Smooth varieties up to $\mathbb{A}^1$-homotopy and algebraic $h$-cobordisms

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Abstract

We start to study the problem of classifying smooth proper varieties over a field $k$ from the standpoint of $\mathbb{A}^1$-homotopy theory. Motivated by the topological theory of surgery, we discuss the problem of classifying up to isomorphism all smooth proper varieties having a specified $\mathbb{A}^1$-homotopy type. Arithmetic considerations involving the sheaf of $\mathbb{A}^1$-connected components lead us to introduce two different notions of connectedness in $\mathbb{A}^1$-homotopy theory. We provide concrete links between these two notions, connectedness of points by chains of affine lines, and various rationality properties of algebraic varieties (e.g., rational connectedness).

We introduce the notion of an $\mathbb{A}^1$-$h$-cobordism, an algebro-geometric analog of the topological notion of $h$-cobordism, and use it as a tool to produce non-trivial $\mathbb{A}^1$-weak equivalences of smooth proper varieties. Also, we give explicit computations of refined $\mathbb{A}^1$-homotopy invariants, such as the $\mathbb{A}^1$-fundamental sheaf of groups, for some $\mathbb{A}^1$-connected varieties. We observe that the $\mathbb{A}^1$-fundamental sheaf of groups plays a central yet mysterious role in the structure of $\mathbb{A}^1$-$h$-cobordisms. As a consequence of these observations, we completely solve the classification problem for rational smooth proper surfaces over an algebraically closed field: while there exist arbitrary dimensional moduli of such surfaces, there are only countably many $\mathbb{A}^1$-homotopy types, each uniquely determined by the isomorphism class of its $\mathbb{A}^1$-fundamental sheaf of groups.
1 Classification problems in algebraic geometry

In this paper, drawing its inspiration from geometric topology, we investigate the problem of classifying smooth proper algebraic varieties over a field using the techniques of $\mathbb{A}^1$-homotopy theory. In geometric topology, one can, without loss of generality, restrict the classification problem by considering connected manifolds, and in this setting classification can be performed most effectively for highly connected spaces (cf. [Wal62]). Similarly, we restrict our algebro-geometric classification problem by imposing $\mathbb{A}^1$-homotopic connectedness hypotheses; these restrictions are highly non-trivial. Indeed, the problem of even defining an appropriate analog of connectedness is subtle, especially as we require the notion to have a close relationship with $\mathbb{A}^1$-homotopy theory. We consider two notions called $\mathbb{A}^1$-connectedness and étale $\mathbb{A}^1$-connectedness; both notions are motivated by homotopic and arithmetic considerations. Contrary to the situation in geometric topology, we will see that one cannot often impose “higher $\mathbb{A}^1$-connectedness” hypotheses for such varieties: strictly positive dimensional smooth proper $\mathbb{A}^1$-connected varieties always have non-trivial $\mathbb{A}^1$-fundamental group (see Proposition 5.4 and Extension 5.5).

Next, we link $\mathbb{A}^1$-connectedness to the birational geometry of algebraic varieties. Over perfect fields $k$, the separably rationally connected smooth proper varieties introduced by Kollár-Miyaoka-Mori are étale $\mathbb{A}^1$-connected (see Definition 2.31 and Theorem 2.33), and, if $k$ furthermore has characteristic 0, retract $k$-rational smooth proper varieties are $\mathbb{A}^1$-connected (see Theorem 2.21). Over any field, Theorem 2.38 provides a geometric characterization of $\mathbb{A}^1$-connectedness for smooth proper varieties, and its subsequent corollaries provide connections with Manin’s notion of $R$-equivalence. These results open vast vistas of new homotopic techniques, and suggest a host of new invariants to study “nearly rational” varieties (see Propositions 2.28 and 4.18).

Finally, we outline a general program for studying the classification problem and provide supporting evidence. We give a detailed study of the classification in low dimensions (see Theorems 1.5 and 1.13). We emphasize the rôle played by the $\mathbb{A}^1$-fundamental (sheaf of) group(s) of a smooth proper variety and provide a number of detailed computations (see Propositions 5.8...
and 5.9). After contemplating these low-dimensional results, we introduce the \( A^1 \)-minimality problem (1.8) and the \( A^1 \)-surgery problem (1.10), motivated by birational geometry and the topological theory of surgery. The first problem explores the structure of \( A^1 \)-homotopy types within a given birational class, while the second explores the isomorphism classes of smooth varieties having a given \( A^1 \)-homotopy type: both problems should be amenable to study by \( A^1 \)-homotopy theory as part of a framework we begin to develop here.

Connectivity restrictions in \( A^1 \)-homotopy: geometry vs. arithmetic

Throughout this paper, the word manifold will mean compact without boundary smooth manifold. Classically, topologists considered two fundamental classification problems: (i) classify \( n \)-dimensional manifolds up to diffeomorphism, and (ii) classify \( n \)-dimensional manifolds up to homotopy equivalence. Problem (i) refines Problem (ii), and the study of both problems breaks down along dimensional lines. Extremely explicit classifications exist in dimensions 1 or 2, and Thurston’s geometrization program provides a classification in dimension 3. While group theoretic decision problems prevent algorithmic solutions to either problem in dimension \( \geq 4 \), the celebrated Browder-Novikov-Sullivan-Wall theory of surgery (cf. [Ran02, §1]) shows that, in dimensions \( \geq 5 \), the problem of identifying diffeomorphism classes of manifolds in a fixed homotopy type can be effectively reduced to computations in homotopy theory.

The problem of classifying smooth algebraic varieties over a field \( k \) up to isomorphism is formally analogous to Problem (i). To state an analog of Problem (ii), one must choose an appropriate notion of “homotopy equivalence” and consider the corresponding homotopy category. We use the \( A^1 \)- (or motivic) homotopy theory developed by the second author and V. Voevodsky in [MV99]. The prefix \( A^1 \)- draws attention to the fact that the affine line in \( A^1 \)-homotopy theory plays the same rôle as the unit interval in ordinary homotopy theory. The resulting \( A^1 \)-homotopy category of smooth varieties over \( k \) is denoted \( \mathcal{H}(k) \), and isomorphisms in \( \mathcal{H}(k) \) are called \( A^1 \)-weak equivalences. One source of examples of \( A^1 \)-weak equivalences is obtained by replacing the unit interval by the affine line in the classical definition of a homotopy equivalence. Another source of examples is provided by the Čech simplicial resolution associated with a Nisnevich covering of a smooth scheme. One of the difficulties of the theory stems from the fact that general \( A^1 \)-weak equivalences are obtained from these two classes by a complicated formal procedure (see, e.g., [Dug01, Proposition 8.1] for development of this point of view).

Compact and without boundary manifolds are akin to algebraic varieties smooth and proper over a field \( k \). Collecting these observations, we suggest a natural analog of Problem (ii) in the context of algebraic varieties over a field \( k \).

Problem 1.1. Classify smooth proper \( k \)-varieties up to \( A^1 \)-weak equivalence.

In order that two manifolds be homotopy equivalent, they must at least have the same number of connected components. However, one usually studies connected manifolds before investigating the disconnected case. A manifold \( M \) is path (or chain) connected if every pair of points lies in the image of a (chain of) map(s) from the unit interval, or equivalently if the
set \( \pi_0(M) \) has exactly one element. In \( \mathbb{A}^1 \)-homotopy theory one defines a (Nisnevich) sheaf \( \pi_0^{\mathbb{A}^1}(X) \) of \( \mathbb{A}^1 \)-connected components (see [MV99, p. 110]). A variety is \( \mathbb{A}^1 \)-connected if it has the same sheaf of connected components as the base field, and \( \mathbb{A}^1 \)-disconnected otherwise.

For varieties that are \( \mathbb{A}^1 \)-disconnected, the classification problem can look drastically different than it does for varieties that are (close to) \( \mathbb{A}^1 \)-connected (see, e.g., Proposition 2.6).

Transposing topological intuition, one might naively imagine that a smooth variety \( X \) is \( \mathbb{A}^1 \)-connected if any pair of \( k \)-points is contained in the image of a morphism from the affine line; this statement is “close” to being true. Motivated by arithmetic considerations we introduce two different notions of connectivity, \( \mathbb{A}^1 \)-connectedness and étale \( \mathbb{A}^1 \)-connectedness, depending on whether one requires such path connectedness properties for all separable field extensions of \( k \) or just separably closed field extensions of \( k \). While the precise definitions (see \$2\) of these connectedness notions have homotopy theoretic origins, the following result provides a geometric characterization of \( \mathbb{A}^1 \)-connectedness for proper schemes.

**Theorem 1.2** (see Theorem 2.38 and Corollary 2.39). If \( X \) is a smooth proper scheme over a field \( k \), \( X \) is \( \mathbb{A}^1 \)-connected if and only if it is \( \mathbb{A}^1 \)-chain connected (see Definition 2.9).

Section 6 is devoted to developing techniques for proving a more general result that implies this one. The following result provides a link between important notions of birational geometry and the aforementioned connectivity properties.

**Theorem 1.3** (see Theorems 2.21 and 2.33). Suppose \( k \) is a perfect field.

- If \( X \) is a separably rationally connected smooth proper variety, then \( X \) is étale \( \mathbb{A}^1 \)-connected.
- If furthermore \( k \) has characteristic 0, and \( X \) is a retract \( k \)-rational variety, then \( X \) is \( \mathbb{A}^1 \)-connected.

A \( k \)-variety \( X \) is retract \( k \)-rational if, e.g., it is \( k \)-rational, stably \( k \)-rational, or factor \( k \)-rational (see Definition 2.18).

We feel the suggested parallels with geometric topology and the direct links with birational geometry justify the importance of these connectivity notions and our subsequent focus on them. We refer the reader to Appendix A for a convenient summary of the relationships between the various notions just mentioned.

**Extension 1.4.** One can study disconnected manifolds by separate analysis of each connected component (however, cf. [Wal99, pp. 34-35] for discussion in the context of classification). In contrast, in \( \mathbb{A}^1 \)-homotopy theory, for any smooth variety \( X \) there is a canonical surjective morphism \( X \to \pi_0^{\mathbb{A}^1}(X) \) that is in general a highly non-trivial epimorphism of sheaves. While separably rationally connected smooth proper varieties are étale \( \mathbb{A}^1 \)-connected, they need not be \( \mathbb{A}^1 \)-connected in general. Indeed, the difference between the \( \mathbb{A}^1 \)-connectedness and étale \( \mathbb{A}^1 \)-connectedness encodes subtle arithmetic information (see Example 2.37). Nevertheless, we expect that \( \mathbb{A}^1 \)-disconnected varieties can be studied by considering the \( \mathbb{A} \)-fundamental (sheaf of) groupoid(s) instead of the \( \mathbb{A}^1 \)-fundamental (sheaf of) group(s).
Low dimensional classification and minimality

For manifolds of dimension 1 or 2 the homotopy classification and the diffeomorphism classification coincide. Each connected component of a 1-dimensional manifold is diffeomorphic to the circle $S^1$. Each connected component of a 2-dimensional manifold is homotopy equivalent to either $S^2$, the connected sum of $g$ copies of $S^1 \times S^1$ for some integer $g \geq 1$, or the connected sum of $g'$ copies of $\mathbb{R}P^2$ for some integer $g' \geq 1$.

Suppose $k$ is an algebraically closed field. There is a unique up to $\mathbb{A}^1$-weak equivalence smooth proper (étale) $\mathbb{A}^1$-connected $k$-variety of dimension 1, namely $\mathbb{P}^1$ (see Proposition 2.6). Smooth proper $k$-rational surfaces are (étale) $\mathbb{A}^1$-connected, and we expect the converse to be true (cf. Conjectures 2.13 and 2.34). In fact Corollary 2.41 shows that $k$-rational smooth proper surfaces are exactly the smooth proper $\mathbb{A}^1$-connected surfaces for fields having characteristic 0. The $\mathbb{A}^1$-homotopy classification of rational smooth proper surfaces, which addresses Problem 1.1 in dimension $\leq 2$ and is strikingly similar to its topological counterpart, can be stated in most elementary terms as follows.

**Theorem 1.5** (See Theorem 3.8). Any rational smooth proper surface over an algebraically closed field $k$ is $\mathbb{A}^1$-weakly equivalent to either $\mathbb{P}^1 \times \mathbb{P}^1$, or a blow-up of some (possibly empty) fixed, finite collection of distinct $k$-points on $\mathbb{P}^2$.

The isomorphism and $\mathbb{A}^1$-homotopy classifications of smooth proper $\mathbb{A}^1$-connected curves coincide. The isomorphism classification of rational smooth proper surfaces is well known. Over an algebraically closed field, any such surface is isomorphic to an iterated blow-up of points of either $\mathbb{P}^2$, or a Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$, i.e., the relatively minimal rational surfaces. Observe that this classification does not coincide with the $\mathbb{A}^1$-homotopy classification. Theorem 1.5 leads us to study the discrepancy between the isomorphism and $\mathbb{A}^1$-homotopy classifications.

**Extension 1.6.** We expect a statement similar Theorem 1.5 to hold more generally. Indeed, if we modify the statement above by replacing the phrase “finite set of points” by “0-dimensional closed subscheme,” we expect the corresponding result to hold for $k$-rational smooth proper surfaces over an arbitrary field $k$. We also expect a classification result analogous to Theorem 1.5 for smooth proper $\mathbb{A}^1$-connected surfaces over an arbitrary field $k$; see Corollary 5.14.

Just as the topological fundamental group obstructs enumeration of homotopy types of manifolds, the $\mathbb{A}^1$-fundamental (sheaf of) group(s) likely obstructs enumeration of $\mathbb{A}^1$-homotopy types of smooth varieties, even if they are $\mathbb{A}^1$-connected. Thus, Problem 1.1 is probably insoluble in dimension $\geq 4$, so we seek further refinements. On the other hand, we expect the sheaf of $\mathbb{A}^1$-connected components to be a birational invariant of smooth proper schemes; we provide evidence for this expectation in §6. We also expect that two smooth proper schemes that are $\mathbb{A}^1$-weakly equivalent are birationally equivalent. These expectations suggest $\mathbb{A}^1$-homotopy types of smooth varieties have additional structure inherited from their birational properties. Indeed, Theorem 1.5 and the discussion preceding it suggest that if a function field $K$ admits a smooth proper representative, it admits a (non-unique) representative that is homotopically minimal in the following sense.
Definition 1.7. An $\mathbb{A}^1$-homotopy type admitting a smooth proper $k$-variety $X$ as a representative is called minimal if given any triple $(X', \psi, \varphi)$ consisting of a smooth proper $k$-variety $X'$, an $\mathbb{A}^1$-weak equivalence $\psi : X' \to X$, and a proper birational morphism $\varphi : X' \to Y$ to a smooth $k$-variety $Y$, $\varphi$ is an isomorphism. Any smooth proper representative of a minimal $\mathbb{A}^1$-homotopy type will be called $\mathbb{A}^1$-minimal.

Given two smooth proper $k$-varieties $X$ and $Y$, we will say that $X$ is an $\mathbb{A}^1$-minimal model for $Y$ if $X$ is $\mathbb{A}^1$-minimal, and there exist a smooth proper $k$-variety $X'$, an $\mathbb{A}^1$-weak equivalence $X' \to Y$, and a proper, birational morphism $X' \to X$.

Problem 1.8 ($\mathbb{A}^1$-minimality). Let $K$ be the function field of a smooth proper $k$-variety of dimension $n$. Describe the set $\mathcal{M}_{\mathbb{A}^1}(K)$ of minimal $\mathbb{A}^1$-homotopy types for smooth proper varieties with function field $K$ by providing explicit $\mathbb{A}^1$-minimal smooth proper $k$-varieties in each $\mathbb{A}^1$-homotopy type in $\mathcal{M}_{\mathbb{A}^1}(K)$. Moreover, for a given smooth proper $k$-variety $Y$ provide an explicit procedure to find an $\mathbb{A}^1$-minimal model of $Y$ from the previous list.

We now proceed to reformulate Theorem 1.5 in a manner suggesting an explanation for the deviation of the isomorphism and $\mathbb{A}^1$-homotopy classifications for surfaces, and further refine the classification problem to study the internal structure of a set $\mathcal{M}_{\mathbb{A}^1}(K)$ as in Problem 1.8.

High dimensions and internal structure of homotopy types via surgery

Surgery theory asserts, roughly speaking, that one can measure the extent to which the homotopy and diffeomorphism classifications for manifolds of a given dimension differ. Indeed, one main application of surgery theory is to the solution (in dimensions $\geq 5$) of Problem (iii): determine the set of diffeomorphism classes of manifolds in a given homotopy type. As Kervaire and Milnor explain in their celebrated work on classification of exotic spheres (see [KM63, Theorem 1.1] and the subsequent remark), arguably still the best introduction to surgery theory, the analysis of Problem (iii) consists of two independent components.

