mathcal{X}$ there is a canonical bijection

$$[\mathcal{X}, BG]_s \simto H^1_{\text{Nis}}(\mathcal{X}, G).$$

Moreover, for any integer $i > 0$, and any smooth scheme $U$, the group $[\Sigma^i U, BG]_s$ is isomorphic to $G(U)$ if $i = 1$ and is trivial if $i > 1$.

Remark 2.6. If $G$ is a linear algebraic group viewed as a Nisnevich sheaf of groups and $X$ is a smooth scheme, the set $H^1_{\text{Nis}}(X, G)$ studied above can actually be identified with the set of Nisnevich locally trivial $G$-torsors on $X$ in the usual sense. For our purposes, it suffices to observe that a Nisnevich locally trivial $G$-torsor on $X$ in the usual sense gives rise to a $G$-torsor on $X$ in the sense above by means of the Yoneda embedding.
Remark 2.7. One immediate consequence of the above discussion is the fact that a Nisnevich sheaf of groups $G$ is strongly $\mathbb{A}^1$-invariant if and only if the classifying space $BG$ is $\mathbb{A}^1$-local; we use this observation repeatedly in the sequel. Using Theorem 2.2, the inclusion of the subcategory of strongly $\mathbb{A}^1$-invariant sheaves of groups into the category of Nisnevich sheaves of groups admits a left adjoint defined by $G \mapsto \pi^i_s(BG)$: this left adjoint creates finite colimits, e.g., amalgamated sums.

If $BG^f$ is a (simplicially) fibrant model of $BG$, then any element of $[\mathcal{X}, BG]$ can be represented by a morphism of simplicial sheaves $X \to BG^f$. If $\mathcal{X}$ is simplicially connected, we can choose a base-point $x \in \mathcal{X}(k)$ making the aforementioned morphism a morphism of pointed simplicial sheaves. Thus, any $G$-torsor on a simplicially connected space $\mathcal{X}$ can be represented by a pointed morphism $X \to BG^f$ for an appropriate choice of base-point. In the sequel, $X$ will be an $\mathbb{A}^1$-connected smooth scheme, in which case the $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}X$ is a simplicially connected space by the unstable $\mathbb{A}^1$-connectivity theorem [MV99, §3 Corollary 3.22].

Corollary 2.8. If $G$ is a Nisnevich sheaf of groups, there is a canonical simplicial weak equivalence $R\Omega^1_sBG \simeq G$.

Proof. For any smooth scheme $U$ we have an adjunction

$$[\Sigma^i_sU, R\Omega^1_sBG]_s \sim \rightarrow [\Sigma^i_sU, BG]_s.$$

By Theorem 2.5 if $i = 0$ the presheaf on the right hand side is precisely $G(U)$, and if $i > 0$ it is trivial. In other words, after sheafification, the morphism of spaces $R\Omega^1_sBG \to \pi^i_s(R\Omega^1_sBG) = G$ is a simplicial weak equivalence. 

Generalities on fiber sequences

We fix some results about $\mathbb{A}^1$-fibration sequences; this material is taken from [Hov99 §6.2]. In any model category, there is a notion of loop object. In the setting of $\mathbb{A}^1$-homotopy theory, if $(\mathcal{X}, x)$ is a pointed $\mathbb{A}^1$-fibrant space, then the simplicial loop space $\Omega^1_s\mathcal{X}$ is precisely the model categorical notion of loop space.

If $p : \mathcal{E} \to \mathcal{B}$ is an $\mathbb{A}^1$-fibration between $\mathbb{A}^1$-fibrant objects (equivalently, by [MV99 §2 Proposition 2.28], simplicially fibrant and $\mathbb{A}^1$-local objects), let $\mathcal{F}$ be the fiber of this map. The general formalism of model categories gives an action of $\Omega^1_s\mathcal{B}$ on $\mathcal{F}$, specified functorially. Here is the construction. Given a space $\mathcal{A}$, an element of $[\mathcal{A}, \Omega^1_s\mathcal{B}]_{\mathbb{A}^1}$ can be represented by a morphism $h : \mathcal{A} \times \Delta^1_s \to \mathcal{B}$ (where $\Delta^1_s$ is the simplicial interval). If $u : \mathcal{A} \to \mathcal{F}$ is a morphism, by composition we get a morphism $u' : \mathcal{A} \to \mathcal{E}$. Let $\alpha : \mathcal{A} \times \Delta^1_s \to \mathcal{E}$ be a lift in the diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{u'} & \mathcal{E} \\
\downarrow i_0 & & \downarrow p \\
\mathcal{A} \times \Delta^1_s & \xrightarrow{h} & \mathcal{B},
\end{array}
$$

where $i_0$ is the inclusion at 0. One then defines $[u] \cdot [h] = [w]$, where $w : \mathcal{A} \to \mathcal{F}$ is the unique map such that $i \circ w = \alpha \circ i_1$. By [Hov99 Theorem 6.2.1], this construction gives a well-defined right
action of \( [\mathcal{A}, \Omega \mathcal{B}]_{\mathbb{A}^1} \) on \( [\mathcal{A}, \mathcal{F}] \). In this situation, there is a boundary morphism \( \delta : \Omega^1_0 \mathcal{B} \to \mathcal{F} \) defined as the composite

\[
\delta : \Omega^1_0 \mathcal{B} \longrightarrow \mathcal{F} \times \Omega^1_0 \mathcal{B} \longrightarrow \mathcal{F},
\]

where the first map is the product of the inclusion of the base-point and the identity map, and the second map is the action map just constructed.

**Definition 2.9.** An \( \mathbb{A}^1 \)-fibration sequence is a diagram of pointed spaces \( \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \) together with a right action of \( \Omega^1 \mathcal{Z} \) on \( \mathcal{X} \) that is isomorphic in \( \mathcal{H}(k) \) to a diagram \( \mathcal{F} \to \mathcal{E} \overset{p}{\longrightarrow} \mathcal{B} \) where \( p \) is an \( \mathbb{A}^1 \)-fibration of (pointed) \( \mathbb{A}^1 \)-fibrant spaces with (\( \mathbb{A}^1 \)-homotopy) fiber \( \mathcal{F} \) together with the action of \( \Omega^1 \mathcal{F} \) on \( \mathcal{F} \) discussed above.

Fibration sequences in topology give rise to long exact sequences in homotopy groups; this result can be generalized to the context of an arbitrary pointed model category. Applying this observation in the context of \( \mathbb{A}^1 \)-homotopy theory the next result is a consequence of (the dual of) \([Hov99, \text{Proposition 6.5.3}]\) together with a sheafification argument. (Note: a corresponding result holds for fiber sequences in the simplicial homotopy category as well.)

**Lemma 2.10.** If \( \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \) is an \( \mathbb{A}^1 \)-fibration sequence, then there is a long exact sequence in \( \mathbb{A}^1 \)-homotopy sheaves

\[
\cdots \longrightarrow \pi^{\mathbb{A}^1}_{i+1}(\mathcal{Z}) \overset{\delta_*}{\longrightarrow} \pi^{\mathbb{A}^1}_{i}(\mathcal{X}) \longrightarrow \pi^{\mathbb{A}^1}_{i}(\mathcal{Y}) \longrightarrow \pi^{\mathbb{A}^1}_{i}(\mathcal{Z}) \longrightarrow \cdots,
\]

where \( \delta_* \) is the map on \( \mathbb{A}^1 \)-homotopy sheaves induced by the morphism \( \delta \) of Equation 2.1 and the sequence terminates with \( \pi^{\mathbb{A}^1}_0(\mathcal{Z}) \).

**Postnikov towers in \( \mathbb{A}^1 \)-homotopy theory**

**Definition 2.11.** An \( \mathbb{A}^1 \)-Postnikov tower for a pointed space \( (\mathcal{X}, x) \) consists of a sequence of pointed spaces \( (\mathcal{X}^{(n)}, x) \), together with (pointed) maps \( i_n : \mathcal{X} \to \mathcal{X}^{(n)} \), and pointed maps \( p_n : \mathcal{X}^{(n)} \to \mathcal{X}^{(n-1)} \) such that the following properties hold.

i) The morphisms \( p_n \) are all \( \mathbb{A}^1 \)-fibrations.

ii) The morphisms \( i_n : \mathcal{X} \to \mathcal{X}^{(n)} \) induce isomorphisms of sheaves \( i_{n*} : \pi^{\mathbb{A}^1}_k(\mathcal{X}) \to \pi^{\mathbb{A}^1}_k(\mathcal{X}^{(n)}) \) for \( k \leq n \).

iii) The sheaves \( \pi^{\mathbb{A}^1}_k(\mathcal{X}^{(n)}) \) are trivial for \( k > n \).

iv) The induced map \( \mathcal{X} \to \text{holim}_n \mathcal{X}^{(n)} \) is an \( \mathbb{A}^1 \)-weak equivalence.

**Theorem 2.12** (Morel-Voevodsky). An \( \mathbb{A}^1 \)-Postnikov tower exists, functorially in the input space \( \mathcal{X} \).

**Sketch of proof.** The proof of this fact is really a construction. The simplicial Postnikov tower is constructed for any space \( \mathcal{X} \) in \([MV99, \text{§2 pp. 57-61}]\); roughly speaking this provides all of the statements above with “\( \mathbb{A}^1 \)” replaced by simplicial. Applying this construction to \( L_{\mathbb{A}^1} \mathcal{X} \) provides
the $\mathbb{A}^1$-Postnikov tower. A simplicial fibration of $\mathbb{A}^1$-local spaces is precisely an $\mathbb{A}^1$-fibration, and thus the $(\mathbb{A}^1$-homotopy-)fibers of the morphisms $p_n$ are spaces of the form $K(\pi_1^{\mathbb{A}^1}(\mathcal{X}), n)$. There induced maps $k_n : \mathcal{X} \to K(\pi_1^{\mathbb{A}^1}(\mathcal{X}), n)$, are cohomology classes called the $k$-invariants of $\mathcal{X}$. Just as in topology, the $\mathbb{A}^1$-homotopy type of a space is completely determined by the $\mathbb{A}^1$-homotopy groups and $k$-invariants.

**Theorem 2.13** ([Mor06b, Theorem 3.28]). If $(\mathcal{X}, x)$ is a pointed, simplicially 0-connected space, then $\mathcal{X}$ is $\mathbb{A}^1$-local if and only if $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ is strongly $\mathbb{A}^1$-invariant and $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ is strictly $\mathbb{A}^1$-invariant whenever $i \geq 2$.

**Sketch of proof.** If $(\mathcal{X}, x)$ is $\mathbb{A}^1$-local, then Theorems 2.2 and 2.3 show that the conditions on $\mathbb{A}^1$-homotopy groups hold. For the other direction, suppose $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ is strongly $\mathbb{A}^1$-invariant and $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ is strictly $\mathbb{A}^1$-invariant for $i \geq 2$. Then the spaces $B_\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ and $K(\pi_1^{\mathbb{A}^1}(\mathcal{X}, x), j)$ are $\mathbb{A}^1$-local spaces for all $i \geq 2$ and arbitrary positive integers $j$. By induction, one deduces that $\mathcal{X}^{(n)}$ is $\mathbb{A}^1$-local since in a simplicial fibration sequence in which both base and fiber are $\mathbb{A}^1$-local, then Theorems 2.2 and 2.3 show that the conditions on $\mathbb{A}^1$-local, then $\mathcal{X}$ must be $\mathbb{A}^1$-local as well.

**Looping and $\mathbb{A}^1$-localization**

If $\mathcal{X}$ is a simplicially fibrant (but not necessarily $\mathbb{A}^1$-local space), then there is a canonical pointed map $\mathcal{X} \to L_{\mathbb{A}^1}\mathcal{X}$ that induces a morphism of spaces

$$\Omega_s^1\mathcal{X} \to L_{\mathbb{A}^1}\Omega_s^1\mathcal{X} \to \Omega_s^1L_{\mathbb{A}^1}\mathcal{X}.$$  

While the first morphism is an $\mathbb{A}^1$-weak equivalence by its very definition, the second morphism is not an $\mathbb{A}^1$-weak equivalence in general. Since $\pi_1^{\mathbb{A}^1}(\Omega_s^1L_{\mathbb{A}^1}\mathcal{X}) = \pi_1^{\mathbb{A}^1}(\mathcal{X})$, by Theorem 2.2, a necessary condition that this morphism be an $\mathbb{A}^1$-weak equivalence is that $\pi_0^{\mathbb{A}^1}(\Omega_s^1\mathcal{X})$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups. Morel proves that this condition is also sufficient.

**Theorem 2.14** ([Mor06b, Theorem 3.46]). The morphism $L_{\mathbb{A}^1}\Omega_s^1\mathcal{X} \to \Omega_s^1L_{\mathbb{A}^1}\mathcal{X}$ is an $\mathbb{A}^1$-weak equivalence if and only if the sheaf of groups $\pi_0^{\mathbb{A}^1}(\Omega_s^1\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant.

**Remark 2.15.** As remarked above, the necessity of the condition in the hypotheses is clear. The proof of the sufficiency uses a sheafified version of the Kan loop group construction (see, e.g., [GJ99, V Theorem 5.10]), Theorems 2.2, 2.3 and 2.13 and some facts about the Borel construction.

**$\mathbb{A}^1$-covering spaces**

One consequence of the existence of the $\mathbb{A}^1$-Postnikov tower is the existence of an $\mathbb{A}^1$-universal covering space and a corresponding collection of results one refers to as $\mathbb{A}^1$-covering space theory. For more details about the construction below, see [Mor06b, §4.1]. If $\mathcal{X}$ is an $\mathbb{A}^1$-connected space, then the definition of the $\mathbb{A}^1$-Postnikov tower shows that $\mathcal{X}^{(0)}$ is $\mathbb{A}^1$-weakly equivalent to a point, and $\mathcal{X}^{(1)}$ is $\mathbb{A}^1$-weakly equivalent to $B\pi_1^{\mathbb{A}^1}(X)$. This observation allows us to make the following definition.
Definition 2.16. If $\mathcal{X}$ is an $\mathbb{A}^1$-connected space, the $\mathbb{A}^1$-universal cover of $\mathcal{X}$, denoted $\tilde{\mathcal{X}}$, is the $\mathbb{A}^1$-homotopy fiber of the map $\mathcal{X} \to \mathcal{X}^{(1)}$.

Remark 2.17. Suppose $\mathcal{X}$ is an $\mathbb{A}^1$-connected space. Fix a base-point $x \in \mathcal{X}(k)$ and set $\pi := \pi^\mathbb{A}^1_1(\mathcal{X}, x)$. Passing to the $\mathbb{A}^1$-universal cover $\tilde{\mathcal{X}} \to \mathcal{X}$ gives rise to an $\mathbb{A}^1$-simply connected space equipped with a free right action of $\pi$. By functoriality of the Postnikov tower, the Postnikov tower of $\tilde{\mathcal{X}}$ admits a $\pi$-action. Keeping track of the action of $\pi$ gives rise to a twisted Postnikov tower of $\mathcal{X}$, which will be necessary for doing obstruction theory.

To justify the name $\mathbb{A}^1$-universal cover recall [Mor06b, Definition 4.1] that a morphism $f : \mathcal{Y}' \to \mathcal{Y}$ is called an $\mathbb{A}^1$-cover if it has the unique right lifting property with respect to morphisms that are simultaneously monomorphisms and $\mathbb{A}^1$-weak equivalences. As a consequence, $\mathbb{A}^1$-covers are $\mathbb{A}^1$-fibrations by their very definition. If $f : (\mathcal{Y}', \mathcal{Y}) \to (\mathcal{Y}, y)$ is a pointed map that is also an $\mathbb{A}^1$-cover (pointed $\mathbb{A}^1$-cover for short), then $f$ induces a long exact sequence in $\mathbb{A}^1$-homotopy groups of a fibration by Lemma 2.10. If $\mathcal{X}$ is an $\mathbb{A}^1$-connected space, the morphism $\tilde{\mathcal{X}} \to \mathcal{X}$ is an $\mathbb{A}^1$-cover in this sense by [Mor06b, Theorem 4.8] and it is universal among $\mathbb{A}^1$-connected $\mathbb{A}^1$-coverings of $\mathcal{X}$.

Lemma 2.18. If $f : (\mathcal{Y}', \mathcal{Y}) \to (\mathcal{Y}, y)$ is a pointed $\mathbb{A}^1$-cover, then the induced map
\[ \pi^\mathbb{A}^1_i(\mathcal{Y}') \to \pi^\mathbb{A}^1_i(\mathcal{Y}) \]
is an isomorphism for $i > 1$.

Proof. We can assume that $\mathcal{Y}$ is $\mathbb{A}^1$-connected by replacing $\mathcal{Y}$ by the $\mathbb{A}^1$-connected component of the base-point $y$, denoted $\mathcal{Y}^0$: the base-point $y$ induces a morphism $y \to \pi^\mathbb{A}^1_0(\mathcal{Y})$, and we define $\mathcal{Y}^0$ to be the $\mathbb{A}^1$-homotopy fiber product of the diagram
\[ y \to \pi^\mathbb{A}^1_0(\mathcal{Y}) \leftarrow \mathcal{Y}. \]

We can assume all spaces in question are simplicially fibrant and $\mathbb{A}^1$-local. The morphism $y \to \pi^\mathbb{A}^1_0(\mathcal{Y})$ is an $\mathbb{A}^1$-weak equivalence. Since the pullback of $f$ along the morphism $\mathcal{Y}^0 \to \mathcal{Y}$ is still an $\mathbb{A}^1$-covering, it follows that the induced morphism $\mathcal{Y}^0 \times_\mathcal{Y} \mathcal{Y}^0 \to \mathcal{Y}'$ is again an $\mathbb{A}^1$-weak equivalence by right properness of the $\mathbb{A}^1$-local model structure.

Now, if $\mathcal{Y}$ is assumed $\mathbb{A}^1$-connected, any pointed morphism $\mathcal{Y}' \to \mathcal{Y}$ factors through the $\mathbb{A}^1$-connected component of $\mathcal{Y}'$ containing $\mathcal{Y}$. Thus, we can replace $\mathcal{Y}'$ by the connected component of the basepoint $y'$, and since the inclusion of the connected component is again an $\mathbb{A}^1$-weak equivalence it induces an isomorphism on $\pi^\mathbb{A}^1_i$ for $i > 0$.

By the discussion of the previous two paragraphs, we can assume that $\mathcal{Y}' \to \mathcal{Y}$ is a pointed morphism of $\mathbb{A}^1$-connected spaces. In that case, the morphism $\mathcal{Y}' \to \mathcal{Y}$ factors the universal covering morphism $\tilde{\mathcal{Y}} \to \mathcal{Y}$. The morphism $\tilde{\mathcal{Y}} \to \mathcal{Y}$ induces an isomorphism on $\pi^\mathbb{A}^1_i$ for $i \geq 2$ by the very definition of $\mathbb{A}^1$-universal covering space. Likewise, the morphism $\tilde{\mathcal{Y}} \to \mathcal{Y}'$ is again the $\mathbb{A}^1$-universal covering, and this morphism also induces an isomorphism on $\pi^\mathbb{A}^1_i$ for $i \geq 2$.

