Remarks on $\mathbb{A}^1$-homotopy groups of smooth toric models

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Abstract

We extend previous results on $\mathbb{A}^1$-homotopy groups of (smooth proper) toric varieties to the case of smooth proper toric models in characteristic 0 (i.e., smooth proper equivariant compactifications of possibly non-split tori).

1 Statement of results

Fix a field $k$ having characteristic 0, let $\mathcal{S}m_k$ denote the category of schemes that are separated, smooth and have finite type over $k$. Suppose $X$ is a smooth proper $k$-scheme. Let $H(k)$ denote the $\mathbb{A}^1$-homotopy category of $k$-schemes as constructed in [MV99, §3.2]. Assume $X(k)$ is non-empty, and fix $x \in X(k)$. One can study the $\mathbb{A}^1$-homotopy (sheaves of) groups $\pi_1^{\mathbb{A}^1}(X,x)$ (denoted $a_1^{\mathbb{A}^1}(X,x)$ on [MV99, p. 110]). Our aim in this short note is to show that the “geometric” decomposition of $\mathbb{A}^1$-homotopy (sheaves of) groups of smooth proper “split” toric varieties (i.e., equivariant compactifications of $\mathbb{G}_m \times \mathbb{A}^n$) studied in [AD09] and [Wen07] extends to “non-split” toric varieties (i.e., equivariant compactifications of tori $T$ over $k$). We will refer to equivariant compactifications of tori $T$ over $k$ as toric $T$-models [MP97, §5].

Let $k^s$ denote a fixed separable closure of $k$ and let $G_k$ denote the Galois group of $k^s$ over $k$. For a $k$-scheme $Y$, let $Y^s$ denote the variety obtained by extending scalars to $k^s$. Suppose $X$ is a smooth proper toric $T$-model. One knows that $Pic(X^s)$ is a finitely generated $G_k$-module, and we denote the associated dual $k$-torus—the Neron-Severi torus—by $T_{NS(X)}$. With any toric $T$-model, one can associate a fan $\Sigma$ in $X^*(T^s)$ that is $G_k$-invariant. Cox’s construction [Cox95] realizing any “split” smooth proper toric variety as a geometric quotient of an open subscheme of affine space by a free action of $T_{NS(X)}$ can be generalized to the non-split case: if $X$ is a smooth proper toric $T$-model, there are a $T_{NS(X)}$-torsor $f : U \to X$ and an open immersion $U \hookrightarrow \mathbb{A}^n_k$ ($n = \dim T + \dim T_{NS(X)}$) [MP97].

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Proposition 2.1. Let \( \text{cofibrations} \) are simultaneously \( A \)-weak equivalences and monomorphisms of sheaves, i.e., \( A \)-acyclic cofibrations.

Theorem 1.1. Assume \( k \) is a field having characteristic 0 and \( T \) is a \( k \)-torus. Suppose \( X \) is a smooth proper toric \( T \)-model, and let \( x \) denote the \( k \)-rational point of \( X \) corresponding to \( 1 \in T(k) \). The \( T_{NS(X)} \)-torsor \( f : U \to X \) above is an \( \mathbb{A}^1 \)-cover. In particular, if \( \tilde{x} \) is any lift of \( x \), there is a short exact sequence (of Nisnevich sheaves of groups)

\[
1 \longrightarrow \pi_1^\text{Nis}(U, \tilde{x}) \longrightarrow \pi_1^\text{Nis}(X, x) \longrightarrow T_{NS(X)} \longrightarrow 1,
\]

and, for each integer \( i > 1 \), there are isomorphisms \( \pi_i^\text{Nis}(U, \tilde{x}) \cong \pi_i^\text{Nis}(X, x) \). Finally, \( f \) induces a morphism of sheaves \( \pi_0^\text{Nis}(X) \to H^1_\text{ét}(T_{NS(X)}) \) that is an isomorphism on sections over finitely generated separable extensions \( L/k \).

Remark 1.2. There are examples of \( k \)-tori \( T \) and smooth proper toric \( T \)-models \( X \) for which \( \pi_0^\text{Nis}(X)(k) \) is non-trivial. Thus, over non separably closed fields, we have the interesting phenomenon that a smooth proper \( \mathbb{A}^1 \)-disconnected space can have \( \mathbb{A}^1 \)-connected covering spaces! For a manifestation of this phenomenon for non-proper smooth varieties, one can consider the morphism \( \mathbb{A}^m \setminus 0 \to \mathbb{A}^m \setminus 0/\mu_n \) \[\text{AD09}\] Remark 3.13.