The first component of the investigation is provided by Smale’s theory of $h$-cobordisms that relates diffeomorphism to more homotopy theoretic notions (e.g., cobordism theory) via Morse theory. An $h$-cobordism $(W, M, M')$ between smooth manifolds $M$ and $M'$ is a cobordism such that the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences. The fundamental group of $M$ plays a central role in the theory of $h$-cobordisms. Smale’s celebrated $h$-cobordism theorem showed that an $h$-cobordism between simply connected manifolds of dimension $\geq 5$ is necessarily trivial, i.e., diffeomorphic to a product of the form $M \times I$. An $h$-cobordism between non-simply connected manifolds of dimension $\geq 5$ need not be diffeomorphic to a product, but Barden, Mazur and Stallings formulated an appropriate generalization, now called the $s$-cobordism theorem, giving necessary and sufficient conditions for triviality of such $h$-cobordisms (see, e.g., [Ran02, Chapter 8] for discussion of these results in the context of classification).

Given a CW complex $X$, let $\mathcal{S}(X)$ denote the structure set of $X$, i.e., the set of $h$-cobordism classes of manifolds homotopy equivalent to $X$. At least in dimensions $\geq 5$, Problem (iii) reduces via the $h$-cobordism (or $s$-cobordism) theorem to determining if the structure set $\mathcal{S}(X)$ is non-empty, and, when it is, providing a description of its elements. The second component
of the analysis of Problem (iii), contained in the beautiful work of Browder, Novikov, Sullivan and Wall, provides a description of $\mathcal{P}(X)$. In its modern formulation, these authors construct a two-stage obstruction theory effectively reducing determination of $\mathcal{P}(X)$ to computations in homotopy theory; we outline this approach at the end of §5. The literature on surgery theory is famously dense and the standard references are [Bro72] and [Wal99]; other references, each having different emphasis, include [Ran02] MM79 Rog.

Mimicking this development in algebraic geometry, we see the strengths of the $A^1$-homotopy category. M. Levine and the second author developed a natural algebro-geometric analog of cobordism theorem, called algebraic cobordism (see [LM07]). In this spirit, we introduce a notion of $h$-cobordism in algebraic geometry (see Definition 3.1), motivated by Morse theory, that we call $A^1$-cobordism. An $A^1$-cobordism between smooth proper varieties $X$ and $X'$ consists of a pair $(W, f)$, with $W$ a smooth variety, and $f : W \to A^1$ a proper, surjective morphism such that $X = f^{-1}(0)$, $X' = f^{-1}(1)$, and the inclusions $X \hookrightarrow W$ and $X' \hookrightarrow W$ are $A^1$-weak equivalences. We think of the pair $(W, f)$ as a cobordism together with a choice of “Morse function.” In analogy with the situation in geometric topology, we suggest the following definition and problem.

**Definition 1.9.** Given a space $\mathcal{X}$ (see the conventions), a *scheme structure* on (or homotopy smoothing of) $\mathcal{X}$ is a pair $(X, s)$, consisting of a smooth proper scheme $X$ and an $A^1$-weak equivalence $s : X \to \mathcal{X}$. The $A^1$-structure set of $\mathcal{X}$, denoted $\mathcal{S}_{A^1}(\mathcal{X})$, is the set of scheme structures on $\mathcal{X}$ subject to the equivalence relation generated by $(X, s) \sim (X', s')$ if there exists a triple $(W, f, H)$ consisting of an $A^1$-cobordism $(W, f)$ between $X$ and $X'$, and a morphism $H : W \to \mathcal{X}$ that upon composition with the morphism $X \hookrightarrow W$ (resp. $X' \hookrightarrow W$) gives $f$ (resp. $f'$). Two scheme structures on a space $\mathcal{X}$ equivalent in $\mathcal{S}_{A^1}(\mathcal{X})$ will be called $A^1$-block equivalent.

**Problem 1.10 ($A^1$-surgery problem).** Given an $A^1$-connected space $\mathcal{X}$, determine if the set $\mathcal{S}_{A^1}(\mathcal{X})$ is non-empty, and, if it is, determine the set of $A^1$-block equivalence classes.

To show that Problem 1.10 is extremely rich, we give techniques for constructing $A^1$-h-cobordisms of smooth proper schemes (see Propositions 3.4 and 3.6). The proof of Theorem 1.5 requires a detailed study of $A^1$-h-cobordisms between rational smooth proper surfaces. Extending this discussion, the results of §4 show that $A^1$-h-cobordisms constructed by these means are abundant and lays some technical foundation for a general investigation.

Reflecting on some basic computations highlights central differences between topology and algebraic geometry. First, $A^1$-h-cobordisms between smooth schemes are “rarely” trivial (i.e., isomorphic to products of the form $X \times A^1$), even when the associated “Morse function” has no critical values. Following topological ideas, we look to the $A^1$-fundamental group, investigated in great detail in [Mor06b], for an explanation of the deviation between $A^1$-block equivalence and isomorphism as schemes. Second, there are arguably few interesting invariants available to distinguish “nearly rational” varieties. For example, given a unirational smooth proper complex variety $X$, Serre famously showed [Ser59 Proposition 1] that the set $X(\mathbb{C})$, viewed as a complex manifold, is simply connected. In stark contrast to the topological situation, the $A^1$-fundamental group of any strictly positive dimensional rational smooth proper complex variety...
is always a highly non-trivial invariant (see Proposition 5.4 and Extension 5.5 for precise and more general statements)! Said differently, using Example 2.4, one can see that the only $\mathbb{A}^1$-connected and $\mathbb{A}^1$-simply connected smooth proper variety over a field is a point. We record the following result mainly for the sake of amusement.

**Scholium 1.11 ($\mathbb{A}^1$-$h$-cobordism theorem).** Any $\mathbb{A}^1$-$h$-cobordism between smooth proper $\mathbb{A}^1$-connected and $\mathbb{A}^1$-simply connected varieties over a field is trivial.

**Extension 1.12.** Extension 5.6 provides a stronger “étale” version of this result involving étale $\mathbb{A}^1$-connected and étale $\mathbb{A}^1$-simply connected varieties.

The $\mathbb{A}^1$-fundamental group of $\mathbb{P}^1$, determined in [Mor06b], plays a distinguished rôle in $\mathbb{A}^1$-homotopy theory, and we review aspects of this computation here. The $\mathbb{A}^1$-fundamental groups of projective spaces, $SL_n$, and smooth proper toric varieties have also been studied ([Mor06b §4], and [AD07a, Wen07]). One main computational result of this paper is the determination of the $\mathbb{A}^1$-fundamental group of various rational smooth proper varieties (see Propositions 5.8 and 5.9). Combining the computations of this paper with Theorem 1.5 provides a solution to Problem 1.10 for rational smooth proper surfaces.

**Theorem 1.13 (See Corollary 5.14).** Let $k$ be an algebraically closed field. Two rational smooth proper surfaces are $\mathbb{A}^1$-$h$-cobordant if and only if their $\mathbb{A}^1$-fundamental groups are isomorphic. Thus, for any rational smooth proper surface $X$ the set $\mathcal{S}_{\mathbb{A}^1}(X)$ consists of a single element.

**Extension 1.14.** Combining Corollary 2.41 with Theorem 1.13 provides a solution to the $\mathbb{A}^1$-surgery problem for smooth proper $\mathbb{A}^1$-connected varieties of dimension $\leq 2$ over algebraically closed fields having characteristic 0. More generally, we expect that Theorem 1.13 continues to hold for smooth proper $\mathbb{A}^1$-connected surfaces over arbitrary fields.

Theorem 1.13 and the discussion preceding it provides the following lesson: the extent to which the isomorphism and $\mathbb{A}^1$-homotopy classifications differ depends on minimality properties in the sense of Problem 1.8. Indeed, we will see that blowing-up makes the $\mathbb{A}^1$-fundamental group more complicated. At the end of §5 we discuss possible analogs of the $s$-cobordism theorem in $\mathbb{A}^1$-homotopy theory and formulate a general approach to the $\mathbb{A}^1$-surgery problem.

The $\mathbb{A}^1$-homotopy type of a smooth proper variety encodes universal cohomological information about the variety, and, in particular, information about Hodge structures on cohomology, étale homotopy type, (higher) Chow groups, algebraic K-theory, or Hermitian K-theory. The eventual goal of this kind of study of smooth proper schemes is to understand the *arithmetic building blocks*, or *motivic skeleton*, of smooth proper varieties over a field using a surgery-style obstruction theory. The introduction to each section contains more detailed discussion of the results contained therein.

**Conventions and notation**

Throughout this paper, $k$ denotes a field. Henceforth, we use the word scheme as a synonym for separated scheme having essentially finite type over $k$, i.e., a filtering limit of $k$-schemes.
having finite type over \( k \) with smooth affine bonding morphisms. The word *variety* means integral scheme having finite type over \( k \). Using this terminology, let \( \text{Sm}_k \) denote the category of smooth schemes (having finite type) over \( k \). The words map and morphism are used synonymously through this paper, and we denote them by solid arrows. Rational maps, where they occur, are denoted by dashed arrows. Rational maps, where they occur, are denoted by dashed arrows. The words map and morphism are used synonymously through this paper, and we denote them by solid arrows.

We let \( \text{Sp}_{k} \) (\( \text{Sp}_{k} \)) stand for the category of (pointed) spaces over \( k \), i.e., the category of (pointed) simplicial \( \text{Nis} \) sheaves of sets on \( \text{Sm}_k \). The word *sheaf* means sheaf in the \( \text{Nis} \) topology (cf. [MV99 §3.1]), unless otherwise indicated. We designate schemes by upper case Roman letters (e.g., \( X,Y \)), spaces by upper case calligraphic letters (e.g., \( \mathcal{X}, \mathcal{Y} \)), and pointed (simplicial) spaces by explicit specification of the base-point. We write \( X_m \) for the sheaf of \( n \)-simplices of a space \( \mathcal{X} \). The Yoneda embedding induces a fully-faithful functor \( \text{Sm}_k \to \text{Sp}_{k} \); we systematically abuse notation and write \( X \) for the space associated with \( X \in \text{Sm}_k \) via this functor. If \( k \) is clear from context, we write \( * \) interchangeably for the space \( \text{Spec} \ k \) or a base-(\( k \))-point.

We refer the reader to [MV99] for the foundations of \( \Lambda^1 \)-homotopy theory. Let \( \mathcal{H}^\text{Nis}(k) \) (\( \mathcal{H}^\text{Nis}_*(k) \)) denote the (pointed) simplicial homotopy category ([MV99 §2.1], especially Definition 1.2 and Theorem 1.4), and \( \mathcal{H}(k) \) (\( \mathcal{H}_*(k) \)) denote the (pointed) \( \Lambda^1 \)-homotopy category (see [MV99 p. 106]). In particular [MV99 §3.2.1] defines the notion of \( \Lambda^1 \)-weak equivalence. Given two pointed spaces \( (\mathcal{X},x) \) and \( (\mathcal{Y},y) \), set: \([\mathcal{X},\mathcal{Y}]_s \) (resp. \( [(\mathcal{X},x),(\mathcal{Y},y)]_s \)) to be the (pointed) simplicial homotopy classes of maps from \( \mathcal{X} \) to \( \mathcal{Y} \). Likewise, set \([\mathcal{X},\mathcal{Y}]_{\Lambda^1} \) (resp. \( [(\mathcal{X},x),(\mathcal{Y},y)]_{\Lambda^1} \)) to be the set of (pointed) \( \Lambda^1 \)-homotopy classes of maps from \( \mathcal{X} \) to \( \mathcal{Y} \).

Sections 2 and 3 do not explicitly require deep properties of the \( \Lambda^1 \)-homotopy category. We use the fact that a Zariski (\( \text{Nis} \)) locally trivial smooth morphism of smooth schemes with fibers isomorphic to affine spaces is an \( \Lambda^1 \)-weak equivalence (cf. [MV99 §3 Example 2.3] or [AD07b]), some formal properties of \( \Lambda^1 \)-weak equivalences, and the existence of an \( \Lambda^1 \)-resolution functor \( E_{\Lambda^1} \) that commutes with formation of finite limits (cf. [MV99 §2 Corollary 3.2] and [AD07a Remark 2.5]). For later sections, recall the \( \Lambda^1 \)-homotopy groups of a pointed space \( (\mathcal{X},x) \), denoted \( \pi^\Lambda_1(\mathcal{X},x) \), are the sheaves associated with the presheaves \( U \mapsto [S^1_\mathbb{A} \wedge U_+,(\mathcal{X},x)]_{\Lambda^1} \); the spheres \( S^1_\mathbb{A} \) are studied in [MV99 §3.2.2]. Finally, one word of caution is in order: we (almost) never use stable homotopy theoretic considerations in this paper, so a lowercase super- or subscript \( s \) is always short for *simplicial*.

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2 Connectedness in $\mathbb{A}^1$-homotopy theory

In this section, we discuss several notions of connectedness in $\mathbb{A}^1$-homotopy theory, point out the fundamental differences between these notions and the usual notion of (path) connectedness for a topological space, and relate our notions of connectedness to birational geometry of algebraic varieties. While classification of disconnected manifolds reduces to classification of connected components, the corresponding fact in $\mathbb{A}^1$-homotopy theory is more subtle (see Remark 2.2, Lemma 2.3 and Proposition 2.6). Definition 2.9 and Proposition 2.12 provide a geometric condition that guarantees $\mathbb{A}^1$-connectedness of a space. Theorem 2.38 and Corollary 2.39 complement these results by providing a (partial) converse to Proposition 2.12 and gives a geometric characterization of $\mathbb{A}^1$-connectedness for smooth proper schemes over a field.

Definition 2.15 and Lemma 2.16 give large classes of smooth $\mathbb{A}^1$-connected varieties over fields having arbitrary characteristic. Theorem 2.21 demonstrates, in particular, that retract $k$-rational varieties over a field $k$ having characteristic 0 are necessarily $\mathbb{A}^1$-connected. Propositions 2.27 and 2.28 state some fundamental cohomological properties of $\mathbb{A}^1$-connected smooth schemes. While separably rationally connected smooth proper varieties over a perfect field are not $\mathbb{A}^1$-connected in general, Theorem 2.33 shows such varieties are étale $\mathbb{A}^1$-connected in the sense of Definition 2.31. Theorem 2.38 provides a purely geometric characterization of $\mathbb{A}^1$-connectedness for smooth proper varieties. Finally, we note here that Appendix A provides a summary of the various notions of connectivity and rationality introduced in this section and used in the rest of the paper.

Reviewing $\mathbb{A}^1$-connectedness and $\mathbb{A}^1$-rigidity

Definition 2.1. Suppose $\mathcal{X} \in \text{Spc}_k$. The sheaf of $\mathbb{A}^1$-connected components of $\mathcal{X}$, denoted $\pi^\mathbb{A}^1_0(\mathcal{X})$, is the sheaf associated with the presheaf

$$U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1},$$

for $U \in \mathcal{S}m_k$. We will say that $\mathcal{X}$ is $\mathbb{A}^1$-connected if $\pi^\mathbb{A}^1_0(\mathcal{X})$ is the trivial sheaf, i.e., the sheaf associated with Spec $k$, and $\mathbb{A}^1$-disconnected otherwise.

For any $\mathcal{X}$ as above, there is a canonical morphism $X_0 \to \pi^\mathbb{A}^1_0(\mathcal{X})$. According to [MV99, §2 Corollary 3.22], the so-called unstable $\mathbb{A}^1$-0-connectivity theorem, this morphism $X_0 \to \pi^\mathbb{A}^1_0(\mathcal{X})$ is an epimorphism of sheaves. In particular, $X$ is a smooth $\mathbb{A}^1$-connected $k$-scheme, the map $X(\text{Spec } k) \to \pi^\mathbb{A}^1_0(X)(\text{Spec } k)$ is surjective, and we conclude that $X$ necessarily has a $k$-rational point.

Remark 2.2. In topology, if $M$ is a manifold, then we can study $M$ by analyzing each connected component separately, since each such component will again be a manifold. In $\mathbb{A}^1$-homotopy theory, given a morphism $* \to \pi^\mathbb{A}^1_0(X)$ corresponding to a $k$-rational point $x \in X(k)$, the “connected component of $X$ containing the point $x$” is only a space, and not necessarily a smooth scheme.
Recall that $X \in Sm_k$ is called $\mathbb{A}^1$-rigid (see [MV99, §3 Example 2.4]) if for every $U \in Sm_k$, the map
\[ Hom_{Sm_k}(U, X) \longrightarrow Hom_{Sm_k}(U \times \mathbb{A}^1, X) \]
induced by pullback along the projection $U \times \mathbb{A}^1 \to U$ is a bijection. The next lemma can be straightforwardly deduced from the fact that smooth $\mathbb{A}^1$-rigid schemes are $\mathbb{A}^1$-local and simplicially fibrant (cf. [MV99, §2 Remark 1.14]).