The following result, which is a straightforward consequence of obstruction theory and the fact that $\pi^\mathbb{A}^1_1(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant, provides a universality property of $\pi^\mathbb{A}^1_1(\mathcal{X})$; see [Mor07, Lemma B.2.2].
Theorem 2.19. Suppose $(\mathcal{X}, x)$ is a pointed $\mathbb{A}^1$-connected space. If $G$ is any strongly $\mathbb{A}^1$-invariant sheaf of groups, the morphism

$$[(\mathcal{X}, x), BG]_{\mathbb{A}^1} \rightarrow \text{Hom}_{Gr^A_1}(\pi^}\mathbb{A}^1_1(\mathcal{X}), G)$$

induced by evaluation on $\pi^}\mathbb{A}^1_1$ is a bijection.

One consequence of this theorem is the following result that is referred to as the *unstable $\mathbb{A}^1$-connectivity theorem*: a simplicially $i$-connected space is $\mathbb{A}^1$-$i$-connected.

Corollary 2.20 ([Mor06b Theorem 3.38]). Suppose $i \geq 0$ is an integer. If $(\mathcal{X}, x)$ is a pointed simplicially $i$-connected space (i.e., $\pi^j(\mathcal{X}, x) = 0$ for all $j \leq i$), then $L_{\mathbb{A}^1}(\mathcal{X})$ is simplicially $i$-connected.

Proof. If $\mathcal{X}$ is a pointed $0$-connected space, then $L_{\mathbb{A}^1}(\mathcal{X})$ is pointed and $0$-connected by [MV99 §2 Corollary 3.22]. Therefore, suppose $\mathcal{X}$ is a pointed and simplicially $1$-connected space. Since $L_{\mathbb{A}^1}(\mathcal{X})$ is simplicially $0$-connected, it suffices to prove $\pi^}\mathbb{A}^1_1(\mathcal{X}, x) = \pi^i(\mathbb{A}^1_1, \mathcal{X})$ is trivial. Equivalently, it suffices by Theorem 2.19 and the Yoneda lemma to prove that for any strongly $\mathbb{A}^1$-invariant sheaf of groups $G$, the set $[(\mathcal{X}, x), BG]_{\mathbb{A}^1}$ is trivial. If $G$ is strongly $\mathbb{A}^1$-invariant, $BG$ is $\mathbb{A}^1$-local, so the canonical map $[(\mathcal{X}, x), BG]_{\mathbb{A}^1} \rightarrow [(\mathcal{X}, x), BG]_{\mathbb{A}^1}$ is an isomorphism. However, since $\mathcal{X}$ is simplicially $1$-connected, it follows that the canonical map $[(\mathcal{X}, x), BG]_{\mathbb{A}^1} \rightarrow \text{Hom}(\pi^}\mathbb{A}^1_1(\mathcal{X}, x), G)$ is a bijection as well (the homomorphisms on the right-hand side are taken in the category of Nisnevich sheaves of groups). Since $\mathcal{X}$ is simplicially $1$-connected, this pointed set is trivial.

There are a number of ways to prove the result for $i \geq 2$, but all the methods we know implicitly involve Theorem 2.19. We proceed by induction on $i$. Suppose $L_{\mathbb{A}^1}(\mathcal{X})$ is simplicially $(i - 1)$-connected and $\mathcal{X}$ is simplicially $i$-connected. It suffices by the Yoneda lemma to show that if $A$ is an arbitrary strictly $\mathbb{A}^1$-invariant sheaf of groups, then $\text{Hom}_{Gr^A_1}(\pi^}\mathbb{A}^1_1(\mathcal{X}), A)$ is trivial. Indeed, using the fact that $L_{\mathbb{A}^1}(\mathcal{X})$ is simplicially $(i - 1)$-connected, one can show that the induced map

$$[\mathcal{X}, K(A, i)]_{\mathbb{A}^1} \rightarrow \text{Hom}_{Gr^A_1}(\pi^}\mathbb{A}^1_1(\mathcal{X}), A)$$

is a bijection (one argues using obstruction theory; see again [Mor07 Lemma B.2.2]). In that case, the assumption that $A$ be strictly $\mathbb{A}^1$-invariant is equivalent to $K(A, i)$ being $\mathbb{A}^1$-local. Therefore, the map $[\mathcal{X}, K(A, i)]_{\mathbb{A}^1} \rightarrow [\mathcal{X}, K(A, i)_{\mathbb{A}^1}]$ is a bijection, and the set on the left-hand side is trivial since $\mathcal{X}$ is simplicially $i$-connected.

Properties of the $\mathbb{A}^1$-fundamental group

If $T$ is a split torus, $T$ is strongly $\mathbb{A}^1$-invariant. Indeed, $T$ is $\mathbb{A}^1$-invariant since there are no non-constant maps from $\mathbb{A}^1$ to $T$ (in fact, $T$ is $\mathbb{A}^1$-rigid in the sense of [MV99 §3 Example 2.4]), and $H_{\mathbb{N}^1}(\cdot, T)$ is $\mathbb{A}^1$-invariant by homotopy invariance of the Picard group. The following result encodes some facts about the role of torus torsors in $\mathbb{A}^1$-covering space theory.

Proposition 2.21. Assume $(X, x)$ is a pointed $\mathbb{A}^1$-connected smooth scheme, and let $T$ be a split torus.

i) There is a canonical isomorphism $H_{\mathbb{N}^1}(X, T) \sim \text{Hom}_{Gr^A_1}(\pi^}\mathbb{A}^1_1(X, x), T)$.  


ii) If \( f : \bar{X} \to X \) is a \( T \)-torsor with \( \bar{X} \) also \( \mathbb{A}^1 \)-connected, then for \( \bar{x} \in \bar{X}(k) \) satisfying \( f(\bar{x}) = x \), there are a short exact sequence

\[
1 \to \pi^{\mathbb{A}^1}_1(\bar{X}, \bar{x}) \to \pi^{\mathbb{A}^1}_1(X, x) \to \pi^{\mathbb{A}^1}_0(T) \to 1, \text{ and}
\]

iii) there are isomorphisms \( \pi^{\mathbb{A}^1}_i(\bar{X}, \bar{x}) \cong \pi^{\mathbb{A}^1}_i(X, x) \) for all \( i > 1 \).

**Proof.** The first statement is a consequence of Theorem 2.19 using the fact that \( T \) is an abelian sheaf of groups. In that case, the set \( H^1_{\text{Nis}}(X, T) = [X, BT]_{\mathbb{A}^1} \) is the quotient of \([X, BT]_{\mathbb{A}^1}\) by the induced conjugation action of \( T(k) \), which is trivial since \( T \) is abelian. The second statement follows immediately from Lemma 2.10 and Theorem 2.5. The third statement follows immediately from Lemma 2.18. \( \Box \)

**Remark 2.22.** An immediate consequence of the first observation is that the \( \mathbb{A}^1 \)-fundamental group of a smooth proper \( \mathbb{A}^1 \)-connected scheme is non-trivial: take \( T = G_m \) and use the fact that the Picard group of a smooth proper \( k \)-scheme is always non-trivial.

**Geometry and the \( \mathbb{A}^1 \)-fundamental group**

While the \( \mathbb{A}^1 \)-universal cover \( \bar{X} \) of a general \( \mathbb{A}^1 \)-connected smooth scheme \( X \) can be quite complicated, the condition that it again be a smooth scheme is surprisingly restrictive, due to the interaction between representability and strong \( \mathbb{A}^1 \)-invariance for a sheaf of groups, which was explored in some detail in \([AM10] \, \S 4.4\]

**Lemma 2.23.** Assume \( k \) is a perfect field, and \( X \) is an \( \mathbb{A}^1 \)-connected smooth \( k \)-scheme. The \( \mathbb{A}^1 \)-universal cover \( \bar{X} \) is a smooth scheme if and only if \( \pi^{\mathbb{A}^1}_1(X) \) is a torus. Moreover, this torus can be canonically identified with Neron-Severi torus \( T_{\text{NS}(X)} \), i.e., the torus Cartier dual to the Neron-Severi group of \( X \).

**Proof.** We know that \( \pi^{\mathbb{A}^1}_1(X) \) is strongly \( \mathbb{A}^1 \)-invariant and since \( \bar{X} \) is a scheme, it is necessarily a representable strongly \( \mathbb{A}^1 \)-invariant sheaf of groups. By \([AM10] \, \text{Proposition 4.4.2}\] the identity connected component \( G^0 \) of any representable strongly \( \mathbb{A}^1 \)-invariant sheaf of groups \( G \) is an extension of an abelian variety \( A \) by a \( k \)-torus (here, we mean an extension in the sense of algebraic groups, i.e., as étale sheaves of groups). Let \( \Gamma \) be the finite étale group scheme of connected components of \( G \). The morphism \( \pi^{\mathbb{A}^1}_1(X) \to \Gamma \) determines a Nisnevich locally trivial cover of \( X \) with group \( \Gamma \). Since \( X \) is \( \mathbb{A}^1 \)-connected, this cover is necessarily trivial and the homomorphism \( \pi^{\mathbb{A}^1}_1(X) \to \Gamma \) is the trivial map. The composite morphism \( \pi^{\mathbb{A}^1}_1(X) \to A \) determines an element of \( H^1_{\text{Nis}}(X, A) \). Since abelian varieties are Nisnevich flasque (this follows because they are birational and \( \mathbb{A}^1 \)-invariant sheaves), it follows that this homomorphism is trivial. Since \( A \) is \( \mathbb{A}^1 \)-rigid, and in particular \( \pi^{\mathbb{A}^1}_0(A) = A \), the triviality of this torsor contradicts the \( \mathbb{A}^1 \)-connectedness of \( \bar{X} \). Thus, \( \pi^{\mathbb{A}^1}_1(X) \) must be a \( k \)-torus.

Again, since \( X \) is \( \mathbb{A}^1 \)-connected, \( \text{Hom}_{\text{Gr}^{\mathbb{A}^1}_k}((\pi^{\mathbb{A}^1}_1(X), G_m) = \text{Pic}(X) \). Since \( \pi^{\mathbb{A}^1}_1(X) \) is a torus, it follows that \( \text{Pic}(X) \) is a finitely generated free abelian group (it cannot have torsion since torsion would correspond to finite covers of \( X \), which are necessarily trivial again using the fact that \( X \) is \( \mathbb{A}^1 \)-connected). It follows that \( \text{Pic}(X) \) coincides with the Neron-Severi group, i.e., that the
character group of $T$ is precisely the Neron-Severi group and the statement follows from the definition of Cartier duality (see \cite{SGA70} Expose VIII Définition 1.1).

\textbf{Remark 2.24.} A complete characterization of smooth proper toric varieties with $\mathbb{A}^1$-fundamental group a torus in terms of the associated fan is given in \cite[Theorem 6.4]{AD09}, and an extension of this condition to toric models (i.e., equivariant compactifications of possibly non-split tori) is given in \cite[Theorem 1]{Aso11}.

\section*{$\mathbb{A}^1$-homotopy theory of projective spaces and punctured affine spaces}

We now recall some results, due to Morel, regarding the structure of the $\mathbb{A}^1$-homotopy groups of projective space. Taking the $\mathbb{A}^1$-homotopy colimit of the presentation of $\mathbb{P}^1$ by two copies of the affine line intersecting in a copy of $\mathbb{G}_m$ identifies $\mathbb{P}^1$ as $\mathbb{A}^1$-weakly equivalent to $\Sigma_1^* \mathbb{G}_m$ \cite{MV99} §3 Corollary 2.18.

Using induction and a similar open covering by two sets, one shows that $\mathbb{A}^{n-1} \mathbb{G}_m \wedge \mathbb{A}^n$ \cite{MV99} §3 Example 2.20. It follows that $\mathbb{A}^n \setminus 0$ is simplicially $(n-2)$-connected. By the unstable $\mathbb{A}^1$-connectivity theorem (see Corollary 2.20), it follows that $\mathbb{A}^n \setminus 0$ is also $(n-2)$-$\mathbb{A}^1$-connected.

Morel then gives a description of the $(n-1)$st $\mathbb{A}^1$-homotopy group of $\mathbb{A}^n \setminus 0$ in terms of so-called Milnor-Witt K-theory sheaves (see \cite{Mor06a} §4 or \cite{Mor06b} §2 for details regarding this theory). One could take the following result as a definition of Milnor-Witt K-theory sheaves, and this point of view will suffice for much of the paper. However, the results of \cite{Mor06b} §2 actually provide a concrete description of the sections of this sheaf over fields, which completely determine the sheaf since it is strictly $\mathbb{A}^1$-invariant.

\textbf{Proposition 2.25 (\cite{Mor06b} Corollary 3.40).} For every integer $n \geq 2$, there is a canonical isomorphism $\pi_{\mathbb{A}^1, n-1} (\mathbb{A}^n \setminus 0) \cong K_{n+1}^{MW}$.

Combining Propositions 2.21 and 2.25 one can deduce the following computations of the $\mathbb{A}^1$-homotopy groups of projective spaces; these properties will be used repeatedly in Section 4.

\textbf{Lemma 2.26.} Suppose $n$ is an integer $\geq 2$. There are isomorphisms

$$\pi_{\mathbb{A}^1, i} (\mathbb{P}^n) = \begin{cases} \mathbb{G}_m & \text{if } i = 1 \\ 0 & \text{if } 1 < i < n, \text{ and} \\ K_{n+1}^{MW} & \text{if } i = n. \end{cases}$$

\textbf{Remark 2.27.} It is also true that the $i$-th $\mathbb{A}^1$-homotopy group of a product of $\mathbb{A}^1$-connected spaces is the product of the $i$-th $\mathbb{A}^1$-homotopy groups of the factors: this fact follows immediately from the fact that $L_{\mathbb{A}^1}$ commutes with formation of finite products and the corresponding fact for simplicial sets.

To describe the $\mathbb{A}^1$-fundamental group of $\mathbb{P}^1$ one has to work harder. The homotopy colimit description of $\mathbb{P}^1$ above gives the identification $\pi_{\mathbb{A}^1, 1} (\mathbb{P}^1) = \pi_{\mathbb{A}^1, 1} (\Sigma_1^* \mathbb{G}_m)$. Any space of simplicial dimension 0 (e.g., a sheaf of groups) is simplicially fibrant by \cite{MV99} §2 Proposition 1.13. Using this fact, Corollary 2.8 and the loops-suspension adjunction there are a sequence of bijections

$$\Hom_{\text{Sp}_{\mathbb{A}^1} \mathbb{Z}} (\mathbb{G}_m, G) \rightarrow [\mathbb{G}_m, G]_s \rightarrow [\mathbb{G}_m, \Omega^1_\mathbb{A} BG]_s \rightarrow [\Sigma_1^* \mathbb{G}_m, (BG, *)]_s.$$
for any Nisnevich sheaf of groups $G$.

If $G$ is, in addition, strongly $\mathbb{A}^1$-invariant, there is an identification $[\Sigma^1_\mathbb{S} \mathbb{G}_m, (BG, \ast)]_s = [\Sigma^1_\mathbb{X} \mathbb{G}_m, (BG, \ast)]_{\mathbb{A}^1}$. Precomposing the (bijective) morphism of Theorem 2.19 with the sequence of maps described in the previous paragraph gives the following result.

**Lemma 2.28.** If $G$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups, then the morphism

$$\text{Hom}_{\text{Sp}_{\mathbb{S} \mathbb{K}}}(\mathbb{G}_m, G) \longrightarrow \text{Hom}_{\text{Gr}_{\mathbb{A}^1}}(\pi_{\mathbb{A}^1}^\mathbb{A}_1(\mathbb{G}_m), G)$$

described in the previous two paragraphs is a bijection, functorially in $G$.

This identification can be interpreted as saying $\pi_{\mathbb{A}^1}^\mathbb{A}_1(\mathbb{P}^1)$ is the free strongly $\mathbb{A}^1$-invariant sheaf of groups generated by $\mathbb{G}_m$, and for that reason one uses the following notation.

**Notation 2.29.** Set $\mathbb{F}_{\mathbb{A}^1}(1) := \pi_{\mathbb{A}^1}^\mathbb{A}_1(\mathbb{P}^1)$.

Later, we will need more precise information about this sheaf of groups.

**Lemma 2.30.** There is a short exact sequence of sheaves of groups of the form

$$1 \longrightarrow \mathbb{K}^\mathbb{MW}_2 \longrightarrow \mathbb{F}_{\mathbb{A}^1}(1) \longrightarrow \mathbb{G}_m \longrightarrow 1$$

The second to last arrow on the right admits a set-theoretic splitting, i.e., a (sheaf-theoretic) transversal.

**Proof.** The usual $\mathbb{G}_m$-torsor $\mathbb{A}^2 \setminus 0 \longrightarrow \mathbb{P}^1$ is an $\mathbb{A}^1$-covering space and thus gives rise to an $\mathbb{A}^1$-fibration sequence. The first statement then follows immediately by combining Propositions 2.21 and 2.25.

Taking $G = \mathbb{F}_{\mathbb{A}^1}(1)$ in Lemma 2.28, the identity map $\mathbb{F}_{\mathbb{A}^1}(1) \longrightarrow \mathbb{F}_{\mathbb{A}^1}(1)$ corresponds to a pointed morphism of sheaves of sets $\mathbb{G}_m \rightarrow \mathbb{F}_{\mathbb{A}^1}(1)$. Functoriality of the construction in Lemma 2.28 applied to the morphism of strongly $\mathbb{A}^1$-invariant sheaves of groups $\mathbb{F}_{\mathbb{A}^1}(1) \rightarrow \mathbb{G}_m$ shows that the composite map $\mathbb{G}_m \rightarrow \mathbb{G}_m$ is the identity map and thus the claimed morphism is in fact a set-theoretic splitting.

The set-theoretic section gives an isomorphism of sheaves of sets $\psi : \mathbb{K}^\mathbb{MW}_2 \times \mathbb{G}_m \overset{\sim}{\longrightarrow} \mathbb{F}_{\mathbb{A}^1}(1)$.

The group structure on $\mathbb{F}_{\mathbb{A}^1}(1)$ is then specified by a factor set, i.e., a morphism of sheaves $\Phi : \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{K}^\mathbb{MW}_2$ (satisfying an appropriate cocycle condition) by means of the formula

$$\psi(a, x)\psi(b, y) = (ab\Phi(xy), xy).$$

Morel identifies this factor set explicitly. To this end, he constructs a symbol morphism $\sigma_2 : \mathbb{G}_m \wedge \mathbb{G}_m \rightarrow \mathbb{K}^\mathbb{MW}_2$ (see [Mor06b, Theorem 2.37]) that can be precomposed with the epimorphism $\mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m \wedge \mathbb{G}_m$ (collapse $\mathbb{G}_m \vee \mathbb{G}_m$); we refer to this composite morphism also as the symbol morphism.

**Theorem 2.31** ([Mor06b, Theorem 4.29]). *The sheaf $\mathbb{F}_{\mathbb{A}^1}(1)$ is a central extension of $\mathbb{G}_m$ by $\mathbb{K}^\mathbb{MW}_2$ with corresponding factor set $\Phi : \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{K}^\mathbb{MW}_2$ equivalent to the symbol morphism.*
The sheaf of automorphism groups of $F_{A^1}(1)$ admits a rather explicit description. Indeed, we know that $\text{Hom}_{\mathcal{G}^1_k}(F_{A^1}(1), F_{A^1}(1)) = \text{Hom}_{\mathcal{G}^1_k}((G_m, F_{A^1}(1)))$ since $F_{A^1}(1)$ is the free strongly $A^1$-invariant sheaf of groups on $G_m$. The whole sheaf of homomorphisms from $G_m$ to $F_{A^1}(1)$ admits a particular nice description in terms of “contractions.” If $\mathcal{F}$ is a sheaf of pointed sets, sets $\mathcal{F}_1 := \text{Hom}_{\mathcal{G}^1_k}((G_m, \mathcal{F}))$ (see [Mor04 Definition 4.3.10]). This construction is functorial by its very definition and by [Mor06b] Lemma 4.33] respects exact sequences of strongly $A^1$-invariant sheaves of groups. Using these observations, one can prove the following result.