2 Torus torsors as \( \mathbb{A}^1 \)-covering spaces

The word \( \text{space} \), will mean “object of \( \Delta \circ \text{Shv}_{Nis}(Sm_k) \)” (the category of simplicial Nisnevich sheaves on \( Sm_k \)); we use caligraphic letters (e.g., \( \mathcal{X}, \mathcal{Y} \)) to denote such objects. We set \([\mathcal{X}, \mathcal{Y}]_s := \text{hom}_{\mathcal{H}_s((Sm_k)_{Nis})} (\mathcal{X}, \mathcal{Y})\), where \( \mathcal{H}_s((Sm_k)_{Nis}) \) is as on \[\text{MV99}\] p. 49 and \([\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1} := \text{hom}_{\mathcal{H}(k)} (\mathcal{X}, \mathcal{Y})\). A morphism \( f : \mathcal{X} \to \mathcal{Y} \) of \( k \)-spaces is an \( \mathbb{A}^1 \)-cover (cf. \[\text{Mor06} \text{ Section 4.1}\]) if it has the unique right lifting property with respect to morphisms that are simultaneously \( \mathbb{A}^1 \)-weak equivalences and monomorphisms of sheaves, i.e., \( \mathbb{A}^1 \)-acyclic cofibrations.

Proposition 2.1. Let \( T \) be a multiplicative group over a field \( k \) having characteristic 0. If \( X \) is a smooth scheme, and \( \pi : U \to X \) is a \( T \)-torsor locally trivial in the étale topology, then \( \pi \) is an \( \mathbb{A}^1 \)-cover and, in particular, an \( \mathbb{A}^1 \)-fibration.

Let \( BT \) denote the simplicial classifying space of \( T \) viewed as a Nisnevich sheaf of groups, and let \( BT_\text{ét} \) denote the simplicial classifying space of \( T \) viewed as an étale sheaf of groups. Let \( \alpha : (Sm_k)_{\text{ét}} \to (Sm_k)_{Nis} \) be the morphism of sites induced by the identity functor. Set \( B_\text{ét} T := \text{R} \alpha_* BT_\text{ét} \); see \[\text{MV99} \text{§4.1}\] for more details.

Lemma 2.2 (cf. \[\text{AM09} \text{Lemma 4.2.4}\]). The space \( B_\text{ét} T \) is \( \mathbb{A}^1 \)-local.

Proof. By adjunction, one has canonical bijections

\[
\text{hom}_{\mathcal{H}_s((Sm_k)_{Nis})}(U, B_\text{ét} T) \cong \text{hom}_{\mathcal{H}_s(k)}(U, BT_\text{ét}).
\]

Choosing a fibrant model for \( BT_\text{ét} \), and using \[\text{MV99} \text{§2 Proposition 3.19 and §4 Proposition 1.16}\], to check that \( B_\text{ét} T \) is \( \mathbb{A}^1 \)-local, it suffices to prove that that the maps

\[
H^i_\text{ét}(U, T) \longrightarrow H^i_\text{ét}(U \times \mathbb{A}^1, T)
\]
are bijections for \( i = 0, 1 \). For \( i = 0 \), this a consequence of étale descent: if \( k'/k \) is a separable extension splitting \( T \), then it suffices to observe that any morphism \( U \times \mathbb{A}^1 \rightarrow \mathbb{G}_m^\times \) factors through a morphism \( U \rightarrow \mathbb{G}_m^\times \). For \( i = 1 \) one could apply [AM09] Lemma 4.3.7 and Proposition 4.4.3. For a direct proof, observe that [CTS87] Lemma 2.4, establishes the result for affine \( X \) (Grothendieck showed that étale and flat cohomology coincide \textit{Ibid.} p.159). We reduce the case of general \( X \) to the affine case by comparing the exact sequences of low degree terms for the Leray spectral sequences associated with an open affine cover \( u : U \rightarrow X \) and the corresponding cover \( u \times id : U \times \mathbb{A}^1 \rightarrow X \times \mathbb{A}^1 \).

\[ \tag*{\Box} \]

Proof of Proposition \[ 2.1 \] After Lemma \[ 2.2 \], the proof is essentially [Mor06] Lemma 4.5(2)]; here are the details. Start with an \( \mathbb{A}^1 \)-acyclic cofibration \( j : A \rightarrow B \) fitting into a diagram

\[
\begin{array}{ccc}
A & \rightarrow & U \\
\downarrow j & & \downarrow \pi \\
B & \rightarrow & X.
\end{array}
\]

Now, since \( B_{\text{et}}T \) is \( \mathbb{A}^1 \)-local, the natural maps \([B, B_{\text{et}}T]_s \rightarrow [A, B_{\text{et}}T]_s\) and \([B, T]_s \rightarrow [A, T]_s\) are bijections. The pullback of \( \pi \) to \( A \) admits a section and is therefore a trivial torsor. By the first bijection just mentioned, it follows that the pullback of \( \pi \) to \( B \) is also trivial, and thus also admits a section, which we denote by \( s \). The composite morphism \( j \circ s \) need not be equal to \( s_0 \), but if it is not, then there is an element \( t_0 \in [A, T]_s \) such that \( t_0 \cdot s = s_0 \). By the second bijection mentioned at the beginning of this paragraph, the element \( t_0 \) determines a unique element \( t \) of \([B, T]_s\). The product \( t^{-1} \cdot s \) is a new section of \( \pi \) pulled back to \( B \). By construction this new section gives back \( s_0 \) upon restriction to \( A \) and thus provides the necessary (unique) lift.

\[ \tag*{\Box} \]

Proof of Theorem \[ 1.1 \] We return to the notation of the introduction: \( X \) is a smooth proper toric \( T \)-model, \( T_{NS(X)} \) is the associated Neron-Severi torus and \( f : U \rightarrow X \) is the \( T_{NS(X)} \)-torsor constructed in [MP97, Proposition 5.6].