**Lemma 2.3.** Suppose $X \in Sm_k$ is $\mathbb{A}^1$-rigid. Then, for any $Y \in Sm_k$, the canonical map
\[ Hom_{Sm_k}(Y, X) \longrightarrow [Y, X]_{\mathbb{A}^1} \]
is a bijection. Consequently, the canonical map $X \to \pi^\mathbb{A}^1_0(X)$ is an isomorphism of sheaves.

**Example 2.4.** There are many well-known examples of schemes that are $\mathbb{A}^1$-rigid. Any 0-dimensional scheme over an arbitrary field $k$ is $\mathbb{A}^1$-rigid. Indeed, abelian $k$-varieties are $\mathbb{A}^1$-rigid, and smooth complex varieties that can be realized as quotients of bounded Hermitian symmetric domains by actions of discrete groups are also $\mathbb{A}^1$-rigid. From the above collection of $\mathbb{A}^1$-rigid varieties, one can produce new examples by taking (smooth) subvarieties or taking products.

Lemma 2.3 implies that the classification of smooth $\mathbb{A}^1$-rigid schemes up to $\mathbb{A}^1$-weak equivalence coincides with the isomorphism classification. Furthermore, we will now see that smooth $\mathbb{A}^1$-rigid schemes are $\mathbb{A}^1$-minimal.

**Proposition 2.5.** The $\mathbb{A}^1$-homotopy type of a smooth proper $\mathbb{A}^1$-rigid $k$-variety $X$ is $\mathbb{A}^1$-minimal in the sense of Definition 1.7.

**Proof.** Suppose $X$ is a smooth proper $\mathbb{A}^1$-rigid variety, and $f : X \to Y$ is a proper birational morphism with $Y$ smooth and proper. By [KM98 Proposition 1.3], for any point $y$ of $Y$, either $f^{-1}(y)$ is a point or $f^{-1}(y)$ contains and is covered by rational curves. Now, the exceptional set of $f$ is of pure codimension 1 ([KM98 Corollary 2.63]). If the exceptional set of $f$ is empty, $f$ is an isomorphism. Otherwise, we can find a non-trivial morphism $\mathbb{A}^1 \to X$ contradicting $\mathbb{A}^1$-rigidity of $X$. □

Using the classification of curves over a field $k$, one can show that any (open subscheme of a) curve of genus $g \geq 1$ is $\mathbb{A}^1$-rigid, and similarly that (any open subscheme of) $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid. Combining these facts with the observation that smooth $\mathbb{A}^1$-connected $k$-schemes have a $k$-rational point, we deduce the following result (cf. [MV99, §3 Remark 2.5]).

**Proposition 2.6.** Two smooth proper curves of genus $g \geq 1$ are $\mathbb{A}^1$-weakly equivalent if and only if they are isomorphic. A smooth proper curve is $\mathbb{A}^1$-connected if and only if it is isomorphic to $\mathbb{P}^1$.

**Remark 2.7.** Generalizing Proposition 2.6, one can show that for arbitrary fields the $\mathbb{A}^1$-homotopy classification and isomorphism classification of curves coincide. Indeed, one can give a relatively elementary proof of the fact that two smooth proper curves over $k$ are $\mathbb{A}^1$-weakly equivalent if and only if they are isomorphic. We have deferred this proof to a later paper for considerations regarding its length.
\(A^1\)-chain connectedness

We now recall a geometric criterion that guarantees \(A^1\)-connectedness. Given \(X \in \mathcal{S}_{m/k}\), \(L\) a finitely generated separable extension of \(k\), and points \(x_0, x_1 \in X(L)\) an elementary \(A^1\)-equivalence between \(x_0\) and \(x_1\) is a morphism \(f : A^1 \to X\) such that \(f(0) = x_0\) and \(f(1) = x_1\). We will say that two points \(x, x' \in X(L)\) are \(A^1\)-equivalent if they are equivalent for the equivalence relation generated by elementary \(A^1\)-equivalence.

**Notation 2.8.** We write \(X(L)/\sim\) for the quotient of the set of \(L\)-rational points for the above equivalence relation and refer to this quotient as the set of \(A^1\)-equivalence classes of \(L\)-points.

**Definition 2.9 (cf. [AD07a] Definition 2.6).** We say that \(X \in \mathcal{S}_{m/k}\) is \(A^1\)-chain connected if for every finitely generated separable field extension \(L/k\) the set of \(A^1\)-equivalences classes of \(L\)-points \(X(L)/\sim\) consists of exactly 1 element.

Given \(X \in \mathcal{S}_{m/k}\), let \(\text{Sing}_{*}^{A^1}(X)\) denote the Suslin-Voevodsky singular construction of \(X\) (see [MV99, p.88 and p. 107]). By construction, there is a canonical morphism \(X \to \text{Sing}_{*}^{A^1}(X)\) that is an \(A^1\)-weak equivalence ([MV99, §2 Corollary 3.8]).

**Definition 2.10.** For \(X \in \mathcal{S}_{m/k}\), let \(\pi_{0}^{ch}(X)\) denote the sheaf associated with the presheaf \(U \mapsto [U, \text{Sing}_{*}^{A^1}(X)]\). We refer to \(\pi_{0}^{ch}(X)\) as the sheaf of \(A^1\)-chain connected components of \(X\).

**Remark 2.11.** One can check that the stalks of \(\pi_{0}^{ch}(X)\) over the Henselization \(\mathcal{O}_{Y,y}^{\text{h}}\) at any point \(y\) of a smooth variety \(Y\) (e.g., finitely generated separable extensions \(L/k\)) coincide with \(A^1\)-equivalence classes of \(\mathcal{O}_{Y,y}^{\text{h}}\)-points on \(X\). The induced map

\[
\pi_{0}^{ch}(X) \longrightarrow \pi_{0}^{A^1}(\text{Sing}_{*}^{A^1}(X)) \cong \pi_{0}^{A^1}(X)
\]

is an epimorphism of sheaves by the unstable 0-\(A^1\)-connectivity theorem. The next result shows that this epimorphism is an isomorphism if \(\pi_{0}^{ch}(X)\) is reduced to a point.

**Proposition 2.12 (cf. [AD07a] Proposition 2.8).** If \(X \in \mathcal{S}_{m/k}\) is \(A^1\)-chain connected, then \(X\) is \(A^1\)-connected.

**Sketch of proof.** The proof of the previous proposition requires two inputs. First, one uses the existence of a functor \(\text{Ex}_{A^1}\) that associates with any space \(X\) a simplicially fibrant and \(A^1\)-local space \(\text{Ex}_{A^1}(X)\) together with an \(A^1\)-acyclic cofibration \(X \to \text{Ex}_{A^1}(X)\), and commutes with formation of finite products (see the conventions). Second, one proves that \(A^1\)-connectedness can be checked over finitely generated separable extensions of \(k\), as opposed to all stalks (see [Mor05, Lemma 6.1.3]); the proof of this fact uses the homotopy purity theorem ([MV99, §3 Theorem 2.23]).

**Conjecture 2.13.** The epimorphism of \(\pi_{0}^{ch}(X) \to \pi_{0}^{A^1}(X)\) of Remark 2.11 is always an isomorphism. In particular, an object \(X \in \mathcal{S}_{m/k}\) is \(A^1\)-chain connected if and only if it is \(A^1\) connected.
Remark 2.14. Theorem \textup{2.38} provides evidence for this conjecture. Also, Corollary \textup{2.39} gives a converse to Proposition \textup{2.12} for proper schemes over perfect fields. More generally, we will see there that under these additional hypotheses the epimorphism of Remark \textup{2.11} is a bijection on sections over finitely generated separable extensions \( L \) of \( k \); this observation provides a positive solution to a problem about the structure of the set \( [\text{Spec} \, k, X]_{\mathbb{A}^1} \) (cf. [Mor04a, p. 386]).

Definition 2.15. We will say that an \( n \)-dimensional smooth \( k \)-variety \( X \) is covered by affine spaces if \( X \) admits an open affine cover by finitely many copies of \( \mathbb{A}^n_k \).

Lemma 2.16 (cf. [AD07a] Lemma 2.9). If \( X \) is a smooth \( k \)-variety that is covered by affine spaces, then \( X \) is \( \mathbb{A}^1 \)-chain connected.

Example 2.17. For simplicity assume that \( k \) is an algebraically closed field. Smooth \( k \)-varieties covered by affine spaces are all rational as algebraic varieties. However, the collection of such varieties includes all smooth proper toric varieties, or generalized flag varieties for connected reductive groups over \( k \). Generalizing both of these examples, recall that a normal variety on which a connected reductive group \( G \) acts is said to be spherical if a Borel subgroup \( B \subset G \) acts with a dense orbit. Using the local structure theory of Brion-Luna-Vust, one can check that any smooth proper spherical variety over an algebraically closed field having characteristic \( 0 \) is covered by affine spaces (see [BLV86, 1.5 Corollaire]). On the other hand, we will see that even over \( \mathbb{C} \), there are smooth proper varieties that are \( \mathbb{A}^1 \)-connected yet not covered by affine spaces (see Example \textup{2.24}).

Near rationality and \( \mathbb{A}^1 \)-connectedness

We now proceed to link \( \mathbb{A}^1 \)-connectedness to rationality properties of algebraic varieties. Recall that two \( k \)-varieties \( X \) and \( Y \) are \( k \)-birational or \( k \)-birationally equivalent if the function fields \( k(X) \) and \( k(Y) \) are isomorphic as \( k \)-algebras.

Definition 2.18. A \( k \)-variety \( X \) is called

i) \( k \)-rational if it is \( k \)-birational to \( \mathbb{P}^n \),

ii) stably \( k \)-rational if there exists an integer \( n \geq 0 \) such that \( X \times \mathbb{P}^n \) is \( k \)-rational,

iii) a direct factor of a \( k \)-rational variety, or simply factor \( k \)-rational, if there exists a \( k \)-variety \( Y \) such that \( X \times Y \) is \( k \)-rational, and finally

iv) retract \( k \)-rational if there exists an open subscheme \( U \) of \( X \) such that the identity map \( U \to U \) factors through an open subscheme \( V \) of an affine space (over \( k \)).

We will say that a \( k \)-variety is rational if it is \( \bar{k} \)-rational for an algebraic closure of \( k \). Similar conventions could be made for the other definitions, but we will not use these notions in this paper.
Lemma 2.19 (cf. [CTS07] Proposition 1.4). If \( X \) is a smooth \( k \)-variety, each condition of Definition 2.18 implies the subsequent one.

**Proof.** The first two implications of the statement are clear from the definitions, so it remains to prove the last one. Assume \( Y \) is a \( k \)-variety such that \( X \times Y \) is \( k \)-rational. Let \( U \subset X \times Y \) be a non-empty open subscheme which is isomorphic to an open subscheme of an affine space. Let \( (x_0, y_0) \in U(k) \). Let \( X_1 \) be the non-empty open subscheme of \( X \) defined by \( X_1 \times \{y_0\} := U \cap (X \times \{y_0\}) \). The open set \( U_1 = U \cap (X_1 \times Y) \) is still isomorphic to an open set of affine space. The composite map \( X_1 \to U_1 \to X_1 \), with the first map induced by \( x \mapsto (x, y_0) \) and the second map induced by projection onto \( X \) provides the necessary retraction. \( \square \)

**Remark 2.20.** The Zariski cancellation problem, sometimes called the birational cancellation problem, asked whether stably \( k \)-rational varieties are necessarily \( k \)-rational. A negative solution to this problem (even over \( \mathbb{C} \)) was provided in the celebrated work [BCTSSD85]; see Example 2.24 for more details. It is known that if \( k \) is not algebraically closed, there exist varieties that are factor \( k \)-rational yet not stably \( k \)-rational ([CTS77]); see Example 2.25 for more details. The notion of retract \( k \)-rationality was introduced and studied by Saltman (cf. [Sal84, Definition 3.1]) in relation to Noether’s problem regarding rationality of fields of invariants.

We will say that *weak factorization holds over \( k \) in dimension \( n \)* if given any two \( k \)-birationally equivalent smooth proper varieties \( X \) and \( X' \) of dimension \( n \), there exist a sequence of smooth proper varieties \( Z_1, \ldots, Z_n, X_1, \ldots, X_n \) of dimension \( n \), and a diagram of the form

\[
X \leftarrow Z_1 \rightarrow X_1 \leftarrow Z_2 \rightarrow \cdots \leftarrow Z_{n-1} \rightarrow X_n \leftarrow Z_n \rightarrow X',
\]

where each morphism with source \( Z_i \) is a blow-up at a smooth center.

**Theorem 2.21.** Suppose \( k \) is a perfect field, and assume weak factorization holds over \( k \) in dimension \( n \).

i) If \( X \) and \( X' \) are \( k \)-birationally equivalent smooth proper varieties of dimension \( n \), then \( X \) is \( \mathbb{A}^1 \)-chain connected if and only if \( X' \) is \( \mathbb{A}^1 \)-chain connected.

Suppose further that \( k \) has characteristic 0.

ii) If \( X \) is a retract \( k \)-rational smooth proper variety, then \( X \) is \( \mathbb{A}^1 \)-chain connected and thus \( \mathbb{A}^1 \)-connected.

**Proof.** For (i) using weak factorization it suffices to check the statement for blow-ups at smooth centers; we do this in Proposition 2.23 below.

For (ii), we know that there exists an open subscheme \( U \subset X \) and an open subscheme \( V \) of \( \mathbb{A}^m \) such that \( id : U \to U \) factors through \( V \). Thus, there are a rational smooth proper variety \( Z \), and rational maps \( X \dashrightarrow Z \dashrightarrow X \) factoring the identity map. By resolution of indeterminacy, we can assume that there exists a rational smooth proper variety \( Z \) dominating \( Z \) and a proper birational morphism \( Y \to X \). Again using resolution of indeterminacy, we can
assume there is a smooth proper variety $X'$ and proper birational morphisms $X' \to X$ and $X' \to Y$ such that the morphism $X' \to X$ restricts to the identity on $U$.

Note that [AKMW02, Theorem 0.1.1] establishes weak factorization in the sense above for any field $k$ having characteristic 0 and any integer $n \geq 0$. By the result of (i), $X$ is $\mathbb{A}^1$-chain connected if and only if $X'$ is $\mathbb{A}^1$-chain connected. For any finitely generated separable extension $L/k$, composition induces maps of $\mathbb{A}^1$-equivalence classes of $L$-points

$$X'(L)/\sim \to Y(L)/\sim \to X(L)/\sim.$$ 

Since $Y$ is $k$-rational and $\mathbb{P}^n$ is $\mathbb{A}^1$-chain connected, again using (i), we deduce that $Y(L)/\sim$ consists of exactly 1 element. Since the composite map is a bijection, it follows that $X(L)/\sim$ must also consist of a single element. Applying Proposition 2.12 finishes the proof.

Weak factorization in the above sense for surfaces over perfect fields $k$ having arbitrary characteristic is well known (see, e.g., [Bea96, Theorem II.11 and Appendix A]). Thus, we have deduced the following result.

**Corollary 2.22.** If $k$ is a perfect field, any $k$-rational smooth proper surface is $\mathbb{A}^1$-connected. If $k$ is a field having characteristic 0, then any stably $k$-rational, or factor $k$-rational smooth proper variety is $\mathbb{A}^1$-connected.

**Proposition 2.23** (cf. [CTS77, Proposition 10]). Let $k$ be a perfect field. Suppose $f : X \to Y$ is a blow-up of a smooth proper $k$-scheme at a smooth closed subscheme $Z$ of codimension $r + 1$. For any finitely generated separable field extension $L/k$, $f$ induces a map of $\mathbb{A}^1$-equivalence classes of $L$-points $X(L)/\sim \to Y(L)/\sim$ that is a bijection. Moreover, $Y$ is $\mathbb{A}^1$-chain connected if and only if $X$ is $\mathbb{A}^1$-chain connected.

**Proof.** We know that $\mathbb{P}^r$ is $\mathbb{A}^1$-chain connected since it is covered by affine spaces. By assumption, every fiber of $f$ is either a projective space of dimension $r$ or of dimension 0, so $f$ induces a surjective map on the level of $L$-rational points for any finitely generated extension $L/k$. Furthermore, any two $L$-rational points in a fiber are contained in the image of a morphism from $\mathbb{A}^1_L$. Thus, composition induces a map on the sets of $\mathbb{A}^1$-equivalence classes of $L$-points $X(L)/\sim \to Y(L)/\sim$ that is surjective.