**Lemma 2.32 ([Mor06b] Corollary 4.34]).** There is a canonical isomorphism $F_{A^1}(1) \sim \mathbb{Z} \oplus K^1_{MW}$, and $\text{Aut}(F_{A^1}(1))$ is a sheaf of abelian groups.

**Remark 2.33.** A direct description of the ring structure on $\mathbb{Z} \oplus K^1_{MW}$ requires some information about the Hopf map [Mor06b] §4.3. The addition is precisely component-wise addition, but the multiplication admits the following description. The Hopf construction applied to the multiplication map $G_m \times G_m \to G_m$ induces a morphism of spaces $\Sigma_2 G_m \wedge G_m \to \Sigma_2 G_m$. The first space is $A^1$-weakly equivalent to $A^2 \setminus \{0\}$ and its $A^1$-fundamental group is isomorphic to $K^2_{MW}$. The second space is $A^1$-weakly equivalent to $p^1$, and the $A^1$-Hurewicz homomorphism induces a map $\pi^1_{A^1}(p^1) \to K^2_{MW}$. Equivalently, using the $A^1$-homology theory developed in [Mor06b] §3.2], the map on first $A^1$-homology sheaves coming from the morphism $A^2 \setminus \{0\} \to p^1$ provides a morphism

$$\eta : K^2_{MW} \to K^1_{MW}. \quad \eta : K^2_{MW} \to K^1_{MW}.$$ 

The category $A^1_k$ is symmetric monoidal for the so-called $A^1$-tensor product, which is described in [Mor06a] Remark 3.12. We can identify $\mathbb{Z} \otimes A^1 K^1_{MW} \sim K^1_{MW}$, and Morel shows that $K^1_{MW} \otimes A^1 K^1_{MW} \sim K^2_{MW}$. We abuse notation and write $\eta$ for the composite map

$$K^1_{MW} \otimes A^1 K^1_{MW} \sim K^2_{MW} \eta \sim K^1_{MW},$$

which completely determines the ring structure. At the level of symbols, the full multiplication map is specified by the formula $(n, [x]) \cdot (n', [x']) = (nn', n[x'] + n'[x] + \eta[x][x'])$.

### 3 Projective bundles and other $A^1$-fibrations

In this section, we fix our notation for the study of projective bundles. After reviewing some basic facts and terminology regarding projective bundles, we describe a number of results on $A^1$-fibrations associated with torsors. The section ends with a review some results of M. Wendt, drawing heavily on work of F. Morel and L.-F. Moser, demonstrating that Zariski locally trivial $PGL_n$-torsors and associated projective bundles give rise to $A^1$-fibration sequences. The remainder of the material in the section introduces techniques that allow us to effectively study the aforementioned fiber sequences (when the base of the fibration is $A^1$-connected), and these techniques will be used in the computations in Section 4. Much of this discussion involves relating the fibration sequences for $PGL_n$-torsors, $GL_n$-torsors and $SL_n$-torsors.
Zariski locally trivial projective bundles

Suppose \( X \) is a smooth scheme and \( \mathcal{E} \) is a finite rank locally free sheaf of \( \mathcal{O}_X \)-modules on \( X \); we will refer to \( \mathcal{E} \) as a vector bundle on \( X \). We also systematically abuse notation and identify \( \mathcal{E} \) with the corresponding geometric vector bundle \( \text{Spec} \text{Sym}^* \mathcal{E}^\vee \to X \). As usual, write \( \mathbb{P}(\mathcal{E}) \) for the associated projective bundle with projection morphism \( \mathbb{P}(\mathcal{E}) \to X \). Recall the following properties of projective bundles.

**Theorem 3.1** ([Gro61 §4.1-4.2]). Suppose \( X \) is a scheme and \( \mathcal{E} \) is a rank \( n \) vector bundle on \( X \).

- If \( f : X' \to X \) and \( \mathcal{E}' = f^* \mathcal{E} \), then there is a canonical isomorphism \( X' \times_X \mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}') \).
- If \( \mathcal{L} \) is a line bundle on \( X \), there is a canonical isomorphism \( \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \).
- The set of sections of \( \mathbb{P}(\mathcal{E}) \) is canonically in bijection with the set of locally-free subsheaves \( \mathcal{F} \subset \mathcal{E} \) such that \( \mathcal{E}/\mathcal{F} \) is invertible.

**Definition 3.2.** Suppose \( \mathcal{E} \) is a rank \( n \) vector bundle on a smooth \( k \)-scheme \( X \). An \( \mathbb{A}^1 \)-homotopy section of \( \mathbb{P}(\mathcal{E}) \to X \) consists of an \( \mathbb{A}^1 \)-weak equivalence \( f : X' \to X \) and a subsheaf \( \mathcal{F} \subset f^* \mathcal{E} \) with invertible quotient \( f^* \mathcal{E}/\mathcal{F} \).

**Remark 3.3.** It is not clear (to us) whether existence of an \( \mathbb{A}^1 \)-homotopy section is a strictly weaker condition than existence of a section. Since the Picard group is \( \mathbb{A}^1 \)-homotopy invariant, the invertible quotient \( f^* \mathcal{E}/\mathcal{F} \) on \( X' \) gives rise to an invertible sheaf \( \mathcal{L} \) on \( X \). Even if \( f \) is flat (which is not an obvious consequence of the assumptions) the situation is unclear. Indeed, by the assumptions on \( X \) and \( X' \), the theory of faithfully flat descent implies that \( \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{L}) \to \text{Hom}_{\mathcal{O}_{X'}}(f^* \mathcal{E}, f^* \mathcal{E}/\mathcal{F}) \) is injective (identified with the subspace of homomorphism preserving the descent datum), but it is not clear whether the morphism in question descends.

Fix an identification \( \text{Pic}(\mathbb{P}^m) \cong \mathbb{Z} \). Given an \( n \)-tuple of integers \( \mathbf{a} := (a_1, \ldots, a_n) \), consider the vector bundle \( \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n) \) on \( \mathbb{P}^m \). Set \( \ell(\mathbf{a}) = m \) and call it the length of \( \mathbf{a} \).

**Notation 3.4.** Set \( \mathbb{F}_{m,\mathbf{a}} := \mathbb{P}^m(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)) \).

**Projective bundles over an \( \mathbb{A}^1 \)-connected base**

Suppose \( X \) is a smooth scheme. Since \( \text{PGL}_n \) is a smooth group scheme, [Gro68 Theorem 11.8] guarantees that \( \text{PGL}_n \)-torsors are étale locally trivial. There is a short exact sequence of algebraic groups (i.e., a short exact sequence of étale sheaves of groups)

\[
1 \longrightarrow \mathbb{G}_m \longrightarrow \text{GL}_n \longrightarrow \text{PGL}_n \longrightarrow 1.
\]

The associated long exact sequence in (non-abelian) étale cohomology yields the sequence of maps

\[
\cdots \longrightarrow H^1_{\text{et}}(X, \text{GL}_n) \longrightarrow H^1_{\text{et}}(X, \text{PGL}_n) \xrightarrow{\psi} H^2_{\text{et}}(X, \mathbb{G}_m),
\]
where, as usual, the first two terms are pointed sets.

The inclusion map \( H^1_{\text{zar}}(X, GL_n) \to H^1_{\text{et}}(X, GL_n) \) is an isomorphism, i.e., \( GL_n \) is a special group in the sense of Serre. The projective bundles corresponding to \( PGL_n \)-torsors in the image of the first map are precisely the projectivizations of vector bundles on \( X \). As a consequence, a projective bundle on \( X \) is the projectivization of a vector bundle if and only if the image of the corresponding element of \( H^1_{\text{et}}(PGL_n) \) under \( \psi \) is trivial in \( H^2_{\text{et}}(X, G_m) \). The following result explains why we restrict to Zariski locally trivial projective space bundles in the remainder of the paper.

**Proposition 3.5.** If \( X \) is an \( A^1 \)-connected smooth scheme, and \( \text{char}(k) \) is not divisible by \( n \), then every projective bundle on \( X \) of rank \( n - 1 \) (i.e., with \( n - 1 \)-dimensional projective space fibers) is the projectivization of a vector bundle.

**Proof.** Write \( Br'(X) \) for the prime to \( p \) part of the cohomological Brauer group of a smooth scheme \( X \). By [AM10, Proposition 4.1.2], if \( X \) is \( A^1 \)-connected, the canonical map \( Br'(k) \to Br'(X) \) is an isomorphism. Equivalently, for any integer \( n \) coprime to \( \text{char}(k) \) and any element \([\mathcal{P}]\) of \( H^1_{\text{et}}(X, PGL_n) \) represented by a torsor \( P \to X \), the class \( \psi[\mathcal{P}] \) is pulled back from a class in \( H^2_{\text{et}}(\text{Spec} k, G_m) \).

Since \( X \) is \( A^1 \)-connected, \( X(k) \) is non-empty by [MV99, §3 Remark 2.5]. The given \( PGL_n \)-torsor is the torsor of automorphisms of a projective space bundle. The restriction of our projective space bundle to \( x \) is projective space over \( \text{Spec} k \). As a consequence, the pullback of \( P \to X \) along \( \text{Spec} k \to X \) is trivial. By functoriality, we deduce that the original class \( \psi[P] \) must actually be the trivial class and the projective space bundle is the projectivization of a vector bundle. \( \square \)

**A1-fibration sequences and torsors**

Principal \( G \)-bundles are among the most common examples of fibrations in topology. By Theorem 2.5 if \( G \) is a Nisnevich sheaf of groups, the space \( BG \) classifies Nisnevich locally trivial \( G \)-torsors and Morel gives hypotheses under which \( G \)-torsors give rise to \( A^1 \)-fibration sequences.

**Theorem 3.6 ([Mor06b, Theorem 3.50]).** Suppose \( \mathcal{X} \) is a \( k \)-space. If \( G \) is a Nisnevich sheaf of groups such that \( \pi_0^{A^1}(G) \) is strongly \( A^1 \)-invariant, and if \( \mathcal{P} \to \mathcal{X} \) is a (Nisnevich locally trivial) \( G \)-torsor on \( \mathcal{X} \), then

\[
\mathcal{P} \to \mathcal{X} \to BG
\]

is an \( A^1 \)-fibration sequence.

**Remark 3.7.** Under the hypothesis that \( \pi_0^{A^1}(G) \) is strongly \( A^1 \)-invariant, the map \( L_\mathcal{A}^1 R\Omega^1_0 BG \to \Omega^1_0 L\mathcal{A}^1 BG \) is an \( A^1 \)-weak equivalence. Thus, the action map is a morphism \( L\mathcal{A}^1 G \to L\mathcal{A}^1 \mathcal{P} \) induced by the action of \( G \) on \( P \).

**Some A1-homotopy theory of BSL_n and BGL_n**

We now discuss the relationship between the \( A^1 \)-homotopy types of the classifying spaces of \( BSL_n \) and \( BGL_n \). Classifying spaces are reduced simplicial sets and thus have a canonical base-point and groups will always be pointed by their identity element: for these reasons, we systematically suppress base-points. Functoriality of the classifying space construction applied to the
determinant homomorphism $GL_n \to G_m$ and the inclusion $SL_n \to GL_n$ give rise to a sequence of maps

$$BSL_n \to BGL_n \to BG_m.$$  

These maps fit into an $\mathbb{A}^1$-fibration sequence.

**Lemma 3.8.** There is an $\mathbb{A}^1$-fibration sequence of the form $BSL_n \to BGL_n \to BG_m$.

**Proof.** We use the construction from [MV99 §4.2], sometimes called Totaro’s model for the classifying space [Tot99 Remark 1.4]. If $V$ is the standard $n$-dimensional representation of $GL_n$, then $V$ can be viewed as a representation of $SL_n$ by restriction. For any integer $N \geq 0$, $GL_n$ (or $SL_n$) acts scheme-theoretically freely on the open subscheme $V_N \subset \text{Hom}_k(k^{N+n}, V)$ whose points correspond to surjective linear maps. The quotient $V_N/GL_n$ exists as a smooth scheme and is precisely the Grassmannian $Gr_{n,N}$. Likewise, the quotient $V_N/SL_n$ exists as a smooth scheme and is the total space of a $G_m$-torsor over $Gr_{n,N}$. Therefore, taking the colimit gives rise to a $G_m$-torsor over colim$_N Gr_{n,N}$. The map colim$_N V_N/SL_n \to \text{colim}_N Gr_{n,N}$ is therefore an $\mathbb{A}^1$-covering space.

The simplicial homotopy class of maps colim$_N V_N/SL_n \to BSL_n$ classifying the $SL_n$-torsor $V_N \to V_N/SL_n$ is an $\mathbb{A}^1$-weak equivalence by [MV99 §4 Proposition 2.6], and by the same argument, the map colim$_N V_N/GL_n \to BGL_n$ is also an $\mathbb{A}^1$-weak equivalence. Since $BG_m$ is already $\mathbb{A}^1$-local, the action $R\Omega^1 L_{\mathbb{A}^1} BG_m$ on $BSL_n$ is precisely the action of $G_m$ on $L_{\mathbb{A}^1} BSL_n$.

**Proposition 3.9.** The space $BGL_n$ is $\mathbb{A}^1$-connected, and $BSL_n$ is $\mathbb{A}^1$-simply connected. Furthermore, $\pi^A_1(BGL_n) \cong G_m$, and for every $i > 1$, the inclusion map $SL_n \hookrightarrow GL_n$ induces isomorphisms $\pi^A_i(BSL_n) \cong \pi^A_i(BGL_n)$.

**Proof.** The spaces $BSL_n$ and $BGL_n$ are simplicially connected, and the $\mathbb{A}^1$-localization of a simplicially connected space is $\mathbb{A}^1$-connected by means of the unstable $\mathbb{A}^1$-connectivity theorem [MV99 §2 Corollary 3.22]. Consider the long exact sequence in $\mathbb{A}^1$-homotopy groups stemming from the fibration sequence of Lemma 3.8. By [MV99 §4 Proposition 3.8] we know that $\pi^A_1(BG_m) = G_m$ and $\pi^A_i(BG_m) = 0$, for $i > 1$ and therefore $\pi^A_i(BSL_n) \cong \pi^A_i(BGL_n)$ in this range.

Next, let us show that $BSL_n$ is $\mathbb{A}^1$-simply connected. By Corollary 2.8 we know that $R\Omega^1 BG \to G$ is a simplicial weak equivalence. Thus, $\pi^A_0(R\Omega^1 BSL_n) = \pi^A_0(SL_n) = 1$, which is a strongly $\mathbb{A}^1$-invariant sheaf of groups. As a consequence, the hypotheses of [Mor06b Theorem 3.46] are satisfied, and the canonical morphism $L_{\mathbb{A}^1}(R\Omega^1 BGL) \to R\Omega^1(L_{\mathbb{A}^1} BGL)$ is a simplicial weak equivalence. Therefore, $\pi^A_i(BSL_n) = \pi^A_i(SL_n)$ in particular we see that $\pi^A_1(BSL_n) = \pi^A_1(SL_n) = 1$.

The map $\pi^A_1(BGL_n) \to \pi^A_1(BG_m) = G_m$ is therefore an isomorphism either by the long exact sequence or by means of Proposition 2.21.

**Theorem 3.10 ([Mor06b Theorem 4.20]).** Suppose $n$ is an integer $\geq 2$. There are isomorphisms

$$\pi^A_i(SL_n) \cong \begin{cases} K^\text{MW}_2 & \text{if } n = 2, \\
K^\text{M}_2 & \text{if } n \geq 3. \end{cases}$$

Furthermore, for every integer $i \geq 0$ the maps $\pi^A_{i+1}(BSL_n) \to \pi^A_i(SL_n)$ induced by the fibration sequence $SL_n \to ESL_n \to BSL_n$ are isomorphisms.
Remark 3.11. By Theorem 3.6, since \( \pi^A_0(SL_n) = 1 \), which is strongly \( A^1 \)-invariant, it follows that \( SL_n \)-torsors give rise to \( A^1 \)-fibration sequences. As a consequence, there is an \( A^1 \)-fibration sequence of the form

\[
SL_n \longrightarrow SL_{n+1} \longrightarrow SL_{n+1}/SL_n.
\]

Furthermore, projection onto the first column determines an \( A^1 \)-weak equivalence \( SL_{n+1}/SL_n \rightarrow A^{n+1} \setminus 0 \). The computations of Proposition 2.25 show that if \( n \geq 3 \), the map \( \pi^A_1(SL_n) \rightarrow \pi^A_1(SL_{n+1}) \) is an isomorphism. On the other hand the \( A^1 \)-weak equivalence \( SL_2 \rightarrow A^2 \setminus 0 \) gives a description of \( \pi^A_1(SL_2) \) by Proposition 2.25. Theorem 3.10 is really a statement about the morphism \( \pi^A_1(SL_2) \rightarrow \pi^A_1(SL_3) \); we will come back to this point later.

Remark 3.12. The map \( BGL_n \rightarrow BG_m \) is split by the map on classifying spaces induced by the inclusion of the center \( G_m \rightarrow GL_n \). Moreover, the map \( BGL_n \rightarrow BG_m \) identifies \( BG_m \) with the first stage of the \( A^1 \)-Postnikov tower for \( BGL_n \). Thus, \( BSL_n \) is the \( A^1 \)-1-connected cover of \( BGL_n \).

Some \( A^1 \)-homotopy theory of \( BPGL_n \)

We view \( PGL_n \) as a Nisnevich sheaf of groups, and when we write \( BPGL_n \), we mean the classifying space for this Nisnevich sheaf of groups. In particular, since \( BPGL_n \) is simplicially connected, it follows from the same argument as above that \( BPGL_n \) is \( A^1 \)-connected for every integer \( n > 0 \). We now relate the \( A^1 \)-homotopy theory of \( BPGL_n \) with that of \( BSL_n \).

We have a quotient morphism \( SL_n \rightarrow PGL_n \) with kernel \( \mu_n \). This morphism allows us to view \( SL_n \) as an \( \acute{e}tale \) locally trivial \( \mu_n \)-torsor. Consider the classifying space \( B\acute{e}t\mu_n \); this is obtained by pushing forward the classifying space for the \( \acute{e}tale \) sheaf of groups \( \mu_n \) to the Nisnevich topology (see [MV99] §4 for details of this construction). So long as \( n \) is not divisible by \( char \ k \), the space \( B\acute{e}t\mu_n \) is \( A^1 \)-local by [MV99] §4 Proposition 3.1), and for any smooth scheme \( U \) an adjunction argument shows (see [MV99] §4 Proposition 1.16) that

\[
[U, B\acute{e}t\mu_n]_{A^1} = H^1_{\acute{e}t}(U, \mu_n).
\]

The map \( SL_n \rightarrow PGL_n \) is therefore equivalent to an element of \( [PGL_n, B\acute{e}t\mu_n]_{A^1} \).