Since \( X \) is proper, it follows from, e.g., [Cox95] Lemma 1.4 that \( U \) has complement of codimension \( \geq 2 \) in the affine space in which it sits since the same thing is true upon passing to a separable closure. Since \( k \) has characteristic 0 and is thus infinite, it follows that \( U \) is even connected by lines. (In fact, [AD09] Proposition 5.12 gives conditions guaranteeing that this complement has codimension \( \geq d \), depending only on the fan of \( X^s \)). In any case, we can choose a point \( \hat{x} \) lifting \( x \).

By Proposition \[ 2.1 \] \( \pi \) is an \( \mathbb{A}^1 \)-cover and thus an \( \mathbb{A}^1 \)-fibration. Consider the long exact sequence in \( \mathbb{A}^1 \)-homotopy groups of \( \pi \), which exists by a formal argument in the theory of model categories (cf. [AD09] Remark 3.2). The higher \( (i > 1) \) homotopy (sheaves of) groups of \( B_{\text{et}}T_{NS(X)} \) are trivial, and \( \pi_i^{\mathbb{A}^1}(B_{\text{et}}T_{NS(X)}) = T_{NS(X)} \) (again, see [MV99 §4 Proposition 1.16]). We then have a long exact sequence of groups (and pointed sets)

\[ 1 \rightarrow \pi_1^{\mathbb{A}^1}(U, \hat{x}) \rightarrow \pi_1^{\mathbb{A}^1}(X, \hat{x}) \rightarrow T_{NS(X)} \rightarrow \pi_0^{\mathbb{A}^1}(U) \rightarrow \pi_0^{\mathbb{A}^1}(X) \rightarrow \pi_0^{\mathbb{A}^1}(B_{\text{et}}T_{NS(X)}), \]

and for each \( i > 1 \), we have isomorphisms \( \pi_i^{\mathbb{A}^1}(U, \hat{x}) \cong \pi_i^{\mathbb{A}^1}(X, \hat{x}) \).  

\[ \tag*{\Box} \]
For the case \( i = 0 \), observe that there is a morphism \( X \to B_{\text{et}}T_{NS(X)} \) classifying \( f \). This induces a morphism \( \pi_0^{A^1}(X) \to \pi_0^{A^1}(B_{\text{et}}T_{NS(X)}) \). Using the \( A^1 \)-weak equivalence \( X \to Sing_{\text{et}}^1(X) \), there is an induced epimorphism \( \pi_0^s(Sing_{\text{et}}^1(X)) \to \pi_0^{A^1}(X) \) by [MV99, \S 2 Corollary 3.22]. Again using the fact that \( X \) is proper, we conclude \( \pi_0^s(Sing_{\text{et}}^1(X))(L) \) is \( X(L)/R \).

Since \( B_{\text{et}}T_{NS(X)} \) is \( A^1 \)-local, \( \pi_0^{A^1}(B_{\text{et}}T) = \mathcal{H}^1_{\text{et}}(T_{NS(X)}) \). Taking sections over finitely generated separable extensions \( L/k \) determines a morphism \( X(L)/R \to H^1_{\text{et}}(L, T_{NS(X)}) \) that coincides with the “obvious” such morphism gotten by restricting \( \pi \) to \( L \)-points of \( X \). The torus \( T_{NS(X)} \) is flasque (see, e.g., [CTS77, Proposition 6]) so [CTS77, \S 5 Corollaire 1] implies that the map \( X(L)/R \to H^1_{\text{et}}(L, T_{NS(X)}) \) is an isomorphism on sections over separable finitely generated \( L/k \).

**Remark 2.3.** The statement in Theorem 1.1 involving \( \pi_0^{A^1} \) provides an alternate proof of [AM09, Theorem 2.4.3] in the special case of smooth proper toric models. Furthermore, this statement can be strengthened slightly. Indeed, the multiplication morphism \( T 	imes T \to T \) gives rise to a rational map \( X \times X \to X \). Resolving indeterminacy, we get a morphism \( X' \to X \times X \) (that is a composite of blow-ups). One can check that this induces a composition on \( \pi_0^{A^1}(X)(L) \) for any \( L/k \) (coinciding with the composition on \( R \)-equivalence classes). The map of the proposition is in fact a homomorphism of abelian groups. One would like to show that \( \pi_0^{A^1}(X) \) can be equipped with the structure of a Nisnevich sheaf of abelian groups and that the map \( \pi_0^{A^1}(X) \to \mathcal{H}^1_{\text{et}}(T_{NS(X)}) \) is an isomorphism of sheaves.

**References**