Since $X$ and $Y$ are both proper, to show this function $X(L)/\sim \to Y(L)/\sim$ is injective, it suffices to prove the following fact. Given $y, y' \in Y(L)$, a morphism $h : \mathbb{P}^1_L \to Y$ joining $y$ and $y'$, and lifts $x, x' \in X(L)$ such that $f(x) = y$ and $f(x') = y'$, the points $x$ and $x'$ are $\mathbb{A}^1$-equivalent $L$-points.

If the image of $h$ is disjoint from $Z$, then there exists a unique lift $h' : \mathbb{P}^1_L \to X$. If not, suppose that $y$ and $y'$ are contained in the same open set $U$ of $Z$ such that the fiber product $U \times_Y X$ is $L$-isomorphic to $U \times \mathbb{P}^r_L$. For any fixed $z$ in $\mathbb{P}^r(L)$, consider the morphism $\psi : \mathbb{P}^1_L \to X$ defined on an open set by $t \mapsto (\varphi(t), z)$. Since $Z$ is smooth, it admits a Zariski open cover by subsets having the above property; this construction provides the required lift. The second fact follows immediately from this proof using the definition of $\mathbb{A}^1$-chain connectedness. \qed
Example 2.24. There exist non-$k$-rational smooth proper varieties that are $\mathbb{A}^1$-connected. Indeed, if $k$ is a non-algebraically closed perfect field, there exist stably $k$-rational, non-$k$-rational smooth proper surfaces by [BCTSSD85]. Such varieties are $\mathbb{A}^1$-connected by Corollary 2.22.

More explicitly, over any field $k$ having characteristic unequal to 2, let $P \in k[x]$ be an irreducible separable polynomial of degree 3 and discriminant $a$. Any smooth proper model of the surface $X_a$ given by the affine equation $y^2 - az^2 = P(x)$ has the property that $X_a \times \mathbb{P}^3$ is $k$-birationally equivalent to $\mathbb{P}^5$, though if $a$ is not a square in $k$, then $X_a$ is not $k$-rational (see [BCTSSD85 Théorème 1 p. 293]). If $k$ is algebraically closed, one can consider the above result for $k(t)$ to obtain threefolds that are stably rational yet non-rational (see [BCTSSD85, Théorème 1’ p. 299] for a precise statement).

Example 2.25. If $k$ is not algebraically closed, there exist examples of smooth proper varieties that are a direct factor of a $k$-rational variety without being stably $k$-rational: see [CTS77, Proposition 20 C p. 223]. Indeed, one can construct a pair of tori $T$ and $T'$ over $\mathbb{Q}$ such that $T \times T'$ is $k$-rational while neither $T$ nor $T'$ is $k$-rational. Taking smooth proper models of these tori provides the required example.

Remark 2.26. The birational geometry of nearly $k$-rational varieties is an incredibly rich subject (see, e.g., [CT92] or [Kol96]). Chevalley and Manin introduced and studied a class of varieties they called special (see [Man86, §14]) that form a slightly more general class than our smooth schemes covered by affine spaces. If $X$ is a smooth proper $k$-variety over a field $k$ having characteristic 0, then $\mathbb{A}^1$-chain connectedness of $X$ is equivalent to the notion of $R$-triviality of the base change of $X$ along all finitely generated separable extensions of $k$. As was pointed out to us by Colliot-Thélène, it is an open problem whether $\mathbb{A}^1$-chain connectedness in the sense of Definition 2.9 is equivalent to retract $k$-rationality.

Cohomological properties of smooth $\mathbb{A}^1$-connected schemes

Recall that a $k$-variety $X$ is (separably) $k$-unirational if $k(X)$ is a subfield of a purely transcendental extension of $k$ (separable over $k(X)$), i.e., if there exists a dominant rational map $\mathbb{P}^n \dasharrow X$. Each of the conditions of Definition 2.18 implies $k$-unirationality. Campana, Kollár, Miyaoka, and Mori introduced a slightly weaker notion of near rationality, implied by $k$-unirationality, they called rational connectedness (see [Kol96 IV.3]); we will investigate this notion in more detail shortly. For algebraically closed fields having characteristic 0, it was initially hoped that $\mathbb{A}^1$-connectedness in the sense studied above would be equivalent to rational connectedness. However, we will see here that there exist $k$-unirational varieties that are not $\mathbb{A}^1$-connected.

As pointed out to the second author by B. Bhatt, if $k$ is an algebraically closed field having characteristic exponent $p$, and $X$ is an $\mathbb{A}^1$-connected scheme over $k$, the cohomological Brauer group of $X$ is a $p$-group (see Proposition 2.28 for a proof). Even over $\mathbb{C}$, Artin and Mumford (cf. [Man86 Appendix 4.1]) constructed examples of conic bundles over 2-dimensional rational surfaces that are unirational but have non-trivial cohomological Brauer group. Nevertheless, we will prove in the next section that (separably) rationally connected varieties over a perfect field $k$ are almost $\mathbb{A}^1$-connected in a sense we will make precise.
We discuss here some cohomological properties of smooth $A^1$-connected schemes; we defer the proofs of these results to §4. The following result shows, in particular, that for algebraically closed fields having characteristic 0, the étale fundamental group of a smooth $A^1$-connected scheme is trivial (see Proposition 4.18 and the subsequent discussion for a proof of a more general result). Let us emphasize that neither of the next two results require properness assumptions.

**Proposition 2.27** (cf. [Mor06a] Remark 3.9). Suppose $k$ is a separably closed field having characteristic exponent $p$. If $X \in Sm_k$ is $A^1$-connected, then $X$ admits no non-trivial finite étale Galois covers of order coprime to $p$.

The following result was communicated to us by Bhargav Bhatt. We provide a slightly different proof of a more general result than the one he suggested (see Proposition 4.21); our proof is very similar to the proof of Proposition 2.27.

**Proposition 2.28** (B. Bhatt (private communication)). Let $k$ be a separably closed field having characteristic exponent $p$. If $X \in Sm_k$ is $A^1$-connected, and $x \in X(k)$ is a base-point, then the Brauer group $Br(X)$ is $p$-torsion.

**Example 2.29.** K3 surfaces over a field $k$ are $A^1$-disconnected because they have non-trivial cohomological Brauer group. Suppose $k$ is an algebraically closed field of characteristic exponent $p$, $\ell$ is a prime number not equal to $p$, and $X$ is a smooth proper variety over $k$. One can show (see [Gro68, Theorem 3.1 p. 80]) that the $\ell$-torsion subgroup of $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is isomorphic to $(\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{b_2-\rho} \oplus M$ where $b_2$ is the second $\ell$-adic Betti number of $X$, $\rho$ is the rank of the Néron-Severi group of $X$, and $M$ is a finite $\ell$-group.

**Extension 2.30.** There are various generalizations of Propositions 2.27 and 2.28 that we do not consider here. We will see that the techniques used in the proofs of these results are quite robust and one can show that various “higher unramified invariants” (see, e.g., [CT95, §4.1] and [CTO89]) vanish for smooth $A^1$-connected schemes.

**Rational connectivity and étale $A^1$-connectedness**

Our goal in this section is to link the “étale version” of $A^1$-connectedness to separable rational connectivity. Recall (cf. [Kol96, IV.3.2]) that a smooth $k$-variety $X$ is called separably rationally connected if there is a $k$-variety $Y$ and a morphism $u : U = Y \times \mathbb{P}^1 \to X$ such that the map $u^{(2)} : U \times_Y U \to X \times X$ is dominant and smooth at the generic point.

Let $Sp_{\text{ét}}^k := \Delta^2Shv_{\text{ét}}(Sm_k)$ be the category of étale simplicial sheaves of sets on $Sm_k$; we refer to objects of this category as étale $k$-spaces. (Note: this notion has nothing to do with the espace étale of a sheaf.) The Yoneda embedding provides a fully-faithful functor $Sm_k \to Sp_{\text{ét}}^k$ by which we identify smooth schemes with their corresponding étale $k$-spaces.

As in [MV99, §2 Definition 1.2], we equip this category with the Joyal-Jardine (injective local) model structure (i.e., cofibrations are monomorphisms, and weak equivalences are defined stalkwise). We write $\mathcal{H}_{\text{ét}}(k)$ for the homotopy category of this model category (see [MV99, §2 Theorem 1.4]), and we refer to $\mathcal{H}_{\text{ét}}(k)$ as the étale simplicial homotopy category. An étale
\(A^1\)-local space is an étale \(k\)-space that is \(A^1\)-local in the sense of [MV99] §2 Definition 2.1. We can define the notion of étale \(A^1\)-weak equivalence using ibid. Definition 2.2. We can localize \(\mathcal{H}^\text{ét}_k\) at the class of \(A^1\)-weak equivalences to obtain the étale \(A^1\)-homotopy category of smooth \(k\)-schemes; we denote this category by \(\mathcal{H}^\text{ét}_k\) (see [MV99] §2 Theorem 3.2). One naturally makes the following definition.

**Definition 2.31.** For an object \(X \in \mathcal{S}_{pc}^{\text{ét}}\), let \(\pi^\text{ét}_{0}(X)\) denote the étale sheaf associated with the presheaf 
\[
U \mapsto \text{Hom}_{\mathcal{H}^\text{ét}_k}(U, X)
\]
for \(U \in \mathcal{S}_{m}^{k}\). We refer to this object as the étale sheaf of \(A^1\)-connected components. An étale \(k\)-space \(X\) will be called étale \(A^1\)-connected if the canonical map \(\pi^\text{ét}_{0}(X) \to \text{Spec} \, k\) induced by the structure morphism \(X \to \text{Spec} \, k\) is an isomorphism of étale sheaves and étale \(A^1\)-disconnected otherwise.

**Example 2.32.** Suppose \(k\) is a field and \(L/k\) is a non-trivial finite separable extension. We can consider \(\text{Spec} \, L\) as a smooth \(k\)-scheme. For every \(U \in \mathcal{S}_{m}^{k}\), we know that \(\text{Hom}_{\mathcal{S}_{m}^{k}}(U, \text{Spec} \, L) \to \text{Hom}_{\mathcal{S}_{m}^{k}}(U \times \mathbb{A}^1, \text{Spec} \, L)\) is a bijection. Using [MV99] §2 Remark 2.14, we conclude that \(\text{Spec} \, L\) is in fact étale \(A^1\)-local and simplicially fibrant. Thus, \(\pi^\text{ét}_{0}(\text{Spec} \, L) = \text{Spec} \, L\), and \(\text{Spec} \, L\) is not étale \(A^1\)-connected.

In many ways, the proof of the next result is simpler (and more general) than the proof of the corresponding result (i.e., Theorem 2.21) in the Nisnevich topology.

**Theorem 2.33.** Suppose \(k\) is a perfect field. If \(X\) is a separably rationally connected smooth proper \(k\)-variety, then \(X\) is étale \(A^1\)-connected.

**Proof.** The unstable 0-\(A^1\)-connectivity theorem ([MV99] §2 Corollary 3.22]) for étale \(k\)-spaces states if \(X \in \mathcal{S}_{pc}^{\text{ét}}\), then the canonical map
\[
\mathcal{X}_0 \longrightarrow \pi^\text{ét}_{0}(X)
\]
is an epimorphism. For \(X \in \mathcal{S}_{m}^{k}\), consider the étale version of the Suslin-Voevodsky singular construction \(\text{Sing}^\text{ét}_{k}(X)\) (see [MV99] p. 88). Let \(\pi^\text{ét, ch}_{0}(X)\) denote the étale sheaf associated with the presheaf 
\[
U \mapsto \text{Hom}_{\mathcal{H}^\text{ét}_k}(U, \text{Sing}^\text{ét}_{k}(X)).
\]
One might call this the sheaf of étale \(A^1\)-chain connected components. The canonical map \(X \to \text{Sing}^\text{ét}_{k}(X)\) is an étale \(A^1\)-weak equivalence. Using this fact, we deduce that the canonical map 
\[
\pi^\text{ét, ch}_{0}(X) \longrightarrow \pi^\text{ét}_{0}(X)
\]
is an epimorphism of étale sheaves, or equivalently, it is an epimorphism on stalks. Now, the “local rings” in the étale topology on \(\mathcal{S}_{m}^{k}\) are strictly Henselian local \(k\)-schemes. More
precisely, to prove our result, it suffices to establish that if $S$ is a strictly Henselian local $k$-scheme, the stalk $\pi_0^{\text{ét, ch}}(X)(S)$ is trivial. This latter set can be identified with $X(S)/\sim$ (see Notation 2.8).

Now, note that separably closed extensions $L$ of $k$ are strictly Henselian local $k$-schemes. Thus, if the sheaf $\pi_0^{\text{ét, ch}}(X)$ is trivial, this means the sets $X(L)/\sim$ are trivial. We now claim that the converse to this statement holds, i.e., that if $X(L)/\sim$ is trivial for any separably closed extension $L/k$, then $\pi_0^{\text{ét, ch}}(X)$ is itself trivial. The proof of this result is identical to the proof of [Mor04a, Lemma 3.3.6] once one observes that the homotopy purity theorem [MV99, §3 Theorem 3.23] holds in the étale $\mathbb{A}^1$-homotopy category (also by the identical proof).

Finally, again using the fact that $k$ is perfect, we apply [Ko96] Theorem IV.3.9 that can be interpreted as saying if $X$ is a separably rationally connected smooth proper $k$-variety, then over any separably closed field extension $L/k$, the set $X(L)/\sim$ is trivial. \hfill $\square$

**Conjecture 2.34.** Let $k$ be a perfect field and suppose $X$ is a smooth proper $k$-variety. The variety $X$ is separably rationally connected if and only if it is étale $\mathbb{A}^1$-connected.

**Remark 2.35.** Suppose $X \in Sm_k$. We say that $X$ is étale $\mathbb{A}^1$-chain connected if for every separably closed field extension $L/k$, the set of $\mathbb{A}^1$-equivalence classes of $L$-points $X(L)/\sim = \ast$. The proof of Theorem 2.33 shows more generally that $X$ is étale $\mathbb{A}^1$-connected if it is étale $\mathbb{A}^1$-chain connected.

### Comparison of $\mathbb{A}^1$- and étale $\mathbb{A}^1$-connectedness

Starting with a general $\mathcal{X} \in Spc_k^{\text{ét}}$, while comparison of chain connectivity properties is relatively straightforward, providing an appropriate comparison of $\mathbb{A}^1$-connectedness and étale $\mathbb{A}^1$-connectedness is more technically involved.

Let $\alpha : (Sm_k)_{\text{ét}} \to (Sm_k)_{\text{Nis}}$ denote the comparison of sites map. By the discussion of [MV99, p. 62], there is a derived push-forward functor $R\alpha_* : \mathcal{H}^{\text{ét}}_*(k) \to \mathcal{H}^*_{\text{Nis}}(k)$. If $\mathcal{X}$ is an étale $k$-space, let $R\alpha_*\mathcal{X}$ denote the corresponding Nisnevich space. Abusing notation, we will also denote by $\mathcal{X}$ the object of $Spc_k$ obtained by applying the restriction functor $Spc_k^{\text{ét}} \to Spc_k$. With this notation, observe that the unit of adjunction provides a canonical morphism $\mathcal{X} \to R\alpha_*\mathcal{X}$ in $Spc_k$.

Since $R\alpha_*$ sends étale $\mathbb{A}^1$-local objects to $\mathbb{A}^1$-local objects [MV99, §2 Lemma 3.15], adjointness of pullback and pushforward provides for any $U \in Sm_k$ a canonical bijection:

$$Hom_{\mathcal{H}^{\text{ét}}_*(k)}(U, \mathcal{X}) \xrightarrow{\sim} [U, R\alpha_*\mathcal{X}]_{\mathbb{A}^1}.$$ 

Thus, for any $U \in Sm_k$, we obtain a morphism:

$$[U, \mathcal{X}]_{\mathbb{A}^1} \longrightarrow [U, R\alpha_*\mathcal{X}]_{\mathbb{A}^1}.$$ 

Write $a_0^{\text{ét}}\pi_0^{\mathbb{A}^1}(\mathcal{X})$ for the étale sheafification of the presheaf on the left hand side. Sheafifying both sides for the étale topology, we obtain a morphism

$$a_0^{\text{ét}}\pi_0^{\mathbb{A}^1}(\mathcal{X}) \longrightarrow \pi_0^{\mathbb{A}^1, \text{ét}}(\mathcal{X})$$

that allows us to compare $\mathbb{A}^1$-connectedness and étale $\mathbb{A}^1$-connectedness.
Lemma 2.36. The morphism $a_{\text{et}}\pi_0^{A^1}(\mathcal{X}) \to \pi_0^{A^1,\text{et}}(\mathcal{X})$ of Equation 2.1 is an epimorphism of étale sheaves. Thus, if the space underlying an object $\mathcal{X} \in \text{Spc}_k$ is $A^1$-connected it is also étale $A^1$-connected.