Lemma 3.13. If \( char \ k \) does not divide \( n \), there is an \( A^1 \)-fibration sequence of the form

\[
SL_n \longrightarrow PGL_n \longrightarrow B\acute{e}t\mu_n.
\]

Proof. Under the hypothesis, the map \( E\acute{e}t\mu_n \rightarrow B\acute{e}t\mu_n \) is even an \( A^1 \)-covering space (see [Mor06b] Lemma 4.5) for this result). Therefore, we have an \( A^1 \)-fibration sequence

\[
\mu_n \rightarrow E\acute{e}t\mu_n \rightarrow B\acute{e}t\mu_n.
\]

Pick a fibrant model of \( B\acute{e}t\mu_n \) of \( B\acute{e}t\mu_n \) and choose an explicit map \( PGL_n \rightarrow B\acute{e}t\mu_n \) representing the \( \mu_n \)-torsor \( SL_n \rightarrow PGL_n \). By construction \( SL_n \) is the pullback of \( E\acute{e}t\mu_n \rightarrow B\acute{e}t\mu_n \) along the map \( PGL_n \rightarrow B\acute{e}t\mu_n \), and thus this map is an \( A^1 \)-fibration. The fiber of this map is \( \mu_n \) and being already simplicially fibrant and \( A^1 \)-local is the \( A^1 \)-homotopy fiber of the map \( SL_n \rightarrow PGL_n \).

Since \( B\acute{e}t\mu_n \) is \( A^1 \)-local, it follows that \( \Omega^2_\mu B\acute{e}t\mu_n \) is also \( A^1 \)-local, and we can identify it with the space \( \mu_n \) (this is proven along the same lines as Corollary 2.8 using the discussion at the top of [MV99] p. 131). The action of \( \Omega^2_\mu B\acute{e}t\mu_n \) on \( SL_n \) is precisely the (left or right) action of \( \mu_n \subset SL_n \).
To proceed further, we need to extend the fibration sequence of Lemma \[\text{3.13}\] further to the right. First, let us observe that \(PGL_n \to B_{\text{et}} \mu_n\) can also be realized as part of an \(\mathbb{A}^1\)-fibration sequence. Consider the left action of \(SL_n\) on \(PGL_n\) where the center \(\mu_n \subset SL_n\) acts trivially. If \(ESL_n\) is an \(\mathbb{A}^1\)-contractible space with free \(SL_n\)-action, we can consider the space \((PGL_n \times ESL_n)/SL_n\) (here we take the Nisnevich sheaf quotient). The projection map \(PGL_n \to \ast\) gives rise to a morphism 

\[(PGL_n \times ESL_n)/SL_n \to BSL_n,\]

and by construction, this morphism is a (Zariski locally trivial) \(PGL_n\)-torsor over \(BSL_n\). The homomorphism \(SL_n \to PGL_n\) in the definition gives rise to a morphism 

\[(3.1) \quad ESL_n = (SL_n \times ESL_n)/SL_n \to (PGL_n \times ESL_n)/SL_n\]

that is by definition an étale locally trivial \(\mu_n\)-torsor. Therefore, we can pick a morphism \((PGL_n \times ESL_n)/SL_n \to B_{\text{et}} \mu_n\) classifying this étale locally trivial \(\mu_n\)-torsor.

**Lemma 3.14.** If \(\text{char } k\) does not divide \(n\), any representative of the simplicial homotopy class of maps 

\[(PGL_n \times ESL_n)/SL_n \to B_{\text{et}} \mu_n\]

classifying the \(\mu_n\)-torsor of Morphism \[\text{3.1}\] is an \(\mathbb{A}^1\)-weak equivalence.

**Proof.** The statement above is actually independent of the model of \(ESL_n\) chosen. The projection morphism \(\pi : PGL_n \times ESL_n \to (PGL_n \times ESL_n)/SL_n\) is an epimorphism of Nisnevich sheaves, and thus the map \(\tilde{\pi} : (PGL_n \times ESL_n)/SL_n \to \ast\) is a simplicial weak equivalence by [MV99] \(\S 2\) Lemma 1.15. Now, by definition \(\tilde{\pi} \circ \mu = PGL_n \times ESL_n \times SL_n\).

Let \(E' SL_n\) be another \(\mathbb{A}^1\)-contractible space with free \(SL_n\)-action, and write \(\pi' : PGL_n \times E' SL_n \to (PGL_n \times E' SL_n)/SL_n\) for the associated quotient morphism. Also, let \(\pi'' : PGL_n \times ESL_n \times E' SL_n \to (PGL_n \times ESL_n \times E' SL_n)/SL_n\) be the quotient morphism. Now the projections \(p : PGL_n \times ESL_n \times E' SL_n \to PGL_n \times ESL_n\) and \(p' : PGL_n \times ESL_n \times E' SL_n \to PGL_n \times E' SL_n\) induce morphisms of simplicial sheaves

\[\tilde{\pi} = \tilde{\pi}' \to \tilde{\pi}''.\]

Since both \(ESL_n\) and \(E' SL_n\) are \(\mathbb{A}^1\)-contractible by assumption, all of these morphisms are termwise \(\mathbb{A}^1\)-weak equivalences, and therefore \(\mathbb{A}^1\)-weak equivalences by [MV99] \(\S 2\) Proposition 2.14.

Since we can pick the model of \(ESL_n\), let us use Totaro’s model from the proof of Lemma 3.8. In that case the result is precisely [MV99] \(\S 4\) Lemma 2.5.

**Lemma 3.15.** If \(\text{char } k\) does not divide \(n\), there is an \(\mathbb{A}^1\)-fibration sequence of the form

\[PGL_n \to B_{\text{et}} \mu_n \to BSL_n.\]

**Proof.** Since \(\pi_0^{\mathbb{A}^1}(SL_n) = 1\), and the trivial group is strongly \(\mathbb{A}^1\)-invariant, it follows from Theorem 3.6 that \(SL_n\)-torsors give rise to \(\mathbb{A}^1\)-fibration sequences. Consider the \(\mathbb{A}^1\)-fibration sequence given by the universal \(SL_n\)-torsor \(SL_n \to ESL_n \to BSL_n\).

In Lemma 3.14 we constructed an \(\mathbb{A}^1\)-weak equivalence \((PGL_n \times ESL_n)/SL_n \to B_{\text{et}} \mu_n\). Consider the projection map \((PGL_n \times ESL_n)/SL_n \to ESL_n/SL_n = BSL_n\), and pullback the universal
$SL_n$-torsor over $BSL_n$ by means of an explicit representative of this morphism. The result is still an $SL_n$-torsor and therefore an $\mathbb{A}^1$-fibration sequence of the form

$$PGL_n \times ESL_n \longrightarrow (PGL_n \times ESL_n)/SL_n \longrightarrow BSL_n$$

The middle term is already $\mathbb{A}^1$-weakly equivalent to $B_{\text{et}}\mu_n$ and projection map $PGL_n \times ESL_n \rightarrow PGL_n$ is an $\mathbb{A}^1$-weak equivalence since $ESL_n$ is $\mathbb{A}^1$-contractible. The boundary map $SL_n = R\Omega^1_\mathcal{A}BSL_n \rightarrow PGL_n$ is precisely the morphism $SL_n \rightarrow PGL_n$.

Let $\mathcal{H}^1_{\text{et}}(\mu_n)$ be the Nisnevich sheaf associated with the presheaf $U \mapsto H^1_{\text{et}}(U, \mu_n)$; this sheaf is precisely $\pi^0_0((B_{\text{et}}\mu_n)$ by the discussion prior to Lemma 3.13. As an immediate consequence of the long exact sequence in the homotopy groups of the fibration in Lemma 3.15, and the fact that $BSL_n$ is $\mathbb{A}^1$-connected, we deduce the following result.

**Corollary 3.16.** If char $k$ does not divide $n$, the map $\pi^A_0(PGL_n) \rightarrow \pi^A_0(B_{\text{et}}\mu_n) = \mathcal{H}^1_{\text{et}}(\mu_n)$ induced by the classifying map for $\mu_n$-torsor $SL_n \rightarrow PGL_n$ is an isomorphism. In particular, $\pi^A_0(PGL_n)$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups.

**Proof.** The only statement that does not follow immediately from Lemma 3.15 is the fact that $\mathcal{H}^1_{\text{et}}(\mu_n)$ is strongly $\mathbb{A}^1$-invariant. In fact, this sheaf is a strictly $\mathbb{A}^1$-invariant Nisnevich sheaf with transfers; see [MVW06, Lemmas 9.23-24].

**Proposition 3.17.** If char $k$ does not divide $n$, there is an extension of the form

$$1 \longrightarrow \pi^A_1(SL_n) \longrightarrow \pi^A_1(PGL_n) \longrightarrow \mu_n \longrightarrow 1$$

where the middle term is an abelian sheaf of groups, and for every $i \geq 2$, the maps $\pi^A_i(SL_n) \rightarrow \pi^A_i(PGL_n)$ are isomorphisms.

**Proof.** The definition of $B_{\text{et}}\mu_n$ shows that $\pi^A_1(B_{\text{et}}\mu_n) = \mu_n$ (as a Nisnevich sheaf) and $\pi^A_i(B_{\text{et}}\mu_n) = 0$ for $i > 1$. The long exact sequence in $\mathbb{A}^1$-homotopy groups of the $\mathbb{A}^1$-fibration of Lemma 3.13 then gives the identifications for $i \geq 2$.

The fact that $\pi^A_i(PGL_n)$ is abelian follows from the fact that the former can be identified with $\pi^A_2$ of a space. Indeed, we observed in Corollary 3.16 that $\pi^A_0(PGL_n)$ is strongly $\mathbb{A}^1$-invariant. We know that $\pi^A_1(PGL_n) = \pi^A_1(R\Omega^1_\mathcal{A}BPGL_n)$ and, as a consequence of 2.14, we deduce that $\pi^A_1(R\Omega^1_\mathcal{A}BPGL_n) = \pi^A_1(BPGL_n)$, which is abelian.

With the results proven so far, we can actually extend the fiber sequence of Lemma 3.15 one step further to the right.

**Lemma 3.18.** If char $k$ does not divide $n$, there is an $\mathbb{A}^1$-fibration sequence of the form

$$B_{\text{et}}\mu_n \rightarrow BSL_n \rightarrow BPGL_n.$$
Some $\mathbb{A}^1$-local spaces

Let $\Delta^n = \text{Spec } k[x_0, \ldots, x_n]/(\sum_i x_i - 1)$ be the algebraic $n$-simplex. The space $\Delta^*$ is a cosimplicial scheme and thus a cosimplicial $k$-space. If $\mathcal{X}$ is a $k$-space, recall that the Suslin-Voevodsky singular construction of $\mathcal{X}$ is the diagonal of the bisimplicial object $(i, j) \mapsto \text{Hom}(\Delta^i, \mathcal{X}_j)$, where we write $\text{Hom}$ for the internal hom in the category of Nisnevich sheaves, and as usual, $\text{Hom}(\Delta^1, X)(U) = \text{Hom}(\Delta^1 \times U, X)$. The key properties of the singular construction are listed on [MV99, p. 87]; among them are the statements i) $\text{Sing}_A^* (\cdot)$ commutes with formation of finite limits (e.g., products), ii) for any space $\mathcal{X}$, the canonical map $\mathcal{X} \to \text{Sing}_A^* (\mathcal{X})$ is an $\mathbb{A}^1$-weak equivalence, and (iii) $\text{Sing}_A^*$ preserves $\mathbb{A}^1$-fibrations.

**Proposition 3.19** (Morel, Moser). The spaces $\text{Sing}_A^* (\mathbb{A}^{n+1} \setminus 0)$ and $\text{Sing}_A^* (\mathbb{P}^n)$ are $\mathbb{A}^1$-local.

**Proof.** Moser proved [Mos11 Theorem 1.1] that $\text{Sing}_A^* (SL_2)$ is $\mathbb{A}^1$-local. For $n > 1$, Morel proved that $\text{Sing}_A^* (SL_{n+1})$ is $\mathbb{A}^1$-local [Mor07 Theorem 13]. There is an affine vector bundle torsor $SL_{n+1}/SL_n \to \mathbb{A}^{n+1} \setminus 0$, which happens to be a unipotent group torsor when $n = 1$. Since such morphisms are Zariski locally trivial with fibers isomorphic to affine spaces, the induced morphisms

$$\text{Sing}_A^* (SL_{n+1}/SL_n) \to \text{Sing}_A^* (\mathbb{A}^n \setminus 0)$$

are simplicial weak equivalences, again by [Mor07, Theorem 13].

We deduce the second statement from the first statement. Since the morphism $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a $G_m$-torsor, and in particular an $\mathbb{A}^1$-fibration, and since $\text{Sing}_A^*$ preserves $\mathbb{A}^1$-fibrations, it follows that the induced map $\text{Sing}_A^* (\mathbb{A}^{n+1} \setminus 0) \to \text{Sing}_A^* (\mathbb{P}^n)$ is an $\mathbb{A}^1$-fibration. In fact, the fiber of this map can also be identified with $G_m$, and there is a simplicial fibration sequence

$$\text{Sing}_A^* (\mathbb{A}^{n+1} \setminus 0) \to \text{Sing}_A^* (\mathbb{P}^n) \to BG_m.$$

This simplicial fibration induces a long exact sequence in simplicial homotopy sheaves using the analog of Lemma 2.10 in the simplicial model structure (constructed in an identical fashion).

To decide whether the term in the middle is $\mathbb{A}^1$-local, we can use Theorem 2.13. The space $\text{Sing}_A^* (\mathbb{P}^n)$ is simplicially 0-connected since both $\text{Sing}_A^* (\mathbb{A}^{n+1} \setminus 0)$ and $BG_m$ are simplicially 0-connected. Furthermore, since $\pi_i^s (BG_m) = 0$ for $i > 0$, the result follows immediately once we know that $\pi_i^s (\text{Sing}_A^* (\mathbb{P}^n))$ is strongly $\mathbb{A}^1$-invariant (since we already know that $\pi_i^s (\text{Sing}_A^* (\mathbb{A}^{n+1} \setminus 0))$ is strictly $\mathbb{A}^1$-invariant). This statement is clear if $n > 1$ since in that case $\pi_i^s (\text{Sing}_A^* (\mathbb{A}^{n+1} \setminus 0)) = \pi_1^s (\mathbb{A}^{n+1} \setminus 0) = 0$. In case $n = 1$, recall that the inclusion of the category of strongly $\mathbb{A}^1$-invariant groups into the category of all Nisnevich sheaves of groups admits a left adjoint: $G \to \pi_1^s (L_{\mathbb{A}^1} BG)$ (see Remark 2.7). From this, it follows immediately that an extension of strongly $\mathbb{A}^1$-invariant sheaves of groups is again strongly $\mathbb{A}^1$-invariant.

### $\mathbb{A}^1$-fibration sequences of associated bundles

Suppose $G$ is a Nisnevich sheaf of groups and $\mathcal{F}$ is a space with a left $G$-action. Given a space $\mathcal{X}$ and a (right) $G$-torsor $\mathcal{P} \to \mathcal{X}$, we can form the contracted product

$$\mathcal{P} \times^G \mathcal{F} := \mathcal{P} \times \mathcal{F} / G,$$
where the quotient is taken in the category of Nisnevich sheaves of groups; a space of the form \( \mathcal{P} \times^G \mathcal{F} \) is called an associated fiber bundle.

If \( G \to \mathcal{P} \to \mathcal{X} \) is an \( \mathbb{A}^1 \)-fibration sequence, then we can consider the associated sequence of morphisms

\[
\mathcal{F} \to \mathcal{P} \times^G \mathcal{F} \to \mathcal{X}.
\]

An action of \( G \) on \( \mathcal{F} \) gives a homomorphism \( G \to \text{Aut}(\mathcal{F}) \). Now, by assumption, we have an action of \( \Omega_1^1 \mathcal{X} \) on \( G \), and by composition we get \( \Omega_1^1 \mathcal{X} \times G \to \text{Aut}(\mathcal{F}) \). Define an action \( \Omega_1^1 \mathcal{X} \) on \( \mathcal{F} \) as the composite

\[
\Omega_1^1 \mathcal{X} \times \mathcal{F} \to \text{Aut}(\mathcal{F}) \times \mathcal{F} \to \mathcal{F}.
\]

If we know that \( \mathcal{F} \) is the \( \mathbb{A}^1 \)-homotopy fiber of \( \mathcal{P} \times^G \mathcal{F} \to \mathcal{X} \), then it follows that this sequence is also an \( \mathbb{A}^1 \)-fibration sequence. This can actually be checked in a number of situations of interest, as was demonstrated by Morel and Wendt.

Suppose \((X, x)\) is a connected and pointed smooth scheme, and \( E \to X \) is a rank \( n + 1 \) vector bundle on \( X \) with zero section \( i : X \to E \). We write \( E^\circ \) for the space \( E \setminus i(X) \). The induced morphism \( E^\circ \to X \) is a Zariski locally trivial smooth morphism with fibers isomorphic to punctured affine spaces. Fix a base-point in \( \mathbb{A}^{n+1} \setminus 0 \) and \( \mathbb{P}^n \) making the usual map \( \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n \) into a pointed map.

The usual fiber of \( E^\circ \) (resp. \( \mathbb{P}(E) \)) at \( x \) is \( \mathbb{A}^{n+1} \setminus 0 \) (resp. \( \mathbb{P}^n \)) and we use the image of the chosen base-point in \( E^\circ \) (resp. \( \mathbb{P}(E) \)) to obtain a base-point of the latter space. The sequences of maps

\[
\begin{align*}
\mathbb{A}^{n+1} \setminus 0 & \to E^\circ \to X, \text{ and } \\
\mathbb{P}^n & \to \mathbb{P}(E) \to X
\end{align*}
\]

together with covariant functoriality of \( \text{Sing}_{\mathbb{A}^1}^* (X) \) gives rise to a sequences of morphisms of pointed spaces

\[
\begin{align*}
\text{Sing}_{\mathbb{A}^1}^* (\mathbb{A}^{n+1} \setminus 0) & \to \text{Sing}_{\mathbb{A}^1}^* (E^\circ) \to \text{Sing}_{\mathbb{A}^1}^* (X), \text{ and } \\
\text{Sing}_{\mathbb{A}^1}^* (\mathbb{P}^n) & \to \text{Sing}_{\mathbb{A}^1}^* (\mathbb{P}(E)) \to \text{Sing}_{\mathbb{A}^1}^* (X).
\end{align*}
\]

If a sheaf of groups \( G \) acts on a space \( \mathcal{X} \), then there is an induced action of \( \text{Sing}_{\mathbb{A}^1}^* (G) \) on \( \text{Sing}_{\mathbb{A}^1}^* (\mathcal{X}) \). As a consequence, \( \text{Sing}_{\mathbb{A}^1}^* (\mathbb{A}^{n+1} \setminus 0) \) and \( \text{Sing}_{\mathbb{A}^1}^* (\mathbb{P}^n) \) can be considered as spaces with an action of \( \text{Sing}_{\mathbb{A}^1}^* (G) \) for appropriate \( G \).

Suppose \( f : P \to X \) is a \( G \)-torsor over \( X \) for \( G \) one of \( GL_n, SL_n \) or \( PGL_n \). Assume \( F \) is a smooth scheme with \( G \)-action, and that the quotient \( P \times^G F \) exists as a smooth scheme (call it \( Y \)): in this case the scheme quotient, which is also the étale sheaf quotient, coincides with the Nisnevich sheaf quotient (colimits commute); this last hypothesis is satisfied by construction in all the situations we consider (e.g., vector bundles, projectivizations of vector bundles, or vector bundles with their zero section removed).