Proof. We have to check that the morphism in question is an epimorphism on stalks. To see this we again use the unstable 0-$A^1$-connectivity theorem [MV99, §2 Corollary 3.22] again. If $S$ is a strictly Henselian local scheme, the canonical epimorphism $X(S) \to \pi_0^{A^1,\text{et}}(\mathcal{X})(S)$ factors through the epimorphism $X(S) \to \pi_0^{A^1}(\mathcal{X})(S) = a_{\text{et}}\pi_0^{A^1}(\mathcal{X})(S)$.

Example 2.37. Suppose $a_1, \ldots, a_{2m} \in \mathbb{R}$, and let $S$ be any smooth compactification of the affine hypersurface in $A^3$ defined by the equation $x^2 + y^2 = -2m \prod_{i=1}^{m} (z - a_i)$.

Projection onto $z$ determines a morphism from this hypersurface to $\mathbb{P}^1$ with conic fibers. The compactified surface is birationally ruled over $\mathbb{P}^1$ and therefore rational over $\mathbb{C}$. One can show that the space $S(\mathbb{R})$ has $m$ connected components, and the set of $A^1$-equivalences classes of $\mathbb{R}$-points $S(\mathbb{R})/\sim$ coincides with the set $\pi_0(S(\mathbb{R}))$ (cf. [Kol1 Corollary 3.4 and Theorem 4.6]). Using the topological realization functor ([MV99, §3 Lemma 3.6]) one can show that such $S$ are $A^1$-disconnected. Thus, $\pi_0^{A^1}(S)$ is not necessarily a subsheaf of a point even if $\pi_0^{A^1,\text{et}}(S)$ is a point.

Comparing $A^1$-connectedness and $A^1$-chain connectedness

Suppose $X$ is an arbitrary scheme having finite type over a field $k$. Since the Nisnevich topology is subcanonical, it follows that the functor of points of $X$ is a Nisnevich sheaf. To provide evidence for Conjecture 2.13, we give the following result whose proof we defer to Section 6.

Theorem 2.38. Suppose $X$ is a proper scheme having finite type over a field $k$. The canonical epimorphism of Remark 2.11 induces for every finitely generated separable extension $L/k$ a bijection:

$$\pi_0^{ch}(X)(L) \longrightarrow \pi_0^{A^1}(X)(L).$$

Note we abuse notation and write $X$ for both a scheme (possibly singular) and the space determined by its functor of points.

Corollary 2.39. If $k$ is a field, and $X \in \text{Sm}_k$ is proper over $k$, then $X$ is $A^1$-connected if and only if $X$ is $A^1$-chain connected.

Let us record some extremely useful consequences of this result. The first corollary follows immediately by combining Theorem 2.38 with Theorem 2.21. The second corollary follows from the fact that $A^1$-chain connected varieties are separably rationally connected for fields having characteristic 0.
Corollary 2.40. Suppose $k$ is a field having characteristic 0. If $X$ and $X'$ are two $k$-birationally equivalent smooth proper varieties, then $X$ is $\mathbb{A}^1$-connected if and only if $X'$ is $\mathbb{A}^1$-connected.

Corollary 2.41 (cf. [Kol96] Exercise IV.3.3.5). Suppose $k$ is an algebraically closed field of characteristic 0. A smooth proper $k$-variety of dimension $\leq 2$ is $\mathbb{A}^1$-connected if and only if it is rational.

Remark 2.42. Suppose $k$ is a field and $X \in S_{mk}$. Recall that two $k$-points in $X$ are called directly $R$-equivalent if there exists a morphism from an open subscheme of $\mathbb{P}^1$ to $X$ whose image contains the given points. We write $X(k)/R$ for the quotient of $X(k)$ by the equivalence relation generated by $R$-equivalence. We might say that $X$ is separably $R$-trivial if for every finitely generated separable extension field $L$ of $k$, $X(L)/R = \ast$.

If $X$ is also proper, then the equivalence relations given by $R$-equivalence and $\mathbb{A}^1$-equivalence of points coincide. Corollary 2.39 then implies, e.g., that $X$ is $\mathbb{A}^1$-connected if and only if it is separably $R$-trivial.

Corollary 2.43. Assume $k$ is a field having characteristic 0. Suppose $X \in S_{mk}$ and suppose $j : X \hookrightarrow \tilde{X}$ is an open immersion into a smooth proper variety. For any finitely generated separable extension $L$ of $k$, the image of the map $\pi_{A^1}^X(L) \to \pi_{A^1}^{\tilde{X}}(L) = \tilde{X}(L)/\sim$ coincides with $X(F)/R$. In particular for any $X \in S_{mk}$, the map $X(L) \to X(L)/R$ factors through the surjective map $X(L) \to \pi_{A^1}^X(L)$.

Remark 2.44. A version of this result should be true for fields having arbitrary characteristic. With more work, one can use a compactification of $X$ to establish the result in that case.

3 $A^1$-cobordisms and rational smooth proper surfaces

In this section we study the notion of $A^1$-cobordism of smooth schemes mentioned in §1. Using this notion, Theorem 3.8 provides the $A^1$-homotopy classification for rational smooth proper surfaces; the proof is essentially elementary. Along the way, we prove general results about the $A^1$-homotopy types of iterated blow-ups of points on smooth proper (étale) $A^1$-connected varieties (see Lemma 3.10) and classify the total spaces of $\mathbb{P}^n$-bundles over $\mathbb{P}^1$ up to $A^1$-weak equivalence (see Proposition 3.21).

Suppose $X$ is a smooth $k$-scheme. Specifying a regular function $f \in \Gamma(X, O_X)$ is equivalent to specifying a morphism $f : X \to \mathbb{A}^1$. Note that $\mathbb{A}^1(k)$ has two canonical elements that we denote by 0 and 1. In the remainder of this section, we will write $f^{-1}(0)$ and $f^{-1}(1)$, or just $X_0$ and $X_1$ assuming $f$ is understood, for the scheme-theoretic fibers over the points 0 and 1. We will say that a closed point $x \in \mathbb{A}^1$ is a regular value if the scheme theoretic fiber $f^{-1}(x)$ is a smooth scheme, otherwise $x$ will be called a critical value of $f$. We begin by defining $A^1$-cobordisms and studying their general properties.
Basic definitions and general properties

Definition 3.1. Suppose $X \in Sm_k$, and $f : X \to \mathbb{A}^1$ is a proper, surjective morphism. We will say that $f$ (or the pair $(X,f)$) is an $\mathbb{A}^1$-cobordism if its direct image $F_k$ is a $\mathbb{A}^1$-weak equivalence. Given a $\mathbb{A}^1$-cobordism $f : X \to \mathbb{A}^1$, we will say that $X_0$ and $X_1$ are directly $\mathbb{A}^1$-cobordant. More generally, we will say that two varieties $X$ and $Y$ are $\mathbb{A}^1$-cobordant if they are in the same equivalence class for the equivalence relation generated by direct $\mathbb{A}^1$-cobordance.

Remark 3.2. Two varieties $X$ and $Y$ that are $\mathbb{A}^1$-cobordant are algebraically cobordant in the sense that they give rise to the same class in the algebraic cobordism ring $\Omega^*(k)$ [LM07, Remark 2.4.8, and Definition 2.4.10]; this observation justifies our choice of terminology.

Remark 3.3. Suppose $W$ is an $h$-cobordism between smooth manifolds $M$ and $M'$. In classical topology, one studies $W$ by means of handle decompositions. By choosing a Morse function $f : W \to \mathbb{R}$, one can decompose $f$ into elements pieces corresponding to the critical points of $f$. The handle decomposition theorem shows that an $h$-cobordism admitting a Morse function with no critical values is necessarily trivial. On the contrary, we will see in Example 3.20 that in algebraic geometry there exist non-trivial $\mathbb{A}^1$-cobordisms $(W,f)$ where $f$ is a smooth morphism and thus has no critical values! In fact, all known $\mathbb{A}^1$-cobordisms are of this form.

$\mathbb{A}^1$-cobordant bundles

Proposition 3.4. Suppose $X,Y \in Sm_k$ with $Y$ proper. Assume $g : Z \to X \times \mathbb{A}^1$ is a Nisnevich locally trivial smooth surjective morphism with fibers isomorphic to $Y$. The morphism $f : Z \to \mathbb{A}^1$ induced by composing the morphism $g$ with the projection $X \times \mathbb{A}^1 \to \mathbb{A}^1$ is an $\mathbb{A}^1$-cobordism.

Proof. Since we can apply an automorphism of $\mathbb{A}^1$ that exchanges the fibers over 0 and 1, it suffices to check that the inclusion morphism $Z_0 \to Z$ is an $\mathbb{A}^1$-weak equivalence. To do this, choose an open cover $U_i$ of $X \times \mathbb{A}^1$ over which $g$ trivializes and fix a trivialization. Let $u : U = \coprod U_i \to X \times \mathbb{A}^1$ denote the covering morphism. Our choice of trivialization determines an isomorphism $Y \times U \cong Z \times X \times \mathbb{A}^1 U$. Also, the pull-back of the morphism $g : z \to X \times \mathbb{A}^1$ by $u$ coincides via this isomorphism with the projection morphism $Y \times U \to U$.

Consider the closed immersion $X \hookrightarrow X \times \mathbb{A}^1$ induced by inclusion at 0. The fiber product $X \times X \times \mathbb{A}^1 U$ gives a Nisnevich cover of $X$ that we will call $U_0$; we also denote by $u_0 : U_0 \to X$ the induced covering morphism. One can check as above, that the morphism $Z_0 \to X$ coincides via this isomorphism with the projection $Y \times U_0 \to U_0$. 


Consider now the Čech simplicial scheme $\tilde{C}(u)$ whose $n$-th term is the $(n + 1)$-fold fiber product of $U$ with itself over $X \times \mathbb{A}^1$. By [MV99 §2 Lemma 1.15], the augmentation map $\tilde{C}(u) \to X \times \mathbb{A}^1$ is a simplicial weak equivalence, and thus also an $\mathbb{A}^1$-weak equivalence. Using the chosen trivialization of $g$ along $u$, one constructs an isomorphism from the Čech simplicial scheme associated with the Nisnevich covering map $\mathbb{Z} \times X \times \mathbb{A}^1 U \to \mathbb{Z}$ to the product $Y \times \tilde{C}(u)$; for the same reason this map is an $\mathbb{A}^1$-weak equivalence. Similarly, one checks that the map $\tilde{C}(u_0) \to X$ is an $\mathbb{A}^1$-weak equivalence and, by restriction, one constructs an isomorphism from the Čech simplicial scheme associated with the covering morphism $Z_0 \times X \to Z_0$ to the product $Y \times \tilde{C}(u_0)$.

The construction above provides a Cartesian square of the form
\[
\begin{array}{ccc}
Y \times \tilde{C}(u_0) & \longrightarrow & Y \times \tilde{C}(u) \\
\downarrow & & \downarrow \\
\tilde{C}(u_0) & \longrightarrow & \tilde{C}(u)
\end{array}
\]

If the inclusion morphism $\tilde{C}(u_0) \hookrightarrow \tilde{C}(u)$ is an $\mathbb{A}^1$-weak equivalence, it follows by [MV99 §2 Lemma 2.15] that the product map $Y \times \tilde{C}(u_0) \to Y \times \tilde{C}(u)$ is also an $\mathbb{A}^1$-weak equivalence. Since the map $X \hookrightarrow X \times \mathbb{A}^1$ is an $\mathbb{A}^1$-weak equivalence, the results of the previous paragraph allow us to conclude that $\tilde{C}(u_0) \hookrightarrow \tilde{C}(u)$ is also an $\mathbb{A}^1$-weak equivalence.

Remark 3.5. Note that the proof of the above result never uses properness of $Y$. Non-trivial $\mathbb{A}^1$-$h$-cobordisms produced by this method will be described in Example 3.20.

Blowing up a moving subvariety

**Proposition 3.6.** Let $X \in Sm_k$, and assume $X$ is also proper. Let $Z \subset X$ be a smooth closed subscheme, and assume we have a morphism $i : Z \times \mathbb{A}^1 \to X$. Suppose further that the morphism $i \times p_{\mathbb{A}^1} : Z \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ is a regular closed embedding and let $\Gamma \subset X$ denote the image of $i \times p_{\mathbb{A}^1}$. The projection $X \times \mathbb{A}^1 \to \mathbb{A}^1$ induces a morphism
\[
f : \text{Bl}_\Gamma(X \times \mathbb{A}^1) \to \mathbb{A}^1.
\]
The morphism $f : \text{Bl}_\Gamma(X \times \mathbb{A}^1) \to \mathbb{A}^1$ is an $\mathbb{A}^1$-$h$-cobordism.

**Proof.** Consider the composite $\Gamma \hookrightarrow X \times \mathbb{A}^1 \to \mathbb{A}^1$, and let $\Gamma_t$ denote the fiber of this morphism over $t \in \mathbb{A}^1(k)$. We let $\mathbb{G}_m$ act on $X \times \mathbb{A}^1$ by letting $\mathbb{G}_m$ act trivially on $X$ and (abusing terminology) by the identity character on $\mathbb{A}^1$. The fixed-point locus of this $\mathbb{G}_m$-action is just $X \times \{0\}$, which we identify with $X$. Note that $\Gamma$ is a $\mathbb{G}_m$-stable subvariety of $X \times \mathbb{A}^1$, and thus $\text{Bl}_\Gamma(X \times \mathbb{A}^1)$ is equipped with a $\mathbb{G}_m$-action.

We will check that the inclusion of the fiber over 0 in $\text{Bl}_\Gamma(X \times \mathbb{A}^1)$ is an $\mathbb{A}^1$-weak equivalence. The fiber over 0 can be identified with $\text{Bl}_{\Gamma_0}(X)$. The fixed point locus of a $\mathbb{G}_m$-action on a smooth scheme is always smooth by [Ive72 Proposition 1.3]. The inclusion $\text{Bl}_{\Gamma_0}(X) \hookrightarrow \text{Bl}_\Gamma(X \times \mathbb{A}^1)^{\mathbb{G}_m}$ is an isomorphism. (To check this, we can use the fact that formation of the
fixed point scheme commutes with base extension to reduce to the case where $X$ is affine. In this case, one can use Luna’s slice theorem, which works over fields having arbitrary characteristic for linearly reductive groups, to reduce to the case of an inclusion of affine spaces where it follows by explicit computation.)

We now construct a morphism $\text{Bl}_{\Gamma}(X \times \mathbb{A}^1) \to \text{Bl}_{\Gamma}(X \times \mathbb{A}^1)^{\mathbb{G}_m}$ that is Zariski locally trivial with affine space fibers providing an explicit retraction of the inclusion of the previous paragraph. Use the discussion of Hesselink’s concentrator scheme [Hes81, §4.1 and Theorem 4.5] to show that such a morphism exists: at the level of $k$-points, one sends $x \in X(k)$ to its limit under the $\mathbb{G}_m$-action, which exists by construction. Then, use ibid. Theorem 5.8 to show that the morphism so constructed is Zariski locally trivial with affine space fibers.

To finish, we can apply an automorphism of $\mathbb{A}^1$ that exchanges 0 and 1. Such an automorphism induces a $\mathbb{G}_m$-action on $\mathbb{A}^1$ with 1 as its unique fixed point. The same argument as above can be used to provide a retraction of the inclusion of the blow-up of $X$ at $\Gamma_1$ into $\text{Bl}_{\Gamma}(X \times \mathbb{A}^1)$.

Remark 3.7. Again, the proof of this result does not use properness of $X$. Non-trivial $\mathbb{A}^1$-cobordisms produced by this method will be described in the next section.

$\mathbb{A}^1$-homotopy classification of rational smooth proper surfaces

The isomorphism classification of rational smooth proper surfaces over an algebraically closed field is well known (for proofs see, e.g., [Bea96]). Using the strong factorization theorem for surfaces ([Bea96 Theorem II.11]), one can show that any rational smooth proper surface is isomorphic to an iterated blow-up of some finite collection of points on either $\mathbb{P}^2$ or on $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a))$.