In [Wen10b, Proposition 4.5], Wendt shows that if \( f : Y \to X \) is a Nisnevich locally trivial fiber space with fibers isomorphic to a smooth scheme \( F \) constructed as above, then

\[
\text{Sing}_{\mathbb{A}^1}^* (F) \to \text{Sing}_{\mathbb{A}^1}^* (Y) \to \text{Sing}_{\mathbb{A}^1}^* (X)
\]
is a simplicial fibration sequence. In [Wen10b, TBA], it is established that for a Nisnevich locally trivial \(G\)-torsors, the associated sequence just mentioned (i.e., take \(F = G\)) is an \(\mathbb{A}^1\)-fibration sequence.

We use the fact that \(\mathbb{A}^n\) is a homogeneous space for both \(GL_{n+1}\) and \(SL_{n+1}\), and that \(\mathbb{P}^n\) is a homogeneous space for \(PGL_{n+1}\), \(GL_{n+1}\) or \(SL_{n+1}\). In the situation above, we have fixed a basepoint in \(\mathbb{P}^n\) and then an identification \(PGL_{n+1}/P \cong \mathbb{P}^n\) for some parabolic subgroup \(P \subset PGL_{n+1}\) (or \(GL_{n+1}\) or \(SL_{n+1}\)). More importantly, the projection map \(PGL_{n+1}/P \to \mathbb{P}^n\) is a Zariski locally trivial fiber bundle. Combining this observation with Proposition 3.19 and a Theorem of Morel asserting \(BS\) is an \(\mathbb{A}^1\)-local, Wendt deduces the following result.

**Proposition 3.20 ([Wen10b, TBA]).** Suppose \((X, x)\) is a pointed smooth scheme and \(\mathcal{E}\) is a rank \((n + 1)\)-vector bundle over \(X\). The sequences of pointed spaces

\[
\mathbb{A}^{n+1} \setminus \{0\} \to \mathcal{E}^\circ \to X, \text{ and} \\
\mathbb{P}^n \to \mathbb{P}(\mathcal{E}) \to X, 
\]

(induced by the projections and inclusions discussed above) are \(\mathbb{A}^1\)-fibration sequences.

The existence of this fibration sequence, together with the fact that \(\mathbb{P}^n\) is an \(\mathbb{A}^1\)-connected has the following consequence, which we use without mention in the sequel.

**Corollary 3.21.** If \(\mathcal{E}\) is a vector bundle of rank \(n \geq 2\) on an \(\mathbb{A}^1\)-connected smooth scheme \(X\), then both \(\mathcal{E}^\circ\) and \(\mathbb{P}(\mathcal{E})\) are \(\mathbb{A}^1\)-connected.

The vector bundle \(\mathcal{E}\) over \(X\) of the statement determines a simplicial homotopy class of maps \(X \to BGL_{n+1}\). If we assume \(X\) is \(\mathbb{A}^1\)-connected, we can assume that this map is pointed. In that case, Corollary 2.8 shows that the map \(X \to BGL_{n+1}\) is equivalent to a simplicial homotopy class of maps

\[
\mathbb{R} \Omega^1_{\mathbb{A}^1} X \to GL_{n+1},
\]

and this morphism coincides with the connecting homomorphism for the fiber sequence of the \(GL_{n+1}\)-torsor defined by \(\mathcal{E}\). Here, we have used the fact that \(GL_{n+1}\) being a space of simplicial dimension 0 is automatically simplicially fibrant.

Similarly, the (Zariski and hence Nisnevich locally trivial) \(PGL_{n+1}\)-torsor underlying the associated projective bundle is then induced by a simplicial homotopy class of maps \(X \to BPGl^f_{n+1}\), and by adjointness a simplicial homotopy class of maps \(\mathbb{R} \Omega^1_{\mathbb{A}^1} X \to PGL_{n+1}\). This discussion allows us to give a description of the connecting homomorphism \(\delta\) from Equation 2.1 up to \(\mathbb{A}^1\)-homotopy, the connecting map in the fiber sequence of Proposition 3.20 is induced by the composite map

\[
\mathbb{R} \Omega^1_{\mathbb{A}^1} X \to GL_{n+1} \to PGL_{n+1} \to PGL_{n+1}/P = \mathbb{P}^n.
\]

**Corollary 3.22.** Suppose \((X, x)\) is a pointed smooth scheme and \(\mathcal{E}\) is a rank \(n + 1\) vector bundle over \(X\), there is a long exact sequence in \(\mathbb{A}^1\)-homotopy groups of the form

\[
\cdots \to \mathbb{A}^{n+1}_i(X, X) \overset{\delta}{\to} \mathbb{A}^{n+1}_i(\mathbb{P}^n) \to \mathbb{A}^{n+1}_i(\mathbb{P}(\mathcal{E})) \to \mathbb{A}^{n+1}_i(X, x) \to \cdots.
\]

Furthermore, if \(\mathbb{P}(\mathcal{E})\) admits an \(\mathbb{A}^1\)-homotopy section (see Definition 3.2) then the morphism \(\mathbb{A}^{n+1}_i(\mathbb{P}(\mathcal{E})) \to \mathbb{A}^{n+1}_i(X)\) is split.
Proof. As an \(A^1\)-fibration sequence, Lemma 2.10 shows that the sequence of Proposition 3.20 induces a long exact sequence in \(A^1\)-homotopy groups. It suffices to observe that the morphism \(X \to \text{Sing}_{A^1}(X)\) is an \(A^1\)-weak equivalence.

If \(f: X' \to X\) is an \(A^1\)-weak equivalence of smooth schemes, it follows from right properness of the \(A^1\)-local model structure that the induced morphism \(P(f^*E) \to P(E)\) is again an \(A^1\)-weak equivalence. Thus, up to \(A^1\)-weak equivalence, we can replace the fibration sequence in the statement by the corresponding fibration sequence on any \(X'\) that is \(A^1\)-weakly equivalent to \(X\). In particular, if \(P(E)\) admits an \(A^1\)-homotopy section, this gives a morphism \(X \to P(E)\) in the \(A^1\)-homotopy category, which provides the required splitting.

4 Splitting behavior of bundles and \(A^1\)-homotopy groups

In this section, we study the \(A^1\)-homotopy groups of projectivizations of vector bundles by analyzing the connecting homomorphism and splitting behavior of the attached long exact sequence in \(A^1\)-homotopy groups of Corollary 3.22. We begin by recalling the topological analog of the problem under consideration, which is the study of homotopy groups of the projectivization of a real rank \(n\) \((n \geq 2)\) vector bundle on a smooth (not necessarily orientable) connected manifold. In outline, our analysis of the connecting homomorphism proceeds in analogy with the topological situation, though there are a number of interesting differences stemming from dissimilarities between the classical homotopy theory of \(\mathbb{R}P^n\) and the \(A^1\)-homotopy theory of \(\mathbb{P}^n\). From the point of view of the \(A^1\)-fundamental group, the most interesting case is that of rank 2 bundles: the fundamental group of \(\mathbb{R}P^1 = S^1\) is free abelian, but the \(A^1\)-fundamental group of \(\mathbb{P}^1\), while still a free sheaf of groups (see Notation 2.29), is not abelian. Furthermore, while the higher homotopy groups of \(S^1\) are trivial, the higher \(A^1\)-homotopy groups of \(\mathbb{P}^1\) are not, and this leads to potential higher obstructions to splitting (we discuss this last point at the very end of the section).

We show that in the case of split vector bundles, the higher \(A^1\)-homotopy groups decompose as direct sums (Corollary 4.3). We then investigate the case of not necessarily split bundles: the analysis is different depending on whether the rank of the bundle is equal to 2 or \(\geq 3\). The second case is easier: the \(A^1\)-fundamental group of the projectivization of a vector bundle of rank \(\geq 3\) on an \(A^1\)-connected smooth scheme decomposes as a product in a number of cases; see Theorem 4.6 for more details. For projectivizations of bundles of rank 2, we introduce a (twisted) Euler class in Definition 4.8. In a number of situations, our Euler class turns out to control the connecting homomorphism in the long exact sequence of Corollary 3.22 and existence of a homotopy section up to the first stage of the \(A^1\)-Postnikov tower; these results are contained in Theorem 4.13. Finally, Lemma 4.23 contains some weaker statements about higher \(A^1\)-homotopy groups of projective bundles.

Interlude: topological motivation

Suppose \(E \to M\) is a rank \(n\) real vector bundle on a closed connected manifold \(M\). Forming the projective space of lines in \(E\) gives an \(\mathbb{R}P^{n-1}\)-bundle over \(M\) denoted \(\mathbb{P}(E)\). There is an associated
long exact sequence in homotopy groups of a fibration

\[ \cdots \to \pi_{i+1}(M) \to \pi_i(\mathbb{P}^{n-1}) \to \pi_i(\mathbb{P}(E)) \to \pi_i(M) \to \pi_{i-1}(\mathbb{P}^{n-1}) \to \cdots. \]

The structure of the group \( \pi_i(\mathbb{P}^{n-1}) \) \((i > 0)\) depends on \(n\). If \(n = 2\), then \(\mathbb{P}^{n-1} = \mathbb{S}^1\). In that case, \(\pi_i(S^1)\) is non-vanishing only for \(i = 1\), and the map \(\pi_2(M) \to \pi_1(S^1) = \mathbb{Z}\) is induced by the map \(M \to BO(2)\) by evaluation on \(\pi_2\). If \(n > 2\), then the canonical map \(S^{n-1} \to \mathbb{P}^{n-1}\) is a covering space and gives identifications \(\pi_1(\mathbb{P}^{n-1}) = \mathbb{Z}/2\) and \(\pi_i(S^{n-1}) = \pi_i(\mathbb{P}^{n-1})\) for \(i > 1\). In particular, \(\pi_i(\mathbb{P}^{n-1})\) vanishes in the range \(2 < i < n - 1\).

We focus on the case where \(n = 2\). If \(\tilde{M} \to M\) is the universal cover of \(M\), the composite map \(\tilde{M} \to M \to BO(2)\) induces the same map upon applying the functor \(\pi_2\) as the original map \(M \to BO(2)\). Since \(\tilde{M}\) is 1-connected, one knows that \(\text{Hom}(\pi_2(M), \mathbb{Z}) = H^2(M, \mathbb{Z})\) by the Hurewicz theorem and the universal coefficient theorem. The element in the latter group determined by the map \(\tilde{M} \to BO(2)\) is the Euler class of the bundle \(\tilde{M} \to BO(2)\). Thus, the connecting homomorphism is trivial if and only if the pullback of \(E\) to \(\tilde{M}\) has trivial Euler class. Equivalently, this Euler class can be viewed as a “twisted” Euler class on \(M\). Indeed, \(E\) determines an orientation character \(\omega_E : \pi_1(M) \to \pi_1(BO(2)) = \mathbb{Z}/2\) and a corresponding local system \(\mathbb{Z}[\omega_E]\), and the Euler class above can be viewed as an element of \(H^2(M, \mathbb{Z}[\omega_E])\).

If the pullback of \(E\) to \(\tilde{M}\) has trivial Euler class, we deduce that there are isomorphisms \(\pi_i(\mathbb{P}(E)) \to \pi_i(M)\) for \(i > 1\), and \(\pi_1(\mathbb{P}(E))\) is an extension of \(\pi_1(M)\) by \(\mathbb{Z}\). Non-triviality of the Euler class of \(E\) is an obstruction to existence of a nowhere vanishing section. Note: many expositions of this fact restrict to the oriented situation (see, e.g., [MS74] Property 9.7 and Theorem 12.5), but everything works if we use the orientation local system described in the previous paragraph. Triviality of the Euler class together with an obstruction theory argument provides a splitting of the map \(\pi_1(\mathbb{P}(E)) \to \pi_1(M)\). The group structure on \(\pi_1(\mathbb{P}(E))\) is therefore specified by a homomorphism \(\pi_1(M) \to \text{Aut}(\mathbb{Z}) = \mathbb{Z}/2\), induced by conjugation via the splitting.

The conjugation action just specified has a geometric origin. The classifying map of the vector bundle \(E\) gives a map \(\Omega^1 M \to O(2)\) and each element \(g\) of \(O(2)\) gives a map \(S^1 \to S^1\) that does not necessarily preserve the base-point, but each such map gives a morphism \(\pi_1(S^1) \to \pi_1(S^1)\), and the automorphism of \(\pi_1(S^1)\) so-defined only depends on the class in \(\pi_0(O(2))\) of the element \(g\), which is precisely the value of \(\det g\). In other words, the homomorphism \(\pi_1(M) \to \mathbb{Z}/2\) can be identified with \(\omega_E\).

**The determinant construction and the connecting homomorphism**

Suppose \((X, x)\) is a pointed \(\mathbb{A}^1\)-connected smooth scheme and \(\mathscr{E}\) is a rank \((n + 1)\) vector bundle over \(X\) classified by a pointed simplicial homotopy class of maps \(X \to BGL_{n+1}\). We now associate with \(\mathscr{E}\) an analog of the orientation character discussed in the topological motivation. This morphism gives rise to a simplicial homotopy class of maps \(X \to L_{\mathbb{A}^1}BGL_{n+1}\). By functoriality of the Postnikov tower, such a morphism induces a map \(X(1) \to (BGL_{n+1})(1)\). By means of the fibration sequence of Lemma 3.8 we know that \((BGL_{n+1})(1) = BG_m\), and the map \(BGL_{n+1} \to BG_m\) is induced by the determinant. Taking an open cover of \(X\) on which \(\mathscr{E}\) trivializes, an explicit cocycle computation allows one to deduce the following fact about the induced homomorphism \(\pi_1^X(X) \to G_m\), which can be called the \(\mathbb{A}^1\)-orientation character of \(\mathscr{E}\).
Lemma 4.1. The composite homomorphism \( \pi_1^{A^1}(X) \to G_m \) is a line bundle on \( X \) isomorphic to \( \det \mathcal{E} \).

The above identification allows us to say something about the connecting homomorphism of the long exact sequence in \( A^1 \)-homotopy groups attached to the \( A^1 \)-fibration sequence arising from a \( PGL_n \)-torse. Indeed, this connecting homomorphism is induced by the classifying map \( X \to BGL_n \) by Corollary 2.8 and the loops-suspension adjunction, the aforementioned map gives a morphism \( R\Omega^1_sL_{A^1}X \to GL_{n+1} \) and by composing with the quotient map \( GL_{n+1} \to PGL_{n+1} \to \mathbb{P}^n \) we get a sequence of morphisms

\[
R\Omega^1_sL_{A^1}X \to GL_{n+1} \to PGL_{n+1} \to \mathbb{P}^n
\]

The map \( \pi_1^{A^1}(X) \to \pi_0^{A^1}(GL_{n+1}) \) was just identified in terms of \( \det \mathcal{E} \), and since \( \pi_0^{A^1}(\mathbb{P}^n) = * \), it follows that the map \( \pi_1^{A^1}(X) \to \pi_0^{A^1}(\mathbb{P}^n) \), i.e., the map \( \delta_* \) in Corollary 3.22 for \( i = 0 \), is trivial.

We can also identity the morphism

\[
\delta_* : \pi_1^{A^1}(X) \to \pi_0^{A^1}(PGL_{n+1}).
\]

Corollary 3.16 shows that \( \pi_0^{A^1}(PGL_{n+1}) = \mathcal{H}^{A^1}_{et}(\mu_{n+1}) \). By means of Theorem 2.19 such a homomorphism gives a class in \( H^1_{Nis}(X, \mathcal{H}^{A^1}_{et}(\mu_{n+1})) \), and this group is canonically identified with \( \text{Pic}(X)/(n+1)\text{Pic}(X) \). Since this map factors through the morphism \( \pi_1^{A^1}(X) \to G_m \), it is easy to see that this class is precisely the image of \( \det \mathcal{E} \) in \( \text{Pic}(X)/(n+1)\text{Pic}(X) \); this class will be particular important in the structure of the \( A^1 \)-fundamental group.

Split vector bundles

First, let us rephrase the splitting condition homotopy theoretically. Let \( T \) be a maximal torus in \( GL_{n+1} \), and let \( G_m \subset GL_{n+1} \) be the inclusion of the center. The composite map \( T \to GL_{n+1} \to PGL_{n+1} \) identifies \( T/G_m \) with a maximal torus of \( PGL_{n+1} \).

Proposition 4.2. If \( X \) is an \( A^1 \)-connected smooth scheme, and \( \mathcal{E} \) is a split rank \( n+1 \) vector bundle on \( X \), then the connecting homomorphism \( \pi_i^{A^1}(X) \to \pi_i^{A^1}(\mathbb{P}^n) \) in the long exact sequence of Corollary 3.22 is the trivial map for \( i \geq 1 \). Thus, for every integer \( i \geq 1 \) there are split short exact sequences of the form

\[
1 \to \pi_i^{A^1}(\mathbb{P}^n) \to \pi_i^{A^1}(\mathbb{P}(\mathcal{E})) \to \pi_i^{A^1}(X) \to 1.
\]

Proof. The inclusion \( T \hookrightarrow GL_n \) gives rise to a morphism \( BT \to BGL_{n+1} \). If a rank \( n+1 \) vector bundle \( \mathcal{E} \) on a smooth scheme \( X \) splits as a sum of line bundles, then any morphism \( \varphi : X \to BGL_{n+1} \) representing the vector bundle lifts (up to simplicial homotopy) to a morphism \( \tilde{\varphi} : X \to BT \). The space \( BT \) is \( A^1 \)-local and hence \( \pi_i^{A^1}(BT) = \pi_i^{A^1}(\mathbb{P}^n) \) for all \( i > 0 \). Moreover, we know that \( BT \) is \( A^1 \)-connected, \( \pi_i^{A^1}(BT) = T \), and \( \pi_i^{A^1}(BT) = 0 \) for \( i > 1 \).

We observed before the proof of Corollary 3.22 that the morphism \( R\Omega^1_sX \to \mathbb{P}^n \) inducing the connecting homomorphism comes from the classifying map \( X \to BGL_{n+1} \) by applying the simplicial loops functor. If the morphism \( X \to BGL_{n+1} \) lifts to \( X \to BT \), then the map displayed above factors through a morphism \( R\Omega^1_sX \to T \to PGL_{n+1} \). The result then follows from the computations of the previous paragraph together with the fact that when \( \mathcal{E} \) splits, \( \mathbb{P}(\mathcal{E}) \to X \) admits an actual section. \( \square \)
Corollary 4.3. If $X$ is an $\mathbb{A}^1$-connected smooth scheme, and $\mathcal{E}$ is a split rank $n+1$ vector bundle on $X$, then for every integer $i \geq 2$, there are canonical isomorphisms of the form

$$\pi_{i}^{\mathbb{A}^1}(\mathbb{P}^n) \times \pi_{i}^{\mathbb{A}^1}(X) \sim \pi_{i}^{\mathbb{A}^1}(\mathbb{P}(\mathcal{E})).$$

Proof. Let $f : \tilde{X} \to X$ be the $\mathbb{A}^1$-universal covering. Since $\tilde{X}$ is $\mathbb{A}^1$-simply connected, it follows from Theorem 2.19 that $\text{Pic}(\tilde{X}) := [\tilde{X}, \mathbb{B}G_m]_{\mathbb{A}^1}$ is trivial. Since $\mathcal{E} \equiv \bigoplus l \mathcal{L}_l$ for line bundles $\mathcal{L}_l$ on $X$, it follows that $f^* \mathcal{E} \equiv \bigoplus f^* \mathcal{L}_l$ is trivial. The pullback $Y = \mathbb{P}(\mathcal{E}) \times_X \tilde{X}$ is the projectivization of a trivial bundle and hence a product. By assumption this space is $\mathbb{A}^1$-connected and since the pullback of an $\mathbb{A}^1$-covering space is again an $\mathbb{A}^1$-covering space, it follows that the projection map $Y \to \mathbb{P}(\mathcal{E})$ is an $\mathbb{A}^1$-covering space. The canonical map of the statement is the map on $\mathbb{A}^1$-homotopy groups induced by $Y \to \mathbb{P}(\mathcal{E})$. The stated isomorphisms are then an immediate consequence of Lemma 2.18.

Remark 4.4. We expect that every smooth $\mathbb{A}^1$-connected scheme admits an $\mathbb{A}^1$-covering $X' \to X$ with $X'/\mathbb{A}^1$-connected and $\text{Pic}(X') = 0$. If $X$ is any smooth proper toric variety (e.g., $\mathbb{P}^n$), then we may take $\tilde{X}$ to be its “Cox cover;” this point of view is developed in [AD09, §5-6]. The existence of such a geometric $\mathbb{A}^1$-covering could perhaps be used to strengthen some of the results below.

$\mathbb{A}^1$-fundamental groups of $\mathbb{P}^n$-bundles, $n \geq 2$

Now, we formulate hypotheses under which the connecting homomorphism $\pi_{2}^{\mathbb{A}^1}(X) \to \pi_{1}^{\mathbb{A}^1}(\mathbb{P}^n)$ from Corollary 3.22 is trivial. The analysis breaks into two parts based on the structure of the $\mathbb{A}^1$-fundamental group of $\mathbb{P}^n$: we deal with $n \geq 2$ first, and later with $n = 2$, which is more interesting.

Lemma 4.5. Suppose $n \geq 2$ is an integer, $X$ is an $\mathbb{A}^1$-connected smooth scheme, and $\mathcal{E}$ is a rank $n+1$ vector bundle on $X$. The connecting homomorphism $\pi_{2}^{\mathbb{A}^1}(X) \to \pi_{1}^{\mathbb{A}^1}(\mathbb{P}^n)$ in Corollary 3.22 is trivial.

Proof. By assumption, our $\mathbb{P}^n$-bundle is the projective bundle of a vector bundle. Since the connecting homomorphism in the long exact sequence is induced by the classifying map for this vector bundle, it lifts through a map $\pi_{2}^{\mathbb{A}^1}(X) \to \pi_{1}^{\mathbb{A}^1}(\mathbb{A}^{n+1}\setminus 0)$ (see the fibration sequence for $\mathcal{E}^\circ$ in Proposition 3.20). Since $\mathbb{A}^{n+1}\setminus 0$ is $\mathbb{A}^1$-simply connected if $n \geq 2$, it follows that the connecting homomorphism is trivial.

Theorem 4.6. Let $n \geq 2$ be an integer, and let $\mathcal{E}$ be a rank $n+1$ vector bundle on an $\mathbb{A}^1$-connected smooth scheme $X$. If either i) the $\mathbb{A}^1$-universal cover of $X$ is a split $k$-torus, or ii) $\mathbb{P}(\mathcal{E})$ admits an $\mathbb{A}^1$-homotopy section, then there is an identification

$$\pi_{1}^{\mathbb{A}^1}(\mathbb{P}(\mathcal{E})) \cong G_m \times \pi_{1}^{\mathbb{A}^1}(X).$$

Proof. By Lemma 4.5, the connecting homomorphism $\pi_{2}^{\mathbb{A}^1}(X) \to \pi_{1}^{\mathbb{A}^1}(\mathbb{P}^n)$ is trivial. Assume $\pi_{1}^{\mathbb{A}^1}(X) = T$, where $T$ is a split $k$-torus. The extension

$$1 \to G_m \to \pi_{1}^{\mathbb{A}^1}(\mathbb{P}(\mathcal{E})) \to T \to 1$$

is a (Nisnevich locally trivial) $G_m$-torsor over $T$. Since $T$ is split, and we know that $H^1_{\mathrm{Nis}}(T, G_m) = Pic(T) = 0$, it follows that the extension is trivial.

If $\mathbb{P}(\mathcal{E})$ admits an $\mathbb{A}^1$-homotopy section, then the short exact sequence in question is split. Split extensions of $\pi^A_1(X)$ by $G_m$ are classified by morphisms $\pi^A_1(X) \rightarrow \text{Aut}(G_m)$. We have an isomorphism of sheaves $\text{Aut}(G_m) = Z/2$, and $Z/2$ is strongly $\mathbb{A}^1$-invariant by [MV99, Proposition 3.5]. We claim that all such extensions are trivial. Since $Z/2$ is strongly $\mathbb{A}^1$-invariant, evaluating a pointed homotopy class of maps $X \rightarrow BZ/2$ on $\pi^A_1$ determines a morphism

$$[(X, x), (BZ/2, *)]_{\mathbb{A}^1} \sim \text{Hom}_{\text{Gr}^A_1}(\pi^A_1(X), Z/2).$$

and since $Z/2$ is abelian, we get an identification $[(X, x), (BZ/2, *)]_{\mathbb{A}^1} \sim [X, BZ/2]_{\mathbb{A}^1}$. However, since $X$ is $\mathbb{A}^1$-connected, any Nisnevich locally trivial $Z/2$-torsor is in fact trivial.

**Example 4.7.** Condition (i) of the theorem is satisfied for $X = \mathbb{P}^n$, $n \geq 2$. Every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles (Grothendieck’s theorem [OSS80, Theorem 2.1.1]), and thus the projectivization of any vector bundle on $\mathbb{P}^1$ admits a section (and not just a homotopy section). In other words, condition (ii) is satisfied for $X = \mathbb{P}^1$.

**Euler classes: the construction**

We observed in Remark 3.12 that the $\mathbb{A}^1$-simply connected cover of $BGL_{n+1}$ is precisely $BSL_{n+1}$. Thus, the morphism $X \rightarrow BGL_{n+1}$ gives rise to a morphism of $\mathbb{A}^1$-fibration sequences:

$$\begin{align*}
\tilde{X} & \longrightarrow X \longrightarrow B\pi^A_1(X) \\
BSL_{n+1} & \longrightarrow BGL_{n+1} \longrightarrow BG_m,
\end{align*}$$

and the discussion just prior to the lemma we just proved actually provides an explicit map $\tilde{X} \rightarrow BGL_{n+1}$. The second stage of the $\mathbb{A}^1$-Postnikov tower for this morphism gives a map

$$\tilde{X} \rightarrow K(\pi^A_2(BGL_{n+1}), 2),$$

and this morphism intertwines the $\pi^A_1(X)$-action on $\tilde{X}$ (see Remark 2.17 with the $G_m$-action on $\pi^A_2(BGL_{n+1}) = \pi^A_2(BSL_{n+1})$ (via Proposition 3.9).

This construction provides a morphism $\tilde{X} \rightarrow K(\pi^A_2(BGL_{n+1}), 2)$ that is $\pi^A_1(X)$ equivariant. Forgetting the $\pi^A_1$-equivariance, specifying such a morphism is equivalent to specifying an element of $H^2_{\mathrm{Nis}}(\tilde{X}, \pi^A_2(BGL_{n+1}))$. Using the $\pi^A_1(X)$-equivariance, we can produce a cohomology class on $X$ itself by the following procedure: take the sheaf of sections of the projection morphism $\tilde{X} \times_{\pi^A_1(X)} \pi^A_2(BGL_{n+1})$ to obtain a sheaf $\pi^A_2(BGL_{n+1})(\det(\mathcal{E}))$ on $X$. Just like in [Mor07, Appendix B], we can view the new map as an element of the sheaf $H^2_{\mathrm{Nis}}(X, \pi^A_2(BGL_{n+1})(\det(\mathcal{E})))$ (in reality, this is just shorthand for an equivariant cohomology class). Using the computations of Proposition 3.9 for an explicit description of $\pi^A_2(BGL_{n+1})$, we can make the following definition.
**Definition 4.8.** If $X$ is an $\mathbb{A}^1$-connected smooth scheme, and $\mathcal{E}$ is a rank $2$ (resp. $n + 1$) vector bundle on $X$, the **Euler class**, denoted $e(\mathcal{E})$, is the element in $H^2_{\text{Nis}}(X, K^M_{2}(\det(\mathcal{E})))$ (resp. $H^2_{\text{Nis}}(X, K^M_{n}(\det(\mathcal{E})))$) described above.

**Remark 4.9.** The situation in rank $\geq 3$ is different from rank $2$. The sheaf $K^M_2$ is orientable, in the sense that the Hopf map $\eta$ acts trivially (we will not make this precise here, but see [Mor04, Definition 6.2.5] or [Deg10, Definition 1.2.7]). Using orientability, one can see that the twist by $\det(\mathcal{E})$ in the preceding definition is unnecessary. Since we will not really use this Euler class for vector bundles of rank $\geq 3$, we will ignore the redundancy of the twist.

**Remark 4.10.** The construction of our Euler class is contravariantly functorial for morphisms of $\mathbb{A}^1$-connected smooth schemes. Nevertheless, one would like to make an appropriately functorial definition for smooth schemes that are not necessarily $\mathbb{A}^1$-connected. If $k$ is an infinite perfect field of characteristic unequal to $2$, the group $H^2_{\text{Nis}}(X, K^M_2(\det(\mathcal{E})))$ can be identified with the $\det(\mathcal{E})$-twisted Chow-Witt group $\widetilde{CH}^2(X, \det(\mathcal{E}))$ (see [BM00, Définition 1.2] and [Fas08, §Définition 10.4.6]), but we will not prove or use this here. The stated definition of the Euler class should be equivalent to other definitions that appear in the literature (e.g., [Mor07, Theorem 9] or [Fas08, Définition 13.2.1]), but we will not verify this here.

**Euler classes and the connecting homomorphism**

**Lemma 4.11.** Suppose $X$ is an $\mathbb{A}^1$-connected smooth scheme. Let $\mathcal{E}$ be a rank $2$ vector bundle on $X$ and let $\mathbb{P}(\mathcal{E})$ be the associated projective space bundle. The connecting homomorphism $\pi_2^A(X) \to \pi_1^A(\mathbb{P}^1)$ is trivial if and only if $e(\mathcal{E})$ is trivial.

**Proof.** As we discussed at the beginning of the section, the connecting homomorphism $\delta : R\Omega^1_1L\mathcal{A}_1 X \to \mathbb{P}^n$ is induced by a map $R\Omega^1_1L\mathcal{A}_1 X \to GL_{n+1}$ that comes from the classifying map of the bundle by looping. Let $\tilde{X}$ be the $\mathbb{A}^1$-universal cover of $X$. By definition, the morphism $\pi_2^A(\tilde{X}) \to \pi_2^A(X)$ is an isomorphism and thus the connecting homomorphism in question is trivial if and only if the map $\pi_2^A(\tilde{X}) \to \pi_1^A(\mathbb{A}^1 \setminus 0) = K^M_2$ is trivial. Now, since $\tilde{X}$ is $\mathbb{A}^1$-simply connected, the canonical map

$$H^2_{\text{Nis}}(\tilde{X}, K^M_2) = [\tilde{X}, K^M_2(2)]_{\mathbb{A}^1} \to \text{Hom}_{\text{A}^1_0}(\pi_2^A(\tilde{X}), K^M_2)$$

induced by evaluation on $\pi_2^A(\cdot)$ is a bijection by [Mor07, Lemma B.2.2]. Tracing through our construction of the Euler class shows that the class so obtained is precisely the Euler class. \qed

We state the following lemma for completeness.

**Lemma 4.12.** If $X$ is an $\mathbb{A}^1$-connected smooth scheme, and $\mathcal{E}$ is a split vector bundle on $X$, then $e(\mathcal{E})$ is trivial.

**Proof.** Under the hypothesis, the map $X \to BGL_{n+1}$ lifts through a map $X \to BT$. If $T' = T \cap SL_{n+1}$, then the map $\tilde{X} \to BSL_{n+1}$ lifts through a morphism $\tilde{X} \to BT'$. By Proposition 2.21 there is a bijection

$$[\tilde{X}, BT']_{\mathbb{A}^1} \to \text{Hom}_{\mathbb{A}^1_0}(\pi_2^A(\tilde{X}), T').$$

We state the following lemma for completeness.
However, since $\tilde{X}$ is $\mathbb{A}^1$-simply connected the resulting map must be trivial. Therefore, the defining map of the Euler class factors through an $\mathbb{A}^1$-homotopically trivial map, and the Euler class must itself be trivial.

\[ \square \]

$\mathbb{A}^1$-fundamental groups of $\mathbb{P}^1$-bundles: trivial Euler class

In this section, we describe the $\mathbb{A}^1$-fundamental group of a $\mathbb{P}^1$-bundle with trivial Euler class in a number of situations.

**Theorem 4.13.** Let $\mathcal{E}$ be a rank 2 vector bundle on an $\mathbb{A}^1$-connected smooth scheme $X$. Assume $e(\mathcal{E})$ is trivial and either i) $\pi_1^A(X)$ is a split torus, or ii) $\mathcal{E}$ admits a homotopy section. The short exact sequence

\[ 1 \longrightarrow F_{\mathbb{A}^1}(1) \longrightarrow \pi_1^A(\mathbb{P}(\mathcal{E})) \longrightarrow \pi_1^A(X) \longrightarrow 1. \]

is split, and the group structure on the term in the middle is completely determined by the class of $\det(\mathcal{E})$ in $\text{Pic}(X)/2\text{Pic}(X)$.

**Proof.** First, let us show that if $e(\mathcal{E})$ is trivial, then under assumption (i) the extension is split. We analyze the extension in two stages corresponding to the fact that $F_{\mathbb{A}^1}(1)$ is an extension of $\mathbb{G}_m$ by $K_2^{MW}$. By assumption $\pi_1^A(X)$ is a split torus, call it $T$. Observe that $K_2^{MW}$ is a normal subgroup sheaf of $\pi_1^A(\mathbb{P}(\mathcal{E}))$ and taking the quotient we get a short exact sequence of the form

\[ 1 \longrightarrow G_m \longrightarrow \pi_1^A(\mathbb{P}(\mathcal{E}))/K_2^{MW} \longrightarrow T \longrightarrow 1. \]

Since the Picard group of a split torus is trivial, it follows that this extension is split. The group structure is determined by a homomorphism $T \to \text{Aut}(G_m) = \mathbb{Z}/2$, which is trivial since $X$ is $\mathbb{A}^1$-connected (using the same argument as in the proof of Theorem 4.6). As a consequence, we have $\pi_1^A(\mathbb{P}(\mathcal{E}))/K_2^{MW} \to T \times G_m$.

Next, consider the short exact sequence

\[ 1 \longrightarrow K_2^{MW} \longrightarrow \pi_1^A(\mathbb{P}(\mathcal{E})) \longrightarrow T \times G_m \longrightarrow 1. \]

The action of $T \times G_m$ on $K_2^{MW}$ requires a bit more effort to describe. If $i : X \to \mathcal{E}$ is the zero section of $\mathcal{E}$, by passing to the $G_m$-torsor over $\mathbb{P}(\mathcal{E})$ corresponding to $\mathcal{E}^0 = \mathcal{E} \setminus i(X)$ it will suffice to study the $T$-action on $K_2^{MW}$. This corresponds to the long exact sequence in $\mathbb{A}^1$-homotopy groups associated with the $\mathbb{A}^1$-fibration sequence $\mathbb{A}^2 \setminus 0 \to \mathcal{E}^0 \to X$ of Proposition 3.20.

Let $f : X \to X$ be the $\mathbb{A}^1$-universal covering map: by assumption $X$ is a $T$-torsor over $X$ and thus a smooth scheme. The composite morphism $\tilde{X} \to X \to BGL_2$ factors through the $\mathbb{A}^1$-simply-connected cover of $BGL_2$ (again, see Remark 3.12), which we identified with $BSL_2$. Thus, we have a map $\tilde{X} \to BSL_2$ that is $T$-equivariant for a morphism $T \to G_m = \pi_1^A(BGL_2)$; this class is precisely $\det(\mathcal{E})$ by Lemma 4.1.
Let $\mathcal{E}^\circ|\tilde{X}$ denote the pullback of $\mathcal{E}^\circ$ under the $\mathbb{A}^1$-universal covering map. The map $\mathcal{E}^\circ|\tilde{X} \to \tilde{X}$ is again an $\mathbb{A}^1$-fibration sequence by Proposition 3.20 and we can try to lift along this map. To this end, identify $\tilde{X}(1) = * = \mathcal{E}^\circ|\tilde{X}(0)$, and consider the lifting problem

$$
\begin{array}{ccc}
\mathcal{E}^\circ|^{(1)}_{\tilde{X}} & \rightarrow & * \\
\downarrow & & \downarrow \\
\tilde{X} & \rightarrow & *
\end{array}
$$

coming from the first stage of the $\mathbb{A}^1$-Postnikov tower of $\mathcal{E}^\circ|\tilde{X}$: we would like to know that the dashed arrow exists. Looking at the long exact sequence in $\mathbb{A}^1$-homotopy sheaves of the $\mathbb{A}^1$-fibration $\mathcal{E}^\circ|\tilde{X} \to \tilde{X}$, triviality of the Euler class allows us to conclude that the connecting homomorphism $\pi_{A^1}^0(X) \to \pi_{A^1}^1(\mathbb{A}^2 \setminus 0)$ is trivial. As a consequence, there is an identification $\mathcal{E}^\circ|^{(1)}_{\tilde{X}} = B(K_{MW}^2)$. Via this identification, the Euler class can be viewed as a particular choice of dashed arrow filling in the diagram. Moreover, since the Euler class is equivariant for the action of $\pi_{A^1}^1(X)$ by its very definition, it follows that this class is even an equivariant class. Said differently, triviality of the Euler class provides a splitting of the sequence

$$
1 \longrightarrow K_{MW}^2 \longrightarrow \pi_{A^1}^1(\mathcal{E}^\circ) \longrightarrow T \longrightarrow 1
$$

for the $T$-action on $K_{MW}^2$ induced by its identification with $\pi_{A^1}^1(X)$. This splitting induces a splitting of the short exact sequence from the statement of the theorem.

Suppose the short exact sequence of the statement is split by a homomorphism $\pi_{A^1}^1(X) \to \pi_{A^1}^1(\mathbb{P}(\mathcal{E}))$, and consider the induced conjugation action of $\pi_{A^1}^1(X)$ on the fiber; this action determines a homomorphism

$$
\pi_{A^1}^1(X) \longrightarrow \text{Aut}(F_{A^1}(1))
$$

that completely determines the group structure. Using the splitting, this conjugation action is defined in the same fashion as in topology.