Recall that $\mathbb{F}_a$ has a curve, denoted here $C_a$, corresponding to the inclusion $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)) \hookrightarrow \mathbb{F}_a$ that has self-intersection number $-a$. As is well-known, the variety $\mathbb{F}_a$ blown up at a point $x \in C_a(k)$ is isomorphic to $\mathbb{F}_{a-1}$ blown up at a point $x' \in \mathbb{F}_{a-1} \setminus C_{a-1}(k)$. This construction provides the standard example of non-uniqueness of minimal models for ruled surfaces.

**Theorem 3.8.** Any rational smooth proper surface over an algebraically closed field is $\mathbb{A}^1$-weakly equivalent to either $\mathbb{P}^1 \times \mathbb{P}^1$, or a blow-up of some fixed (possibly empty) finite collection of points on $\mathbb{P}^2$.

**Proof.** We deduce this result immediately from Lemmas 3.10, 3.11, and 3.12 below.

**Remark 3.9.** Since separably rationally connected surfaces over an algebraically closed field $k$ are all rational, Theorem 3.8 conjecturally provides (see Conjectures 2.13 and 2.34) a complete classification of all $\mathbb{A}^1$-connected or étale $\mathbb{A}^1$-connected surfaces over such fields. By Corollary 2.41, the classification of $\mathbb{A}^1$-connected smooth proper surfaces is established if $k$ has characteristic 0. Our proof will also show that any pair of $\mathbb{A}^1$-weakly equivalent smooth proper surfaces are in fact $\mathbb{A}^1$-cobordant by a series of $\mathbb{A}^1$-cobordisms without critical points.
Lemma 3.10. Suppose $X$ is a smooth proper separably rationally connected variety over an algebraically closed field $k$. Suppose $Y_1$ and $Y_2$ are each of the form
\[ \text{Bl}_{y_1}(\text{Bl}_{y_2}(\cdots(\text{Bl}_{y_n}(X)))) \]
for specified collections of $k$-points $y_1, \ldots, y_n$ and $y'_1, \ldots, y'_m$. Then $Y_1$ and $Y_2$ are $\mathbb{A}^1$-weakly equivalent if and only if $n = m$.

Lemma 3.11. Two Hirzebruch surfaces $F_a$ and $F_b$ are $\mathbb{A}^1$-weakly equivalent if and only if $a$ and $b$ are congruent mod 2.

Lemma 3.12. Let $k$ be an algebraically closed field. For any integer $n > 1$, and arbitrary collections of $k$-points $x_1, \ldots, x_n, y_1, \ldots, y_n$, the iterated blow-ups $\text{Bl}_{x_1}(\cdots(\text{Bl}_{x_n}(F_a))\cdots)$ and $\text{Bl}_{y_1}(\cdots(\text{Bl}_{y_n}(F_{a-1}))\cdots)$ are $\mathbb{A}^1$-h-cobordant.

Proof. Using Lemma 3.10, we can always assume that $x_n$ lies on $C_a$. Apply the observation above about non-minimality of Hirzebruch surfaces to identify this iterated blow-up with a corresponding one with $F_a$ replaced by $F_{a-1}$. Another application of Lemma 3.10 then allows one to construct an $\mathbb{A}^1$-h-cobordism from this new iterated blow-up to $\text{Bl}_{y_1}(\cdots(\text{Bl}_{y_n}(F_{a-1}))\cdots))$.

Rational connectivity and the proof of Lemma 3.10

Recall Notation 2.8

Proposition 3.13. Assume $k$ is a perfect field, and suppose $X \in Sm_k$ is also a proper scheme. Assume the set of $\mathbb{A}^1$-equivalence classes of $k$-points $X(k)/\sim$ consists of exactly 1 element. Suppose $f_1 : X_1 \to X$ and $f_2 : X_2 \to X$ are proper birational morphisms that are composites of blow-ups of $k$-points. The schemes $X_1$ and $X_2$ are $\mathbb{A}^1$-weakly equivalent if and only if $\text{rk } \text{Pic}(X_1) = \text{rk } \text{Pic}(X_2)$.

Proof. Note that by Proposition 2.23 if $X(k)/\sim$ consists of exactly 1 element, then the same is true for $X_1(k)/\sim$ and $X_2(k)/\sim$ by our assumptions.

Step 1. Let us first prove that if $x_1$ and $x_2$ are distinct $k$-points on $X$, then $\text{Bl}_{x_1}X$ and $\text{Bl}_{x_2}X$ are $\mathbb{A}^1$-h-cobordant. Indeed, we can assume that $\dim X \geq 2$ as otherwise the blow-up of a point is trivial. Since $X(k)/\sim$ consists of exactly 1 element, we can always choose a chain of subschemes isomorphic to $\mathbb{A}^1$ connecting $x_1$ and $x_2$, so it suffices to prove the statement assuming $x_1$ and $x_2$ both lie on a single copy of $\mathbb{A}^1$. Thus assume we have a smooth rational curve $\mathbb{A}^1 \to X$ containing the images of $x_1$ and $x_2$ and apply Proposition 3.6.

Step 2. There are open subvarieties $U_x \subset X$ and $U_y \subset Y$ over which the iterated blow-ups
\[ \text{Bl}_{x_1}(\text{Bl}_{x_2}(\cdots(\text{Bl}_{x_n}(X))\cdots))) \to X \]
and
\[ \text{Bl}_{y_1} \left( \cdots \left( \text{Bl}_{y_n} \left( X \right) \right) \cdots \right) \rightarrow Y. \]
are isomorphisms. Let \( U = U_x \cap U_y \). Choose \( n \) distinct points \( x'_1, \ldots, x'_n \) in \( U(k) \). In order of increasing \( i \), choose a chain of smooth rational curves \( A^1 \rightarrow \text{Bl}_{x_{i+1}} \left( \cdots \left( \text{Bl}_{x_n} \left( X \right) \right) \cdots \right) \) connecting \( x_i \) to \( x'_i \). By the result of Step 1, these curves can be used to show that \( \text{Bl}_{x_i} \left( \cdots \left( \text{Bl}_{x_{i+1}} \left( \cdots \left( \text{Bl}_{x_n} \left( X \right) \right) \cdots \right) \right) \right) \) and \( \text{Bl}_{x'_i} \left( \cdots \left( \text{Bl}_{x_{i+1}} \left( \cdots \left( \text{Bl}_{x_n} \left( X \right) \right) \cdots \right) \right) \right) \) are \( A^1 \)-cobordant. Since \( x'_i \) are distinct points lying off the exceptional divisors of all of these blow-ups, the blow-ups commute. Doing the same thing for \( \text{Bl}_{y_1} \left( \cdots \left( \text{Bl}_{y_n} \left( X \right) \right) \cdots \right) \) then produces the required family of \( A^1 \)-cobordisms. Thus, if the numbers of points agree, then the resulting varieties are \( A^1 \)-cobordant.

**Step 3.** To finish, let us note that the number of points being blown up is an \( A^1 \)-homotopy invariant. Indeed, the Picard group of a smooth \( k \)-scheme, and thus its rank, is an \( A^1 \)-homotopy invariant by [MV99, §4 Proposition 3.8] and it is well known that blowing up a point on a smooth variety increases the rank of the Picard group. The result then follows by induction on the number of points.

Recall the following result about separably rationally connected varieties (cf. [Kol96 IV.3]).

**Theorem 3.14 ([Kol96 IV.3.3.3, IV.3.9]).** Let \( X \) be a separably rationally connected smooth proper variety over an algebraically closed field \( k \).

i) If \( X' \) is a smooth proper \( k \)-variety \( k \)-birational to \( X \), then \( X' \) is separably rationally connected.

ii) Given two points \( x_1, x_2 \in X(k) \), there is a morphism \( f : \mathbb{P}^1 \rightarrow X \) such that \( x_1 \) and \( x_2 \) lie in the image of \( f \). If \( \dim X \geq 2 \), we can assume \( f \) is an immersion, and if \( \dim X \geq 3 \), we can assume that \( f \) is an embedding.

**Proof of Lemma 3.10.** If \( X \) is a separably rationally connected smooth proper variety over an algebraically closed field \( k \), then Theorem 3.14 can be used to check that the hypotheses of Proposition 3.13 are satisfied.

**Remark 3.15.** There is another way to approach the proof of Proposition 3.13. If \( X \) is a separably rationally connected smooth proper \( k \)-variety, then one can show that the Fulton-MacPherson compactification (see [FM94]) of \( n \)-points on \( X \), often denoted \( X[n] \), is also a rationally connected smooth proper variety (as it is birationally equivalent to \( X \times n \)). One can then use the construction of \( X[n] \) in terms of iterated blow-ups to produce appropriate \( A^1 \)-cobordisms between iterated blow-ups of points on \( X \).

**Torsors over \( \mathbb{P}^1 \) and the proof of Lemma 3.11**

We will deduce Lemma 3.11 from a much more general result regarding torsors over \( \mathbb{P}^1 \) that will also be useful in the next section. For the rest of this section, let \( G \) denote a connected split reductive group (by convention reductive groups will be assumed throughout to be affine.
algebraic). For the purposes of our discussion here, a torsor over a $k$-scheme $X$ under a $k$-group scheme $G$ locally trivial in the étale (resp. Zariski, Nisnevich) topology is a triple $(\mathcal{P}, f, G)$ consisting of a scheme $\mathcal{P}$ equipped with a scheme-theoretically free right $G$-action, a faithfully flat morphism $f : \mathcal{P} \rightarrow X$ that is equivariant for the trivial $G$-action on $X$, such that locally in the étale (resp. Zariski, Nisnevich) topology $f$ is isomorphic to a product. For compactness of notation, such objects will just be called étale (resp. Zariski, Nisnevich) locally trivial $G$-torsors over $X$. We will deduce Lemma 3.11 from a much more general result regarding torsors over $\mathbb{P}^1$ that will also be useful in the next section. For the remainder of this section, let $G$ denote a connected split reductive group (by convention reductive groups will be assumed throughout to be affine algebraic). For the purposes of our discussion here, a torsor over a $k$-scheme $X$ under a $k$-group scheme $G$ locally trivial in the étale (resp. Zariski, Nisnevich) topology is a triple $(\mathcal{P}, f, G)$ consisting of a scheme $\mathcal{P}$ equipped with a scheme-theoretically free right $G$-action, a faithfully flat morphism $f : \mathcal{P} \rightarrow X$ that is equivariant for the trivial $G$-action on $X$, such that locally in the étale (resp. Zariski, Nisnevich) topology $f$ is isomorphic to a product. For compactness of notation, such objects will just be called étale (resp. Zariski, Nisnevich) locally trivial $G$-torsors over $X$.

One knows that all $G$-torsors over $\mathbb{P}^1$ are in fact Zariski locally trivial. Suppose $\lambda : \mathbb{G}_m \rightarrow G$ is a cocharacter. Such a cocharacter induces an action of $\mathbb{G}_m$ on the trivial $G$-torsor $\mathbb{A}^2 \setminus 0 \times G$: act via homotheties on $\mathbb{A}^2 \setminus 0$ and via left multiplication by $\lambda^{-1}$ on $G$. We write $\mathcal{P}_\lambda$ for the right $G$-torsor obtained as a quotient of $\mathbb{A}^2 \setminus 0 \times G$ by this action. If we write $X_\ast(G)$ for the (pointed) set of cocharacters of $G$, the assignment $\lambda \mapsto \mathcal{P}_\lambda$ induces a function

$$X_\ast(G) \longrightarrow H^1_{\text{Zar}}(\mathbb{P}^1, G).$$

preserving the distinguished point. In particular, the map $X_\ast(\mathbb{G}_m) \rightarrow \text{Pic}(\mathbb{P}^1)$ so constructed is a bijection (sending the identity cocharacter to $\mathcal{O}_{\mathbb{P}^1}(1)$).

Let $T$ be a fixed maximal torus on $G$ and let $W$ be the Weyl group of $G$ for this choice. There are induced functions $X_\ast(T) \rightarrow X_\ast(G)$ and thus $X_\ast(T) \rightarrow H^1_{\text{Zar}}(X, G)$. Any (isomorphism class) of $G$-torsors in the image of the last map is said to be obtained by extension of structure group from a $T$-torsor. Using this notation, we can state the isomorphism classification of $G$-torsors over $\mathbb{P}^1$.

**Theorem 3.16** (Grothendieck-Harder, Satz 3.1 and 3.4). Suppose $T \subset G$ is a maximal torus. Every $G$-torsor over $\mathbb{P}^1$ is isomorphic to one obtained by extension of structure group from a $T$-torsor associated with an element of $X_\ast(T)$. The $G$-torsors $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda'}$ associated with cocharacters $\lambda, \lambda' \in X_\ast(T)$ are isomorphic if and only if $\lambda = w\lambda'$ for some $w \in W$. In other words, isomorphism classes of $G$-torsors over $\mathbb{P}^1$ are in bijection with elements of the set $X_\ast(T)/W$.

**Definition 3.17.** Suppose $X$ is a smooth $k$-scheme and $G$ is a group scheme. An elementary $\mathbb{A}^1$-equivalence between two $G$-torsors $\mathcal{P}_0$ and $\mathcal{P}_1$ over $X$ is a $G$-torsor $\mathcal{P}$ over $X \times \mathbb{A}^1$ whose restrictions to $X \times \{0\}$ and $X \times \{1\}$ coincide with $\mathcal{P}_0$ and $\mathcal{P}_1$. We will say that two $G$-torsors over $X$ are $\mathbb{A}^1$-equivalent if they are equivalent for the equivalence relation generated by elementary $\mathbb{A}^1$-equivalence.
Remark 3.18. Recall that if $G$ is a connected reductive group, one can define an algebraic fundamental group that is a finitely generated abelian group coinciding, when $k = \mathbb{C}$, with the topological fundamental group of the analytic space $G(\mathbb{C})$. If $G$ is a semi-simple group over an algebraically closed field the fundamental group is dual to the quotient of the lattice of characters of a maximal torus by the weight lattice.

Theorem 3.19 (Ramanathan). Suppose $k$ is an algebraically closed field, and $G$ is a semi-simple linear algebraic group over $k$ with maximal torus $T$. Given two cocharacters $\mu, \mu' : \mathbb{G}_m \to T$, the $G$-torsors $\mathcal{P}_\mu$ and $\mathcal{P}_{\mu'}$ over $\mathbb{P}^1$ are $\mathbb{A}^1$-equivalent if and only if $\mu$ and $\mu'$ have the same image in $X_*(T)/\Phi$, where $\Phi$ is the coroot lattice of $G$. Thus, $\mathbb{A}^1$-equivalence classes of $G$-torsors over $\mathbb{P}^1$ are in bijection with elements of the algebraic fundamental group of $G$.

Proof. This result follows from Theorem 7.7 and the explicit Construction 8.3 of [Ram83]. $\blacksquare$

Example 3.20. We can be very explicit in the case of $\mathrm{PGL}_n$-torsors, or rather the associated $\mathbb{P}^1$-bundles. Any $\mathbb{P}^n$-bundle over $\mathbb{P}^1$ is the projectivization of a rank $(n + 1)$ vector bundle on $\mathbb{P}^1$. Thus, it suffices for us to study rank $n$ vector bundles on $\mathbb{P}^1 \times \mathbb{A}^1$. Cover $\mathbb{P}^1$ by $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$ and $\mathbb{A}^1 = \mathbb{P}^1 \setminus 0$. Since all vector bundles on an affine space are trivial (cf. [Qui76, Theorem 4]), any rank $(n + 1)$ vector bundle on $\mathbb{A}^1 \times \mathbb{A}^1$ is isomorphic to a trivial bundle. Thus, fix a trivialization of such a bundle over $\mathbb{A}^1 \times \mathbb{A}^1$ and $\mathbb{A}^1 \times \mathbb{A}^1$. The intersection of these two open sets is isomorphic to $\mathbb{G}_m \times \mathbb{A}^1$. Thus, isomorphism classes of rank $(n + 1)$ vector bundles on $\mathbb{P}^1 \times \mathbb{A}^1$ are in bijection with elements of $\mathrm{GL}_{n+1}(k[t, t^{-1}, x])$ up to change of trivialization, i.e., left multiplication by elements of $\mathrm{GL}_{n+1}(k[t, t^{-1}, x])$ and right multiplication by elements of $\mathrm{GL}_{n+1}(k[t, x])$; the required cocycle condition is automatically satisfied.

Suppose $a = (a_1, \ldots, a_{n+1})$, and set $\mathcal{E}(a) = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n+1})$. The pull-back of $\mathcal{E}(a)$ to $\mathbb{P}^1 \times \mathbb{A}^1$ has transition function defined by the matrix whose diagonal entries are given by $(t^{-a_1}, \ldots, t^{-a_{n+1}})$. For notational simplicity, consider the rank 2 case, and consider the transition function defined by

$$
\begin{pmatrix}
t^a & xt \\
0 & 1
\end{pmatrix}.
$$

Over $x = 1$, one can show that this transition function defines the bundle $\mathcal{O}(-a + 1) \oplus \mathcal{O}(-1)$. Over $x = 0$, this transition function defines the bundle $\mathcal{O}(-a) \oplus \mathcal{O}$. This family of bundles provides an explicit $\mathbb{A}^1$-$h$-cobordism between $\mathbb{F}_{a-2}$ and $\mathbb{F}_a$.

More generally, set $\mathbb{F}_a = \mathbb{P}(\mathcal{E}(a))$. By permuting the elements of $a$, we can assume its entries are increasing. Let $a'$ be another increasing sequence of $n + 1$ integers. Using explicit cocycles as above, one can construct $\mathbb{A}^1$-$h$-cobordisms between $\mathbb{F}_a$ and $\mathbb{F}_{a'}$ whenever $\sum_i a_i \equiv \sum_i a'_i \mod n + 1$ (cf. [Ram83, §9.4(i) and (iii)]). Note that these $\mathbb{A}^1$-$h$-cobordisms do not have critical values yet are not trivial.

Proposition 3.21. Let $a = (a_1, \ldots, a_{n+1})$ and $a' = (a'_1, \ldots, a'_{n+1})$ be a sequences of integers with $a_1 \leq \cdots \leq a_{n+1}$ (and similarly for the entries of $a'$). The varieties $\mathbb{F}_a$ and $\mathbb{F}_{a'}$ are $\mathbb{A}^1$-weakly equivalent if and only if $\sum_i a_i \equiv \sum_i a'_i \mod n + 1$. 

4 Classifying spaces and strong $\mathbb{A}^1$-invariance

Proof. Proposition 3.4 or Example 3.20 constructs explicit $\mathbb{A}^1$-$h$-cobordisms between $\mathbb{F}_a$ and $\mathbb{F}_{a'}$ whenever $\sum_i a_i \equiv \sum_i a'_i \mod n + 1$. For the only if part of the statement, we need to write down appropriate $\mathbb{A}^1$-homotopy invariants.

Let us observe that the Chow (cohomology) ring $CH^*(\mathbb{F}_a)$ can be explicitly computed as follows. The Chern polynomial of a rank $n + 1$ vector bundle $E$ over $\mathbb{P}^1$ takes the form $\xi^{n+1} + c_1(E)\xi^n$. If $\sigma$ denotes the hyperplane class on $\mathbb{P}^1$, then we can write $c_1(E) = a\sigma$ for some integer $a$. Let $d = \sum_i a_i$. These identifications give an isomorphism of graded rings

$$CH^*(\mathbb{F}_a) \cong \mathbb{Z}[\sigma, \xi]/\langle \sigma^2, \xi^{n+1} + d\xi^n \sigma \rangle,$$

where $\sigma$ and $\xi$ both have degree 2.

For any integer $m > 1$, we have the identities $(\xi + \sigma)^m \sigma = \xi^m \sigma$. The change of variables $\xi' = \xi + \sigma$, shows that the graded rings $\mathbb{Z}[\sigma, \xi]/\langle \sigma^2, \xi^{n+1} + d\xi^n \sigma \rangle$ and $\mathbb{Z}[\sigma, \xi']/\langle \sigma^2, \xi'^{n+1} + (d - n - 1)\xi'^n \sigma \rangle$ are abstractly isomorphic. Thus, the Chow ring of $\mathbb{F}_a$ depends only on the value of $d \mod n + 1$.

On the other hand, if $d$ and $d'$ are integers that are not congruent mod $n + 1$, we can see by explicit comparison that the resulting graded rings are not abstractly isomorphic. Any graded ring homomorphism is given by $\xi \mapsto a_{11}\xi + a_{12}\sigma$ and $\sigma \mapsto a_{21}\xi + a_{22}\sigma$. In order to be invertible, we require that the matrix with coefficients $a_{ij}$ lies in $GL_2(\mathbb{Z})$. A factorization argument shows that it suffices to treat the case where this element of $GL_2(\mathbb{Z})$ is either upper triangular or lower triangular. The upper triangular case produces the isomorphisms of the previous paragraph, and it is straightforward to check that the lower triangular case doesn’t introduce new isomorphisms.

Next, observe that the motivic cohomology ring is an invariant of the unstable $\mathbb{A}^1$-homotopy type (see, for example, [Voe03, §2 Theorem 2.2]). Finally, we use the fact that the motivic cohomology ring $\bigoplus_i H^{2i,i}(X, \mathbb{Z})$ coincides with the Chow cohomology ring ([Voe02, Corollary 2]). Combining this with the computation of the previous paragraph provides the explicit $\mathbb{A}^1$-homotopy invariants we required.

Proof of Lemma 3.11 This result is now a special case of Proposition 3.21.

4 Classifying spaces and strong $\mathbb{A}^1$-invariance

We now make a general study of $\mathbb{A}^1$-local classifying spaces, comparing the Nisnevich and étale topologies along the way. The techniques discussed here are used in the proofs of Propositions 2.27 and 2.28 provide foundations for further study of étale $\mathbb{A}^1$-connectivity (Definition 2.31), and provide explanation for the “source” of $\mathbb{A}^1$-$h$-cobordisms (Definition 3.1) constructed by means of Proposition 3.4. Furthermore, ideas from this section will be used in the course of the computations of §5.
Motivation

Suppose $X \in \mathcal{S}_m k$, and consider the projection morphism

$$p_X : X \times \mathbb{A}^1 \rightarrow X.$$ 

Suppose $Y \in \mathcal{S}_m k$. By a Nisnevich $Y$-bundle over $X$, we will mean a Nisnevich locally trivial morphism $g : Z \rightarrow X$ with fibers isomorphic to $Y$. Let $Aut(Y)$ denote the subsheaf of $\text{Hom}(Y,Y)$ consisting of automorphisms of $Y$; $Aut(Y)$ is in fact a sheaf of groups. Every Nisnevich $Y$-bundle over $X$, say given by $g$, defines (via Čech cohomology) an element $[g]$ of $H^1_{Nis}(X, Aut(Y))$. The image of $[g]$ under the natural map 

$$p^*_X : H^1_{Nis}(X, Aut(Y)) \rightarrow H^1_{Nis}(X \times \mathbb{A}^1, Aut(Y)).$$

corresponds in geometric terms to the pull-back via $p_X$, i.e., a morphism $p^*_X(g) : Z \times_X (X \times \mathbb{A}^1) \rightarrow X \times \mathbb{A}^1$ that is automatically a Nisnevich $Y$-bundle over $X \times \mathbb{A}^1$. Theorem 3.19 discusses this situation when $X = \mathbb{P}^1$ and $Y$ is a linear algebraic group.

Remark 4.1. For technical reasons, note that as $Aut(Y)$ is only a sheaf of groups, as opposed to a smooth $k$-group scheme, we use here the definition of torsor given in [MV99, p. 127-128]. When $Aut(Y)$ is a smooth $k$-group scheme, this definition coincides with the one mentioned just prior to Theorem 3.19.

Any non-trivial $\mathbb{A}^1$-h-cobordism constructed by means of Proposition 3.4 corresponds to an element of $H^1_{Nis}(X \times \mathbb{A}^1, Aut(Y))$ not lying in the image of $p^*_X$. Theorem 3.19 shows that if $G$ is a connected reductive group that has non-trivial algebraic fundamental group (see Remark 3.18), then $p^*_X$ need not be a bijection. We would like to study conditions on $G$ under which $p^*_X$ is always a bijection.

Strong $\mathbb{A}^1$-invariance and $\mathbb{A}^1$-local classifying spaces

Henceforth, the term Nisnevich (resp. étale) sheaf of groups will be synonymous with Nisnevich (resp. étale) sheaf of groups on $\mathcal{S}_m k$.

Definition 4.2. Suppose $G$ is a Nisnevich sheaf of groups. We will say that $G$ is strongly $\mathbb{A}^1$-invariant if for every $U \in \mathcal{S}_m k$, the canonical maps

$$p^*_U : H^i_{Nis}(U, G) \rightarrow H^i_{Nis}(U \times \mathbb{A}^1, G),$$

induced by pullback along the projection $p_U : U \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ are bijections for $i = 0, 1$. Similarly, if $G$ is an étale sheaf of groups, we will say that $G$ is strongly $\mathbb{A}^1$-invariant in the étale topology if for every $U \in \mathcal{S}_m k$, the maps

$$p^*_U : H^i_{\text{ét}}(U, G) \rightarrow H^i_{\text{ét}}(U \times \mathbb{A}^1, G),$$

defined as above, are bijections for $i = 0, 1$. 
Remark 4.3. Strong $\mathbb{A}^1$-invariance was introduced and extensively studied in [Mor06b]. Many examples of strongly $\mathbb{A}^1$-invariant sheaves of groups that are non-commutative will be provided in §5.

Recall from the end of §2 (just prior to Lemma 2.36) that we write $\mathcal{S}pc_k^{\text{ét}}$ for the category $\Delta^\circ \text{Shv}_{\text{ét}}(Sm_k)$ of étale simplicial sheaves of sets on $Sm_k$, $\mathcal{H}_s^{\text{ét}}(k)$ for the corresponding homotopy category of simplicial sheaves, and $R\alpha_*$ for the derived push-forward functor $\mathcal{H}_s^{\text{ét}}(k) \to \mathcal{H}_s^{\text{Nis}}(k)$. For both the Nisnevich and étale topology, there are corresponding notions of $\mathbb{A}^1$-local object (see [MV99, §2 Definition 3.1]). Furthermore, for a pointed space $(X, x)$ we write $\pi_*(X, x)$ for the Nisnevich sheaf on $Sm_k$ associated with the presheaf on $Sm_k$ defined by $U \mapsto \left[\Sigma_* \cup U, (X, x)\right]$. Using an $\mathbb{A}^1$-fibrant replacement functor, one can construct maps $\pi_*(X, x) \to \pi_{1, \mathbb{A}^1}(X, x)$.

Suppose $G$ is a Nisnevich (resp. étale) sheaf of groups. Together with Voevodsky, the second author has constructed for such $G$, an explicit (étale) simplicial sheaf $BG$, via a “bar construction,” that classifies $G$-torsors locally trivial in the Nisnevich (resp. étale) topology (see [MV99, §4.1, especially Proposition 1.16]). Using this result, [MV99, §2 Proposition 3.19], and the existence of fibrant replacements, we observe that if $G$ is a Nisnevich (resp. étale) sheaf of groups, then $BG$ is $\mathbb{A}^1$-local (resp. étale $\mathbb{A}^1$-local) if and only if $G$ is strongly $\mathbb{A}^1$-invariant (in the étale topology).

Notation 4.4. Suppose $G$ is an étale sheaf of groups. We set

$$B_{\text{ét}}G := R\alpha_* BG$$

where $BG^f$ is an étale simplicially fibrant replacement for $BG$.

Lemma 4.5. If $G$ is an étale sheaf of groups, then $G$ is strongly $\mathbb{A}^1$-invariant in the étale topology if and only if $B_{\text{ét}}G$ is $\mathbb{A}^1$-local. Thus, if $G$ is strongly $\mathbb{A}^1$-invariant in the étale topology, for every $U \in Sm_k$ the canonical maps

$$[\Sigma_* \cup U, (B_{\text{ét}}G,*)]_{\mathbb{A}^1} \to H^1_{\text{ét}}(U, G)$$

induced by adjunction are bijections for $i = 0, 1$.

Proof. The morphism $\alpha$ satisfies condition (iii) of [MV99, §2 Lemma 3.15] and, by the equivalences of (iii) and (i) of that lemma, we conclude that the functor $R\alpha_*$ preserves $\mathbb{A}^1$-local objects. The equivalence of the statement then follows from the discussion of the last paragraph. Now, if $B_{\text{ét}}G$ is $\mathbb{A}^1$-local we conclude that for any pointed space $(X, x)$ the canonical map

$$[(X, x), (B_{\text{ét}}G,*)] \to [(X, x), (B_{\text{ét}}G,*)]_{\mathbb{A}^1}$$

is a bijection. If $(X, x)$ is a pointed étale simplicial sheaf, adjunction gives a canonical bijection

$$\text{Hom}_{\mathcal{H}_s^{\text{ét}}(k)}((X, x), (BG,*)) \cong [(X, x), (B_{\text{ét}}G,*)]_s.$$ 

The final statement follows immediately from this by applying [MV99, §4 Proposition 1.16].
We quote without proof the following deep result of the second author, which we will use without mention in the sequel.

**Theorem 4.6** ([Mor06b] Theorem 3.1). If \((X, x)\) is a pointed space, then \(\pi^A_1(X, x)\) is a strongly \(A^1\)-invariant sheaf of groups for any integer \(i > 0\).

**Corollary 4.7.** If \(G\) is an étale sheaf of groups that is strongly \(A^1\)-invariant in the étale topology, then the Nisnevich sheaf underlying \(G\) is strongly \(A^1\)-invariant.

**Proof.** If \(B \ltimes G\) is \(A^1\)-local, the map \(\pi^s_1(B \ltimes G, \ast) \rightarrow \pi^A_1(B \ltimes G, \ast)\) is an isomorphism. Then, apply the previous theorem together with the identification of \(\pi^s_1(B \ltimes G)\) with \(G\) itself using Lemma 4.5.

The category of strongly \(A^1\)-invariant sheaves of groups

Let \(Gr_k\) denote the category of Nisnevich sheaves of groups on \(Sm_k\). We write \(Gr^A_1\) for the full subcategory of \(Gr_k\) consisting of strongly \(A^1\)-invariant sheaves of groups.

**Lemma 4.8** (cf. [Mor06b] Remark 4.11). The category \(Gr^A_1\) admits finite colimits.

**Proof.** We claim that the inclusion functor \(Gr^A_1 \hookrightarrow Gr_k\) admits a left adjoint defined by the functor \(G \mapsto \pi^A_1(BG, \ast)\). We have maps

\[
\text{Hom}_{Gr_k}(H, G) \leftarrow [(BH, \ast), (BG, \ast)]^s \rightarrow [(BH, \ast), (BG, \ast)]^A_1 \rightarrow \text{Hom}_{Gr^A_1}(\pi^A_1(BH, \ast), G)
\]

where the left-most map is given by applying the functor \(\pi^s_1\), and the right-most map is given by applying the functor \(\pi^A_1\). The map in the middle is a bijection since \(G\) is strongly \(A^1\)-invariant, and the Postnikov tower (cf. [AD07a, 3.10.1]) can be used to show that both the left-most and right-most maps are bijections; this observation establishes adjointness.

Now, any functor that is a left adjoint preserves small colimits (see [ML98, V.5]). The category of presheaves of groups on \(Sm_k\) admits finite colimits (defined sectionwise). Since sheafification is a left adjoint, it follows that \(Gr^1_k\) admits all finite colimits. Finally, using the fact that the functor \(H \mapsto \pi^A_1(BH)\) is a left adjoint, we deduce that \(Gr^A_1\) admits all finite colimits.

**Definition 4.9.** Given a diagram of strongly \(A^1\)-invariant sheaves of groups of the form

\[
G_1 \leftarrow H \rightarrow G_2,
\]

we write \(G_1 \star^A_1 H G_2\) for the colimit of this diagram computed in \(Gr^A_1\). Precisely, \(G_1 \star^A_1 H G_2\) is the strongly \(A^1\)-invariant sheaf of groups \(\pi^A_1(B(G_1 \star H G_2))\), where \(G_1 \star H G_2\) is the coproduct computed in the category \(Gr_k\). We refer to \(G_1 \star^A_1 H G_2\) as the sum of \(G_1\) and \(G_2\) amalgamated over \(H\), or, if \(H\) is trivial, as the amalgamated sum of \(G_1\) and \(G_2\) (and in both cases strong \(A^1\)-invariance is understood).
Definition 4.10. The free strongly $\mathbb{A}^1$-invariant sheaf of groups on a (pointed) sheaf of sets $(S,s)$, denoted $F_{\mathbb{A}^1}(S)$, is the Nisnevich sheaf of groups $\pi^{\mathbb{A}^1}_1(\Sigma_1 S)$.

One can show ([Mor06b, Lemma 4.23]) that if $S$ is a sheaf of pointed sets, then for any strongly $\mathbb{A}^1$-invariant sheaf of groups $G$, the canonical map $S \to \Omega_1 \Sigma_1 S$ induces a bijection

$$\text{Hom}_{\mathcal{G}r^A_1}(F_{\mathbb{A}^1}(S), G) \xrightarrow{\sim} \text{Hom}_{\mathcal{Sp}_{\mathbb{A}^1}}(S, G).$$

Thus, $F_{\mathbb{A}^1}$ is left adjoint to the forgetful functor $\mathcal{G}r^A_1 \to \mathcal{Sp}_{\mathbb{A}^1}$, and this observation justifies our naming convention.