Unwinding the definitions, we see that the conjugation action has geometric origin: we claim that it comes from the morphism $R\Omega^1 X \to PGL_2$ defining the connecting homomorphism. Recall that we have the sequence of maps

$$
R\Omega^1 X \longrightarrow R\Omega^1 X \times \mathbb{P}^1 \longrightarrow PGL_2 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1.
$$

Adjunction gives a morphism from $R\Omega^1 X$ to the space of self-maps of $\mathbb{P}^1$, and this morphism factors through $PGL_2$. Of course, the action of $PGL_2$ on $\mathbb{P}^1$ moves base-points in the fiber. Thus, the splitting shows that the morphism of sheaves $\pi_{A^1}^1(X) \to \text{Aut}(F_{A^1}(1))$ factors through a morphism of sheaves

$$
\pi_{A^1}^0(PGL_2) \to \text{Aut}(F_{A^1}(1))
$$

corresponding to the change of base-point.

This homomorphism is non-trivial since any element coming from the inclusion of the maximal torus $G_m \subset PGL_2$ fixes the base-point in $\mathbb{P}^1$ and [Mor06b, Theorem 4.35] shows that such
homomorphisms can have non-trivial “Brouwer degree.” Combining these two observations, we get a map
\[ \pi_1^{A^1}(X) \to \text{Aut}(F_{A^1}(1)), \]
that is completely specified by an element of \( \text{Pic}(X)/2\text{Pic}(X) \).

**Remark 4.14.** By Lemma \ref{lemma:aut}, we know that \( \text{Aut}(F_{A^1}(1)) \) is a sheaf of abelian groups. As a consequence, any morphism \( \pi_1^{A^1}(X) \to \text{Aut}(F_{A^1}(1)) \) factors through the first \( A^1 \)-homology sheaf of \( X \). Using this observation, one can circumvent the discussion of base-points in the proposition. Furthermore, there is another way to interpret the factorization through \( H^1(\mu_2) \) described above. Multiplication determines a morphism of sheaves \( K_0^{MW} \to \text{Aut}(K_2^{MW}) \). On the other hand, \( K_0^{MW} \) is a quotient of the free strictly \( A^1 \)-invariant sheaf of groups on the sheaf \( H^1(\mu_2) = \mathbb{G}_m/\mathbb{G}_m \times 2 \). Theorem \ref{thm:main} thus asserts that the \( \pi_1^{A^1} \) acts on the \( A^1 \)-fundamental group of the fiber through the induced action on \( A^2 \setminus 0 \).

**Example 4.15.** The hypotheses of the Proposition are satisfied for \( X = \mathbb{P}^n, n \geq 3 \) and this gives part of Theorem \ref{thm:mainintro} from the introduction. As a consequence, in this case, the obstruction to Hartshorne’s conjecture stemming from the \( A^1 \)-fundamental group is trivial.

**Example 4.16.** Again by Grothendieck’s theorem \cite[Theorem 2.1.1]{OSS80}, Theorem \ref{thm:main} applies as well to projectivizations of rank 2 vector bundles over \( \mathbb{P}^1 \), in which case we recover \cite[Proposition 5.3.1]{AM10}.

**Non-trivial Euler classes: an example**

Analogous to the topological situation, the Euler class provides an obstruction to splitting. For bundles that do not split, the Euler class discussed above really can be non-trivial. The next example is closely related to Remark \ref{rem:fl3} since the map \( SL_3/B \to \mathbb{P}^2 \) we study below is, up to toral factors and \( A^1 \)-weak equivalences, the stabilization map for the \( A^1 \)-fibration sequence
\[ SL_3 \to SL_3/SL_2 \to BSL_2 \]
coming from the \( SL_2 \)-torsor \( SL_3 \to SL_3/SL_2 \) via Theorem \ref{thm:fl3}.

**Example 4.17.** Let \( \text{Fl}_3 \) be the variety of complete flags in a 3-dimensional \( k \)-vector space. After fixing a flag, this space can be identified with the homogeneous space \( SL_3/B \), where \( B \subset SL_3 \) is a Borel subgroup. Proposition \ref{prop:fl3} demonstrates that we have a short exact sequence of the form
\[ 1 \to \pi_1^{A^1}(SL_3) \to \pi_1^{A^1}(SL_3/B) \to T \to 1. \]

Furthermore, we know that \( \pi_1^{A^1}(SL_3) = K_2^{MW} \) (and not \( K_2^{MW} \)) by Theorem \ref{thm:fl3}. By forgetting either the 1 or 2-dimensional subspace in a flag, we see that \( SL_3/B \) fibers over \( \mathbb{P}^1 \) with fibers isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). Indeed, the morphism \( SL_3/B \to \mathbb{P}^2 \) is the projectivization of a tautological rank 2 vector bundle over \( \mathbb{P}^2 \) (either the 2-dimensional subspace of \( V \) or the 2-dimensional quotient depending on which projective space we choose), which is also indecomposable.
Examples

Combining Theorems 4.6 and 4.13, and Example 4.7 we can give a complete computation of the \(\mathbb{A}^1\)-fundamental group of a scroll (for brevity, we do not include the computations of Example 4.16).

**Theorem 4.18.** Suppose \(m\) and \(n\) are integers \(\geq 1\), and \(a = (a_1, \ldots, a_{n+1})\) is a sequence of integers. If \(n \geq 2\), then there are isomorphisms

\[
\pi_{\mathbb{A}^1}^1(F_{m,a}) \xrightarrow{\sim} \begin{cases} 
G_m \times F_{\mathbb{A}^1}(1) & \text{if } m = 1 \\
G_m \times G_m & \text{if } m > 1.
\end{cases}
\]

If \(n = 1\) and \(m > 1\), then there are isomorphisms

\[
\pi_{\mathbb{A}^1}^1(F_{m,a}) \xrightarrow{\sim} \begin{cases} 
F_{\mathbb{A}^1}(1) \times G_m & \text{if } a \equiv 0 \mod 2 \\
F_{\mathbb{A}^1}(1) \times G_m & \text{if } a \equiv 1 \mod 2,
\end{cases}
\]

where \(a := a_1 + a_2\).

**Example 4.19 (\(\mathbb{A}^1\)-fundamental groups of some blow-ups).** One particular application of the results above is to computations of \(\mathbb{A}^1\)-fundamental groups of blow-ups of linearly embedded projective subspaces of a given projective space. Recall that the blow-up of a point \(x\) in \(\mathbb{P}^n\) (\(n \geq 3\) for simplicity) is isomorphic to the projectivization of a direct sum of line bundles over \(\mathbb{P}^{n-1}\). The explicit description of these line bundles gives the isomorphism

\[
\pi_{\mathbb{A}^1}^1(\text{Bl}_x \mathbb{P}^n) \xrightarrow{\sim} F_{\mathbb{A}^1}(1) \times G_m.
\]

More generally, if \(\mathbb{P}^{n-k}\) is a linear subvariety of \(\mathbb{P}^n\), with \(k \geq 2\), then there is an \(\mathbb{A}^1\)-weak equivalence \(\mathbb{P}^n \setminus \mathbb{P}^{n-k} \rightarrow \mathbb{P}^{k-1}\) (in fact a vector bundle). This morphism allows us to realize \(\text{Bl}_{\mathbb{P}^{n-k}} \mathbb{P}^n\) as the projectivization of a vector bundle over \(\mathbb{P}^{k-1}\). Thus, Theorem 4.18 shows that

\[
\pi_{\mathbb{A}^1}^1(\text{Bl}_{\mathbb{P}^{n-k}} \mathbb{P}^n) \xrightarrow{\sim} \begin{cases} 
F_{\mathbb{A}^1}(1) \times G_m & \text{if } k = 2 \\
G_m \times G_m & \text{if } 2 < k < n.
\end{cases}
\]

One could also use the \(\mathbb{A}^1\)-van Kampen theorem [Mor06b, Theorem 4.12] to approach these results, but the group structure on the extension in the cases where we blow-up a point or a codimension 2 subvariety is not very transparent.

As a generalization of the question implicit in the previous example, we pose the following problem.

**Problem 4.20.** Suppose \(X\) is an \(\mathbb{A}^1\)-connected smooth proper \(k\)-scheme. If \(Y \subset X\) is an a smooth closed \(\mathbb{A}^1\)-connected subvariety, describe \(\pi_{\mathbb{A}^1}^1(\text{Bl}_Y X)\) in terms of the \(\mathbb{A}^1\)-fundamental group of the projectivization of the normal bundle to \(Y\) in \(X\), the \(\mathbb{A}^1\)-fundamental group of \(X\) and the \(\mathbb{A}^1\)-fundamental group of \(X \setminus Y\).
Higher $\mathbb{A}^1$-homotopy groups and higher obstructions

We close with a very brief discussion of higher obstructions and their implications for the structure of higher $\mathbb{A}^1$-homotopy groups of projectivizations of vector bundles over $\mathbb{A}^1$-connected smooth schemes. Above, we only really used the Euler class in the case of rank 2 bundles. When $X = \mathbb{P}^n$, $n \geq 3$, and $E$ is a rank 2 vector bundle on $X$, we observed that the Euler class $e(E)$ of Definition 4.8 is trivial since $\pi_1^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0)$ vanishes in this situation. Both of these observations factor into our construction of higher obstruction classes, which are only given for rather special situations.

If $X$ is an $\mathbb{A}^1$-connected smooth proper scheme, then $\pi_1^{\mathbb{A}^1}(X)$ is always non-trivial (see Remark 2.22). As a consequence, we cannot easily impose higher $\mathbb{A}^1$-connectivity hypotheses on $X$. Instead, we introduce the following property depending on an integer $i \geq 1$:

(C$_i$) the universal $\mathbb{A}^1$-covering space $\tilde{X}$ is $\mathbb{A}^1$-$i$-connected.

Projective space $\mathbb{P}^n$ satisfies (C$_{n-1}$), and [AD09] Theorem 6.4.2 gives a condition on the fan guaranteeing that a smooth proper toric variety satisfies (C$_i$) for some $i$.

Let $E$ be a rank $r$ vector bundle on $X$. If $f : \tilde{X} \to X$ is the $\mathbb{A}^1$-universal covering morphism, then $f^* E$ is determined by a simplicial homotopy class of maps $\tilde{X} \to BGL_r$. Applying $\pi_1^{\mathbb{A}^1}(-)$ to an explicit representative of this map gives a morphism $\pi_1^{\mathbb{A}^1}(\tilde{X}) \to \pi_1^{\mathbb{A}^1}(BGL_r)$. If $X$ satisfies (C$_i$), using [AD09] Theorem 3.30 we get a bijection

$$H_{\text{Nis}}^{i+1}(X, \pi_1^{\mathbb{A}^1}(BGL_r)) \xrightarrow{\sim} \text{Hom}_{\mathbb{A}^1}(\pi_1^{\mathbb{A}^1}(\tilde{X}), \pi_1^{\mathbb{A}^1}(BGL_r)).$$

The element of $H_{\text{Nis}}^{i+1}(X, \pi_1^{\mathbb{A}^1}(BGL_r))$ corresponding to the morphism $\pi_1^{\mathbb{A}^1}(\tilde{X}) \to \pi_1^{\mathbb{A}^1}(BGL_r)$ induced by the classifying map of $f^* E$ will be denoted $e_i(E)$. If $i+1 < r$, the $\mathbb{A}^1$-homotopy groups of $BGL_r$ are already in the stable range (again, see Remark 3.11) and one can introduce algebraic K-theory into the discussion.

Definition 4.21. Keeping hypotheses as in the preceding paragraph, the class $e_i(E)$ will be called the $i$-ary Euler class of $E$.

Remark 4.22. This definition can be made more similar to the definition of the Euler class. The vector bundle $E$ is classified by a simplicial homotopy class of maps $X \to BGL_r$. The homomorphism $\pi_1^{\mathbb{A}^1}(X) \to \pi_1^{\mathbb{A}^1}(BGL_r)$ therefore determines an action of $\pi_1^{\mathbb{A}^1}(X)$ on $\pi_1^{\mathbb{A}^1}(BGL_r)$ for $i \geq 0$. Using this, one can define an “$\mathbb{A}^1$-local system” on $X$: it is the Nisnevich sheaf (on the small site of $X$) of local sections of the morphism $\tilde{X} \times _{\pi_1^{\mathbb{A}^1}(X)} \pi_1^{\mathbb{A}^1}(BGL_r) \to X$. The $i$-ary Euler class of $X$ can then be viewed as a class on $X$ taking values in this sheaf.

Lemma 4.23. Suppose $X$ is an $\mathbb{A}^1$-connected smooth scheme such that $\tilde{X}$ is $\mathbb{A}^1$-$i$-connected for some integer $i \geq 2$. Assume $E$ is a rank $n+1$ vector bundle on $X$. If $e_i(E)$ is trivial, then connecting homomorphism $\pi_1^{\mathbb{A}^1}(X) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^n)$ in the long exact sequence of Corollary 3.22 is trivial. Therefore, there are induced isomorphisms

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}(E)) \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}(\mathbb{P}^n).$$
Proof. The first statement is valid even for $i = 1$, in which case it is precisely Lemma 4.11. For $i \geq 1$, the proof is identical: one uses the observation that the map $\pi_1^{A^1}(\tilde{X}) \to \pi_1^{A^1}(X)$ is an isomorphism, and by construction, the connecting homomorphism factors through a map $X \to BSL_{n+1}$. The second statement follows from the first using Lemma 2.18 and the $A^1$-connectivity assumption.

Example 4.24. The connectivity of $\tilde{X}$ will force vanishing of higher Euler classes. For example, if $X = \mathbb{P}^n$, $(n \geq 3)$, then $\pi_i^{A^1}(\mathbb{P}^n)$ vanishes for for $1 < i < n$. Given a rank 2 vector bundle $\mathcal{E}$ on $X$, the first non-vanishing higher Euler class is

$$e^n(\mathcal{E}) \in H^0_{\text{Nis}}(\mathbb{A}^{n+1} \setminus 0, \pi^{A^1}_{n+1}(BGL_2)),$$

and vanishing of this class completely controls the connecting homomorphism of Corollary 3.22 at the point $i = n$. One way to guarantee that this higher Euler class vanishes is as follows. As a cohomology class, the Euler class so defined is contravariantly functorial (for morphisms of varieties satisfying $(G_i)$) and by construction it is $A^1$-homotopy invariant.

With more work, the definition of the higher Euler class $e^j(\mathcal{E})$ can be given for varieties that are only $A^1$-connected assuming the vanishing of the classes $e^j(\mathcal{E})$ for $1 \leq j < i$. Say that a vector bundle $\mathcal{F}$ on $X$ is extended from a lower-dimensional variety $Y$ if there is an open subset $U \subset X$, a Zariski locally trivial smooth morphism $\psi : U \to Y$ with $A^1$-contractible fibers such that $\dim Y < \dim X$, and a vector bundle $\mathcal{F}'$ on $Y$ such that $\mathcal{F}|_U = \psi^* \mathcal{F}'$. If $U$, $Y$, and $X$ are all $A^1$-connected, and a bundle $\mathcal{F}$ on $X$ is extended from a lower-dimensional variety $Y$, the $i$-ary Euler class will vanish so long as $\dim Y \leq i$.

Remark 4.25. In order to make the above results more effective, we need more information about the higher $A^1$-homotopy groups of $BGL_{n+1}$, or equivalently by Proposition 3.9, the higher $A^1$-homotopy groups of $SL_{n+1}$. If $n \neq 1$, there is a nice description of these sheaves of groups in terms of so-called unstable Karoubi-Villamayor K-theory groups in [Wen10a] Theorem 1]. The case of $SL_2$, excluded for technical reasons in [Wen10a], can be treated by combining [Wen10a Theorem 2.5] and [Mos11] Theorem 1.1. In some special cases, nicer descriptions are possible, e.g., $\pi_2^{A^1}(SL_2)$ admits a description by means of its identification with $\pi_2^{A^1}(\mathbb{P}^1)$ (see [Wen10b Proposition 5.11]).

Constructing vector bundles

When $X$ is a smooth scheme having the $A^1$-homotopy type of $\mathbb{A}^n \setminus 0$, we can say even more. Indeed, recall that if $Q_{2n-1}$ is the affine quadric defined by the equation $\Sigma_i x_i x_{n+i} = 1$ in $\mathbb{A}^{2n}$ with coordinates $x_1, \ldots, x_{2n}$, then projection onto $x_1, \ldots, x_n$ defines an $A^1$-weak equivalence $Q_{2n-1} \to \mathbb{A}^n \setminus 0$. The following is a result is a consequences of results of Morel and Moser: Morel proved this in [Mor07] Theorem 3] for $n \geq 2$, and using [Mos11] Theorem 4.1] Morel's techniques can be extended to $n = 2$ (the case $n = 1$ is homotopy invariance of the Picard group [MV99 §4 Proposition 3.8] and does not require the assumption $X$ affine that is essential in the statement below).

Theorem 4.26 (Morel, Moser). If $X$ is a smooth affine scheme, and if $\mathcal{V}_r(X)$ denotes the set of isomorphism classes of rank $r$ vector bundles on $X$ ($r \geq 1$), there is a canonical bijection

$$[X, BGL_r]_{A^1} \overset{\sim}{\longrightarrow} \mathcal{V}_r(X).$$
This theorem together with some obstruction theory allows us to study rank \( r \) vector bundles on \( Q_{2n-1} \). Indeed, \( Q_{2n-1} \) is \((n-2)\mathbb{A}^1\)-connected since it is \( \mathbb{A}^1 \)-weakly equivalent to \( \mathbb{A}^n \setminus 0 \). The first non-trivial \( k \)-invariant of an \( \mathbb{A}^1 \)-homotopy class of maps \( Q_{2n-1} \to BGL_r \) comes from the homomorphism \( \pi_{n-1}^A(Q_{2n-1}) \to \pi_n BGL_r \). The next non-trivial \( k \)-invariant is a class in \( H^{n-1}_{\text{Nis}}(Q_{2n-1}, \pi_n^A(BGL_r)) \). Because \( \mathbb{A}^n \setminus 0 \) has dimension \( n \), it follows that all higher \( k \)-invariants of a map \( Q_{2n-1} \to BGL_r \) are trivial. Thus, we have constructed a function

\[
\Psi : [Q_{2n-1}, BGL_r]_{\mathbb{A}^1} \to H^{n-1}_{\text{Nis}}(Q_{2n-1}, \pi_n^A(BGL_r)) \times H^n_{\text{Nis}}(Q_{2n-1}, \pi_n^A(BGL_r)).
\]

We now see how to reconstruct a vector bundle from a class on the right hand side.

**Proposition 4.27.** The function \( \Psi \) is bijective.

**Proof.** We construct an inverse function. To build a rank \( r \) vector bundle on \( Q_{2n-1} \), we specify a homomorphism \( \pi_{n-1}^A(Q_{2n-1}) \to \pi_n^A(BGL_r) \); such a map is equivalent to specifying a class in \( H^{n-1}_{\text{Nis}}(Q_{2n-1}, \pi_n^A(BGL_r)) \). This homomorphism defines an obstruction class in \( H^{n+1}_{\text{Nis}}(Q_{2n-1}, \pi_n^A(BGL_r)) \).

By means of the \( \mathbb{A}^1 \)-weak equivalence \( Q_{2n-1} \to \mathbb{A}^n \setminus 0 \), and the fact that \( \mathbb{A}^n \setminus 0 \) has Nisnevich cohomological dimension \( n \), all Nisnevich cohomology of \( Q_{2n-1} \) in degrees strictly greater than \( n \) (with coefficients in a strictly \( \mathbb{A}^1 \)-invariant sheaf) vanishes. Thus, the obstruction class just mentioned necessarily vanishes. In other words, the map \( \mathbb{A}^n \setminus 0 \to BGL_r^{(n-1)} \) lifts along the fibration \( BGL_r^{(n)} \to BGL_r^{(n-1)} \). There is a free transitive action of \( H^n_{\text{Nis}}(Q_{2n-1}, \pi_n^A(BGL_r)) \) on the space of lifts. Fixing an element of this set therefore defines a choice of lift. Again using the vanishing statement, we observe inductively that all higher obstruction classes and \( k \)-invariants are trivial, and we can lift our map (at each stage the lift is uniquely specified) to a morphism \( Q_{2n-1} \to BGL_r \) in the \( \mathbb{A}^1 \)-homotopy category.

**Remark 4.28.** The class in \( H^{n-1}_{\text{Nis}}(Q_{2n-1}, \pi_n^A(BGL_r)) \) just defined is, by its very construction, the higher Euler class of \( \mathcal{E} \). Thus, there is one further piece of cohomological data required to determine the splitting—up to homotopy—of a rank \( r \) vector bundle on \( \mathbb{P}^n \). In other words, we have almost a complete cohomological description of the \( \mathbb{A}^1 \)-homotopy types of Zariski locally trivial \( \mathbb{P}^m \)-bundles on \( \mathbb{P}^n \). To make this description more explicit, we need more detailed information about the cohomology of the sheaves \( \pi^A_r(BGL_r) \).

## 5 Varieties with isomorphic \( \mathbb{A}^1 \)-homotopy groups

In this section, we use the computations of \( \mathbb{A}^1 \)-homotopy groups of the previous section to investigate varieties that have isomorphic \( \mathbb{A}^1 \)-homotopy groups yet are not \( \mathbb{A}^1 \)-weakly equivalent. All of the examples we construct become isomorphic after performing sequences of blow-ups with smooth centers. Blowing-up changes the \( \mathbb{A}^1 \)-fundamental group, and this is the hardest invariant to control in our examples. In contrast, constructing varieties of dimension \( \geq 4 \) with isomorphic \( \mathbb{A}^1 \)-homotopy groups is not so complicated.

**Remark 5.1.** For the purposes of motivation, we recall that examples of weak-homotopy inequivalent 3-manifolds with abstractly isomorphic homotopy groups go back to the birth of geometric topology. Indeed, fix a prime \( p \), an integer \( q \) coprime to \( p \), and let \( \zeta = e^{2\pi i/q} \). Choose coordinates \( (x_1, x_2) \) on \( \mathbb{C}^2 \), and consider the action of the cyclic group \( \mathbb{Z}/p \) generated by \( \zeta \cdot (x_1, x_2) =
(ξx1, ξax2). This action is free and induces an action on S3 \subset C^2; the quotient of this action is the lens space L(p, q). It has long been known that L(p, q) is homotopy equivalent to L(p, q') if and only if \( pq' \) is a square mod \( p \) [Whi41, p. 1198]. It is also known that 3-dimensional lens spaces can be viewed as Seifert \( S^1 \)-bundles over surfaces. The examples in \( \mathbb{A}^1 \)-homotopy theory we study below are analogous to lens spaces by “taking the real points,” though unlike Seifert \( S^1 \)-bundles they do not possess singular fibers.

\( \mathbb{A}^1 \)-weak equivalences of scrolls

We will need some rather explicit information about the construction of \( \mathbb{A}^1 \)-weak equivalences of scrolls (recall Notation 3.4). To begin, we give a rather general construction of \( \mathbb{A}^1 \)-weak equivalences of projective space bundles in the spirit of the current work; these examples also give rise to \( \mathbb{A}^1 \)-h-cobordisms in the sense of [AM10, Definition 3.1.1]

**Lemma 5.2.** Suppose \( X \) is an \( \mathbb{A}^1 \)-contractible smooth scheme, and fix \( x \in X \). Let \( \mathcal{E} \) be a vector bundle on \( Y \times X \), and let \( \mathcal{E}_x \) be the restriction of \( \mathcal{E} \) to \( Y \times x \). The inclusion \( \mathbb{P}(\mathcal{E}_x) \hookrightarrow \mathbb{P}(\mathcal{E}) \) is an \( \mathbb{A}^1 \)-weak equivalence.

**Proof.** The morphism \( \mathbb{P}(\mathcal{E}) \to Y \times X \) is an \( \mathbb{A}^1 \)-fibration by Proposition 3.20. The morphism \( Y \to Y \times X \) induced by the inclusion \( x \hookrightarrow X \) is an \( \mathbb{A}^1 \)-weak equivalence since \( X \) is \( \mathbb{A}^1 \)-contractible. By right properness of the \( \mathbb{A}^1 \)-local model structure, it follows that

\[ \mathbb{P}(\mathcal{E}_x) \hookrightarrow \mathbb{P}(\mathcal{E}) \]

is also an \( \mathbb{A}^1 \)-weak equivalence.

Fix an integer \( a \), and consider the scroll \( F_{1,a} \). By explicitly writing down a cocycle, one can see that there is a rank \( n+1 \) vector bundle \( \mathcal{E}_a \) on \( \mathbb{P}^1 \times \mathbb{A}^1 \) whose restriction to \( \mathbb{P}^1 \times \{0\} \) is isomorphic to \( \mathcal{O}(a) \oplus \mathcal{O} \) and whose restriction to \( \mathbb{P}^1 \times \{1\} \) is isomorphic to \( \mathcal{O}(a+1) \oplus \mathcal{O}(-1) \). Projectivizing this vector bundle gives a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \times \mathbb{A}^1 \) whose fiber over \( 0 \in \mathbb{A}^1 \) is \( \mathbb{F}_{1,a} \) and whose fiber over \( 1 \in \mathbb{A}^1 \) is isomorphic to \( \mathbb{F}_{1,a-2} \).

Let \( \mathbf{a} = (a_1, \ldots, a_n, 0) \) and \( \mathbf{a}' = (a_1, \ldots, a_n, 1, -1) \). Take the direct sum of a line bundle on \( \mathbb{P}^1 \times \mathbb{A}^1 \) of the form \( \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n-1) \) with the vector bundle \( \mathcal{E}_{a_0} \) on \( \mathbb{P}^1 \times \mathbb{A}^1 \) described in the previous paragraph; write \( \mathcal{E}_a \) for the resulting vector bundle on \( \mathbb{P}^1 \times \mathbb{A}^1 \). Observe that \( \mathbb{P}(\mathcal{E}_a) \) is a \( \mathbb{P}^n \) bundle on \( \mathbb{P}^1 \times \mathbb{A}^1 \) whose fiber over \( 0 \in \mathbb{A}^1 \) is the scroll \( F_{1,a} \), but whose fiber over \( 1 \in \mathbb{A}^1 \) is the scroll \( F_{1,a'} \). These scrolls are non-isomorphic, but \( F_{1,a'} \) is isomorphic to the scroll \( F_{1,b} \) with \( b = (a_1+1, \cdots, a_n-1+1, a_n+2, 0) \). The next result, for which we present a proof in the spirit of this paper, is also consequence of [AM10, Proposition 3.1.5] via [AM10, Example 3.2.9].

**Proposition 5.3.** Fix \( n+1 \)-tuples of integers \( \mathbf{a} = (a_1, \ldots, a_{n+1}) \) and \( \mathbf{a}' = (a'_1, \ldots, a'_{n+1}) \). If \( \sum_i a_i = \sum_i a'_i \mod n+1 \) the scrolls \( F_{1,a} \) and \( F_{1,a'} \) are \( \mathbb{A}^1 \)-weakly equivalent.

**Proof.** Suppose \( \sum_i a_i = \sum_i a'_i \mod n+1 \). We can then find a sequence of vector bundles as described in the paragraph preceding the statement such that the associated projective space bundles link the source and target scrolls by a chain of \( \mathbb{A}^1 \)-weak equivalences (by means of Lemma 5.2).
Motivic cohomology rings of projective bundles

Suppose $E$ is a rank $r$ vector bundle over a projective space $\mathbb{P}^n$. The Chow cohomology ring of the projective bundle $\mathbb{P}_{p^n}(E)$ is computed as follows. Let $\xi \in H^{2,1}(\mathbb{P}^n, \mathbb{Z})$ be the first Chern class of $\mathcal{O}(1)$. Let $P_r(E)$ be the Chern polynomial of $E$ defined by

$$P_r(E) = \sum_i \tau^{d-i} c_i(E).$$

Then,

$$H^{2,1^*}(\mathbb{P}_{p^n}(E), \mathbb{Z}) \to H^{2,1^*}(\text{Spec } k)[\xi, \tau]/\langle \xi^{n+1}, P_r(E) \rangle;$$

here we use the notation of motivic cohomology and implicitly [MVW06 Corollary 19.2] to identify this subring of motivic cohomology and Chow cohomology and [Ful93 Remark 3.2.4]. In order that two algebraic varieties be $\mathbb{A}^1$-weakly equivalent, the associated Chow cohomology rings must be isomorphic as graded rings by [Voe03 §2 Theorem 2.2]. Any such graded ring homomorphism is induced by an action of $GL_2(\mathbb{Z})$ on $H^{2,1}(\mathbb{P}_{p^n}(E), \mathbb{Z})$, with basis $\xi$ and $\tau$.

**Example 5.4.** If $a = (a_1, a_2)$, the Chow cohomology ring of the scroll $F_{1,a}$ takes the form $\mathbb{Z}[\xi, \tau]/(\xi^2, \tau^3 + (a_1 + a_2)\xi \tau^2)$. Similarly, the Chow cohomology ring of the scroll $F_{2,b}$ takes the form $\mathbb{Z}[\xi, \tau]/(\xi^3, \tau^2 + b\xi \tau)$. Any isomorphism of graded rings is induced by a $GL_2(\mathbb{Z})$ action on the generators $\xi, \tau$.

$\mathbb{A}^1$-weakly inequivalent varieties with isomorphic $\mathbb{A}^1$-homotopy groups

**Theorem 5.5.** Let $n \geq 2$ be an integer, and suppose $a = (a_1, a_2, a_3)$ is a sequence of integers. For every $i \geq 1$ there are isomorphisms

$$\pi_1^{\mathbb{A}^1}(F_{1,a}) \cong \pi_i^{\mathbb{A}^1}(\mathbb{P}^n) \times \pi_i^{\mathbb{A}^1}(\mathbb{P}^1).$$

However, if $a' := (a'_1, a'_2, a'_3)$, then $F_{1,a}$ is $\mathbb{A}^1$-weakly equivalent to $F_{1,a'}$ if and only if $\sum_i a_i = \sum_i a'_i \mod 3$.

**Proof.** We will prove that the groups in question are abstractly isomorphic. Under the stated hypotheses, the isomorphism of $\mathbb{A}^1$-fundamental groups is a consequence of Theorem 4.6 (via Example 4.7), and the isomorphisms between higher $\mathbb{A}^1$-homotopy groups are an immediate consequence of 4.3. That the varieties in question are $\mathbb{A}^1$-weakly equivalent under the stated hypothesis is a consequence of Proposition 5.3. If the hypothesis on the $a_i$ is not satisfied, it is easy to see that the corresponding Chow cohomology rings are not isomorphic by explicit computation using the discussion of Example 5.4.

Note: the claimed isomorphism is not, in general, induced by a morphism of spaces, even in the $\mathbb{A}^1$-homotopy category; by the $\mathbb{A}^1$-Whitehead theorem [MV99 §3 Proposition 2.14], it can be induced by a morphism of spaces if and only if the underlying spaces are $\mathbb{A}^1$-weakly equivalent.

**Remark 5.6.** Theorem 5.5 implies the existence of pairs of $\mathbb{A}^1$-connected varieties that are not $\mathbb{A}^1$-weakly equivalent, yet have isomorphic $\mathbb{A}^1$-homotopy groups in every dimension $\geq 3$, e.g., by taking products with a fixed $\mathbb{A}^1$-connected smooth scheme $Y$ (use Remark 2.27). Producing such examples in dimension $\geq 4$ is significantly easier than in dimension 3. For example, suppose
$m$ and $n$ are integers ≥ 2. Using Proposition 2.21 it is easy to check that any two scrolls $F_{m,a}$ and $F_{n,b}$ with $m + \ell(a) = n + \ell(b) \geq 4$ have isomorphic $A^1$-homotopy groups for all $i > 0$. There are examples of such scrolls with non-isomorphic cohomology rings.

**Remark 5.7.** Blowing up the same number of points on each of $F_{1,a}$ and $F_{2,b}$ produces many more examples of varieties with isomorphic $A^1$-homotopy groups. However, $A^1$-weakly inequivalent varieties can become $A^1$-weakly equivalent after blowing-up: recall that $F_1$ is not $A^1$-weakly equivalent to $F_2$, but after blowing-up an appropriately chosen point on each, they become isomorphic.

More generally, all of the examples constructed above are non-singular projective toric varieties. Given two toric projective bundles $X$ and $X'$ it is not difficult to give a non-singular projective toric variety $X''$ birationally equivalent to and dominating both $X$ and $X'$ which is constructed from either of $X$ or $X'$ by means of an explicit sequence of blow-ups at non-singular toric subvarieties. In other words, all of the examples constructed above, while $A^1$-inequivalent, become $A^1$-equivalent after blowing-up.

**Remark 5.8.** In the study of lens spaces, the homotopy classification problem can be attacked by means of obstruction theory (see, e.g., [Olu53, Theorems V and VI] and the references in that paper). Indeed, to perform this classification two invariants are necessary: the (usual) fundamental group and an appropriate notion of “degree” (see also [Fra43, Theorem I]). In the context of $A^1$-homotopy theory of projective bundles, one can introduce various notions of degree and hope that the situation is similar to topology. A related problem is as follows.

**Problem 5.9.** Determine the $A^1$-homotopy classification of all $A^1$-connected, smooth proper 3-folds with fixed $A^1$-fundamental group.

**Example 5.10.** The $A^1$-connected smooth proper 3-folds with $A^1$-fundamental group isomorphic to $G_m$ include $P^3$ and the smooth (split) quadric hypersurface in $P^4$. The $A^1$-connected smooth proper 3-folds with $A^1$-fundamental group $F_{A^1}(1) \times G_m$ include all $P^2$-bundles over $P^1$, and some $P^1$-bundles over $P^2$. In either situation, it is not clear whether there are any “exotic” examples.

**A variant of the above approach**

While we remarked in Example 4.19 that the $A^1$-van Kampen theorem was difficult to use in the study of projective bundles, with some geometry it is sufficiently refined to detect whether varieties have isomorphic $A^1$-fundamental groups in a number of situations. Let us consider the case of projectivizations of a split vector bundle $E$ over $\mathbb{P}^m$. The cases of interest are when the rank of $E$ is 2 or $m = 1$. Let us treat the case where $E$ has rank 2 and $m \geq 2$. In that case, if we pick a linearly embedded $\mathbb{P}^{m-2} \subset \mathbb{P}^m$, then picking a projection $\mathbb{P}^m \setminus \mathbb{P}^{m-2} \to \mathbb{P}^1$ we obtain a vector bundle. Moreover, the restriction of $E \to \mathbb{P}^m \setminus \mathbb{P}^{m-2}$ is pulled back from a (necessarily split) vector bundle $E'$ on $\mathbb{P}^1$. In other words, $E$ is extended from $\mathbb{P}^1$. In that case, we have a diagram of the form

$$
\begin{array}{ccc}
\mathbb{P}(E'|_{\mathbb{P}^m \setminus \mathbb{P}^{m-2}}) & \longrightarrow & \mathbb{P}(E') \\
\downarrow & & \downarrow \\
\mathbb{P}^m \setminus \mathbb{P}^{m-2} & \longrightarrow & \mathbb{P}^1.
\end{array}
$$
Both horizontal maps are $A^1$-weak equivalences. Let us pick a copy of $A^{m+1} \subset \mathbb{P}(\mathcal{E})$ intersecting the given $\mathbb{P}^{m-2}$ in a copy of $A^{m-2}$. The union of $A^{m+1}$ and $\mathbb{P}(\mathcal{E}|_{\mathbb{P}^{m-2}})$ is therefore an open subscheme of $\mathbb{P}(\mathcal{E})$ with complement of codimension $\geq 3$ that we shall call $X$.

**Proposition 5.11.** The morphism $\pi_1 A^1 (X) \to \pi_1 A^1 (\mathbb{P}(\mathcal{E}))$ induces an isomorphism

$$\pi_1 A^1 (\mathbb{P}(\mathcal{E})) \cong \pi_1 A^1 (\mathbb{P}(\mathcal{E})),$$

where $\star A^1$ denotes the (amalgamated) sum in the category of strongly $A^1$-invariant sheaves of groups (see Remark 2.7).

**Idea of proof.** This is a consequence of the $A^1$-van Kampen theorem: the intersection of $A^{m+1}$ and $\mathbb{P}(\mathcal{E}|_{\mathbb{P}^{m-2}})$ is isomorphic to $A^{m+1} \setminus A^{m-1}$, which is $A^1$-weakly equivalent to $A^2 \setminus \{0\}$. The inclusion map $A^{m+1} \setminus A^{m-1} \to A^{m+1}$ is $A^1$-weakly equivalent to the constant map since $A^{m+1}$ is $A^1$-contractible. One can observe that the morphism of $A^1$-fundamental groups induced by the inclusion map $A^{m+1} \setminus A^{m-1} \to \mathbb{P}(\mathcal{E}|_{\mathbb{P}^{m-2}})$ maps epimorphically onto a factor of $K^M_W$ in one of the copies of $F_{A^1}(1)$. To finish, one can use excision to conclude that $\pi_1 A^1 (X) \to \pi_1 A^1 (\mathbb{P}(\mathcal{E}))$ is an isomorphism since both spaces are $A^1$-connected and the codimension of the complement of the image is $\geq 3$. (This can also be seen directly by a van Kampen argument.)

**Example 5.12.** From these examples, we see directly that the $A^1$-fundamental group of $F_{m,a}$, $m \geq 2$ only depends on the value of $a$ modulo 2. If $a$ is congruent to 0 modulo 2, then the $A^1$-homotopy groups of such an example are isomorphic to the $A^1$-homotopy groups of a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Nevertheless, cohomology ring computations can be used to provide examples that are not $A^1$-weakly equivalent.

**Remark 5.13.** There is another point of view on the above examples: all the varieties for which we have produced isomorphic $A^1$-homotopy groups arise as $G_m \times G_m$-quotients of products of pairs of punctured affine spaces. The punctured affine spaces arise as stable points for different linearizations of $G_m \times G_m$-actions on affine space. All the projectivizations of split vector bundles above are realized in this fashion by changing linearizations for fixed actions on a given ambient affine space. The $A^1$-fundamental groups of the quotients can be described by means of Proposition 2.21 and the group structure on the $A^1$-fundamental group, i.e., the extension in question, is completely determined by the linearization, which is determined by a character of $G_m \times G_m$. The space of characters of $G_m \times G_m$ breaks into finitely many chambers such that all the quotients for linearizations in a fixed chamber are isomorphic. When passing through a wall, the birational type of the variety in question remains the same, while the isomorphism class can change; variation of GIT in this particular instance is worked out in considerable detail in [BCZ04, Appendix A]. When the resulting birational transformations are sufficiently “small,” one can use geometric arguments to show that $A^1$-fundamental groups do not change.
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