Strict $\mathbb{A}^1$-invariance and $\mathbb{A}^1$-local Eilenberg-MacLane spaces

There are versions of the results proved above for higher cohomology of sheaves of abelian groups; we give here the corresponding statements together with brief indications of the modifications required in the proofs. For any Nisnevich (resp. étale) sheaf of abelian groups $A$, one can define Eilenberg-MacLane spaces $K(A,i)$ such that, if $U \in \mathcal{S}_{m_k}$, the $H^i_{\text{Nis}}(U,A)$ (resp. $H^i_{\text{ét}}(U,A)$) can be computed in terms of homotopy classes of maps from $U$ to $K(A,i)$ in $\mathcal{H}^{\text{Nis}}_{\text{s}}(k)$ (resp. $\mathcal{H}^{\text{ét}}_{\text{s}}(k)$). See [MV99, pp. 55-60] for more details.

Definition 4.11. Suppose $A$ is a Nisnevich sheaf of abelian groups. We will say that $A$ is strongly $\mathbb{A}^1$-invariant if for every $U \in \mathcal{S}_{m_k}$ the pull-back map

$$H^i_{\text{Nis}}(U,A) \rightarrow H^i_{\text{Nis}}(U \times \mathbb{A}^1, A)$$

is a bijection for every $i \geq 0$. Similarly, given an étale sheaf of abelian groups, we will say that $A$ is strictly $\mathbb{A}^1$-invariant in the étale topology if for every $U \in \mathcal{S}_{m_k}$ the pull-back map

$$H^i_{\text{ét}}(U,A) \rightarrow H^i_{\text{ét}}(U \times \mathbb{A}^1, A)$$

is a bijection for every $i \geq 0$.

Both the étale and Nisnevich topologies are sites of finite type in the sense of [MV99, §2 Definition 1.31]. Combining [MV99, §2 Proposition 1.26], ibid. §2 Theorem 1.34, and ibid. §2 Proposition 3.19, we observe that if $A$ is a Nisnevich (resp. étale) sheaf of abelian groups, then $A$ is strictly $\mathbb{A}^1$-invariant (resp. for the étale topology) if and only if $K(A,i)$ is $\mathbb{A}^1$-local for every $i \geq 0$.

Notation 4.12. Suppose $A$ is an étale sheaf of abelian groups. We set

$$K_{\text{ét}}(A,i) := \mathbb{R}\alpha_* K(A,i)^f$$

where $K(A,i)^f$ is an étale simplicially fibrant replacement of $K(A,i)$.

The proof of the following result is essentially identical to the proof of Lemma 4.5.
Lemma 4.13. Suppose $A$ is an étale sheaf of abelian groups, then $A$ is strictly $\mathbb{A}^1$-invariant in the étale topology if and only if $K_{\text{ét}}(A,i)$ is $\mathbb{A}^1$-local. Thus, if $A$ is strictly $\mathbb{A}^1$-invariant in the étale topology, for every $U \in Sm_k$, the canonical maps

$$[\Sigma^j \wedge U_+, K_{\text{ét}}(A,i)]_{\mathbb{A}^1} \to H^{i-j}_{\text{ét}}(U, A)$$

induced by adjunction are bijections for $0 \leq j \leq i$.

Theorem 4.14 ([Mor06b] Theorem 3.25). If $A$ is a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, then $A$ is strictly $\mathbb{A}^1$-invariant.

Using this Theorem, the proof of the next result is very similar to the proof of Corollary 4.7

Corollary 4.15. If $A$ is an étale sheaf of groups that is strictly $\mathbb{A}^1$-invariant for the étale topology, then the Nisnevich sheaf underlying $A$ is strictly $\mathbb{A}^1$-invariant.

Deducing $\mathbb{A}^1$-invariance properties

Definition 4.16. Recall that a presheaf (resp. sheaf, or étale sheaf) of sets $S$ is said to be $\mathbb{A}^1$-invariant, if for every $U \in Sm_k$, the canonical map

$$S(U) \to S(U \times \mathbb{A}^1)$$

induced by pull-back along the projection $U \times \mathbb{A}^1 \to U$ is a bijection.

The following result gives a way to construct étale sheaves of abelian groups that are strictly $\mathbb{A}^1$-invariant in the étale topology.

Lemma 4.17 (cf. [Voe00] §3.4). Let $k$ be a field having characteristic exponent $p$, and suppose $A$ is an $\mathbb{A}^1$-invariant étale sheaf of $\mathbb{Z}[1/p]$-modules with transfers (in the sense of [MVW06] §6 p. 39), then $A$ is strictly $\mathbb{A}^1$-invariant for the étale topology.

Proof. Given a short exact sequence of étale sheaves of $\mathbb{Z}[1/p]$-modules

$$0 \to A' \to A \to A'' \to 0,$$

the associated long exact sequence in cohomology shows that if any two of the three sheaves are strictly $\mathbb{A}^1$-invariant in the étale topology, then the third must be as well. Using the exact sequence of étale sheaves

$$0 \to A_{\text{tors}} \to A \to A \otimes \mathbb{Q} \to A \otimes \mathbb{Q}/\mathbb{Z} \to 0,$$

one reduces to treating the cases where $A$ is an étale sheaf of $\mathbb{Q}$-vector spaces or, using the assumptions, $A$ is an étale sheaf of torsion prime to $p$. In the first case, one reduces to [MVW06] Theorem 13.8 by using the fact the Nisnevich and étale cohomology coincide (see [MVW06] Proposition 14.23). In the second case, the Suslin rigidity theorem (see [MVW06] Theorem 7.20]) shows that $A$ is in fact a locally constant étale sheaf of groups and one concludes by applying [SGA73] Exposé XV Corollaire 2.2].
Proof of Proposition 2.27
We first prove the following more precise result.

**Proposition 4.18.** Suppose \( X \in S_{m_k} \) is \( \mathbb{A}^1 \)-connected, and \( G \) is an étale sheaf of groups strongly \( \mathbb{A}^1 \)-invariant in the étale topology. For any point \( x \in X(k) \) the restriction map

\[ x^* : H^1_{\text{ét}}(X, G) \to H^1_{\text{ét}}(\text{Spec } k, G) \]

is independent of \( x \) and we denote it by \( \rho \). The natural map

\[ H^1_{\text{Nis}}(X, G) \to H^1_{\text{ét}}(X, G) \]

injects into the inverse image under \( \rho \) of the base-point of the pointed set \( H^1_{\text{ét}}(\text{Spec } k, G) \). In other words, an étale locally trivial \( G \)-torsor over \( X \) whose restriction to a rational point is trivial is Nisnevich locally trivial.

**Proof.** From Lemma 4.5 we know that for any \( U \in S_{m_k} \) the canonical map

\[ [U, B_{\text{ét}}G]_{\mathbb{A}^1} \to H^1_{\text{ét}}(U, G) \]

is a bijection. Given a class in \( \tau \in H^1_{\text{ét}}(X, G) \), choose an explicit representative \( \tau : X \to B_{\text{ét}}G \) (we can do this because \( B_{\text{ét}}G \) is \( \mathbb{A}^1 \)-fibrant). The composite map

\[ X \xrightarrow{\tau} B_{\text{ét}}G \to \pi_{A^1_0}(B_{\text{ét}}G) \]

factors through the canonical map \( X \to \pi_{A^1_0}(X) = \ast \). As the set of sections of \( \pi_{A^1_0}(B_{\text{ét}}G) \) over \( \text{Spec } k \) is exactly \( H^1_{\text{ét}}(\text{Spec } k, G) \), this proves the independence statement.

Now, again by Lemma 4.5 we know that \( B_{\text{ét}}G \) is \( \mathbb{A}^1 \)-local and that \( \pi_{A^1_0}(B_{\text{ét}}G) = G \). Thus, we conclude that the map \( B_{\text{ét}}G \to B_{\text{ét}}G \) induced by adjunction is the inclusion of the (\( \mathbb{A}^1 \)-)connected component of the base-point. If \( X \in S_{m_k} \) is \( \mathbb{A}^1 \)-connected, it follows that the induced map

\[ H^1_{\text{Nis}}(X, G) \simto [X, B_{\text{ét}}G]_{\mathbb{A}^1} \to [X, B_{\text{ét}}G] \]

is an injection whose image can be identified with the set of morphisms \( X \to B_{\text{ét}}G \) that map \( \pi_{A^1_0}(X) \) to the base-point of \( \pi_{A^1_0}(B_{\text{ét}}G) \). By the discussion of the previous paragraphs, this proves our claim. \( \square \)

**Proof of 2.27.** Let \( G \) be a finite étale group scheme of order prime to \( p \). In this situation, \( B_{\text{ét}}G \) is \( \mathbb{A}^1 \)-local by [MV99, §4 Proposition 3.1], so we could just apply Proposition 4.18. Really, we just have to observe that \( G \) is a strongly \( \mathbb{A}^1 \)-invariant sheaf of groups in the étale topology by [SGA73, Exposé XV Corollaire 2.2]. \( \square \)

**Remark 4.19.** We continue with notation as in Proposition 4.18. The map \( \rho : H^1_{\text{ét}}(X, G) \to H^1_{\text{ét}}(\text{Spec } k, G) \) can be reinterpreted as follows. Recall the identification \( H^1_{\text{ét}}(X, G) := [X, B_{\text{ét}}G]_{\mathbb{A}^1} \). Since \( X \) is \( \mathbb{A}^1 \)-connected, “evaluation on \( \pi_{A^1_0} \)” gives a map

\[ [X, B_{\text{ét}}G]_{\mathbb{A}^1} \to Hom_{\text{Spec } k}(\pi_{A^1_0}(X), B_{\text{ét}}G) \simto Hom_{\text{Spec } k}(\ast, B_{\text{ét}}G) \simto [\text{Spec } k, B_{\text{ét}}G]_{\mathbb{A}^1} \]

that coincides with \( \rho \).
Remark 4.20. Given a 1-cocycle of \( k \) with values in \( G \) associated with a class \( \tau \in H^1(\text{Spec} \, k, G) \), one may twist \( G \) by \( \tau \) to get another sheaf of groups that we denote by \( G_\tau \). Using a similar but more involved argument, one can prove that the sheaf \( G_\tau \) is also strongly \( \mathbb{A}^1 \)-invariant in the étale topology, and the fiber of \( \rho \) at \( \tau \) is (the image of) \( H^1_{Nis}(X, G_\tau) \).

Proof of Proposition 4.21. Let \( k \) be a field having characteristic exponent \( p \). Let \( \mathbb{G}_{m'} \) denote the étale sheaf whose sections over \( U \in \text{Sm}_k \) are given by
\[
U \mapsto \mathcal{O}^+(U) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].
\]
We refer to \( \mathbb{G}_{m'} \) as the multiplicative group with characteristic exponent inverted. This étale sheaf of groups is just \( \mathbb{G}_m \) if \( k \) has characteristic 0. By [Gro68, p. 88 Proposition 1.4] we know that there is a canonical injection \( Br(X) \hookrightarrow H^2_{\text{ét}}(X, \mathbb{G}_m) \). We can conclude that the map \( Br(X) \to H^2_{\text{ét}}(X, \mathbb{G}_{m'}) \) induces an injection on \( \ell \)-torsion subgroups for \( \ell \) prime to \( p \).

Proposition 4.21. Let \( k \) be a field having characteristic exponent \( p \), suppose \( X \in \text{Sm}_k \) is \( \mathbb{A}^1 \)-connected, and \( x \in X(k) \). The structure morphism \( X \to \text{Spec} \, k \) induces an isomorphism
\[
H^2_{\text{ét}}(\text{Spec} \, k, \mathbb{G}_{m'}) \to H^2_{\text{ét}}(X, \mathbb{G}_{m'}).
\]
In particular, if \( k \) is separably closed, then \( Br(X) \) is \( p \)-torsion.

Proof. In outline, this proof is essentially identical to the proof of Proposition 2.27 via Proposition 4.18. In this case, we use Lemma 4.13 to reduce to showing that \( \mathbb{G}_{m'} \) is a strongly \( \mathbb{A}^1 \)-invariant in the étale topology; this latter fact follows from Lemma 4.17. Indeed, \( \mathbb{G}_{m'} \) is an étale sheaf of \( \mathbb{Z}[1/p] \)-modules, has transfers given by the “norm” map, and is \( \mathbb{A}^1 \)-invariant since \( \mathbb{G}_m \) itself is \( \mathbb{A}^1 \)-invariant. Then \( \pi^1_{\text{ét}}(K_{\text{ét}}(\mathbb{G}_{m'}), 2) \) is 0 for \( i \geq 3 \), and the Nisnevich sheaf associated with the presheaf \( U \mapsto H^{2-i}_{\text{ét}}(U, \mathbb{G}_{m'}) \) for \( 0 \leq i \leq 2 \). We can use Grothendieck’s version of Hilbert’s Theorem 90 to show that \( \pi^1_{\text{ét}}(K_{\text{ét}}(\mathbb{G}_{m'}), 2) \) is trivial.

Example 4.22. Suppose \( G \) is a simply connected, semi-simple algebraic group over a field \( k \) having characteristic exponent \( p \). Let \( G^+(k) \) denote the subgroup of \( G(k) \) generated by the images of homomorphisms from the additive group \( \mathbb{G}_a(k) \). The quotient \( G(k)/G^+(k) \) is called the Whitehead group of \( G \), often denoted \( W(k, G) \). The Kneser-Tits problem asks for which groups \( W(k, G) = 1 \). Slightly more generally, [Gil08a, Question 1.1] asks whether one can characterize groups such that \( W(L, G) \) is trivial for every extension \( L/k \); such groups are called \( W \)-trivial. If \( G \) is \( W \)-trivial, it is \( \mathbb{A}^1 \)-chain connected and thus \( \mathbb{A}^1 \)-connected by Proposition 2.12. If \( G \) is in addition split then [Gil08a, Proposition 3.1] shows that \( G \) is \( W \)-trivial and hence \( \mathbb{A}^1 \)-connected. Thus, Proposition 4.21 shows that the Brauer group of a \( W \)-trivial group is \( p \)-torsion. S. Gille uses related ideas to study the Brauer group of general simply connected, semi-simple algebraic groups \( G \) (see [Gil10]).
Algebraic groups and strong $\mathbb{A}^1$-invariance

We now study the subcategory of $G_{\mathbb{A}^1}$ consisting of representable objects, i.e., smooth group schemes having finite type over $k$. Throughout this section, if $G$ is a smooth $k$-group scheme, we denote by $G^0$ the connected component of $G$ containing the identity element.

Lemma 4.23. Let $k$ be a perfect field, and suppose $G$ is a smooth affine algebraic $k$-group. The sheaf of groups $G$ is $\mathbb{A}^1$-invariant if and only if $G^0$ is a $k$-torus.

Proof. We use the following dévissage. There is an exact sequence of algebraic groups

$$1 \rightarrow G^0 \rightarrow G \rightarrow \Gamma \rightarrow 1$$

where $\Gamma$ is the (finite) group of connected components. Since the group $\Gamma$ is strongly $\mathbb{A}^1$-invariant by [MV99, §4 Proposition 3.5], proving the statement for $G$ is equivalent to proving it for $G^0$. Thus, we assume $G$ is connected.

As $k$ is perfect, the unipotent radical $R_u$ of $G$ is a smooth unipotent $k$-group scheme. Therefore, $G$ fits into an exact sequence of the form

$$1 \rightarrow R_u \rightarrow G \rightarrow G^{\text{red}} \rightarrow 1,$$

where $G^{\text{red}}$ is a reductive $k$-group scheme. Since $R_u$ is connected and smooth, and $k$ is perfect, by a theorem of Lazard [DG70, Chapter IV §2.3.9] $R_u$ is split, i.e., admits an increasing sequence of normal subgroups with subquotients isomorphic to $G_a$. Thus, if $R_u$ is non-trivial, it possesses a non-trivial group homomorphism from $G_a$. Since $G_a$ is not $\mathbb{A}^1$-invariant, it follows in this case that $R_u$ is not $\mathbb{A}^1$-invariant either. Thus, for $G$ to be $\mathbb{A}^1$-invariant $R_u$ must be trivial, and we may assume $G$ is reductive.

If $G$ is reductive, we have an exact sequence of the form

$$1 \rightarrow R(G) \rightarrow G \rightarrow G^{\text{ss}} \rightarrow 1,$$

where $R(G)$ is a $k$-torus. Now, since $R(G)$ is a $k$-torus, it splits over a finite separable extension $L/k$. By étale descent, it follows that $R(G)$ is $\mathbb{A}^1$-rigid. Thus, $G$ is $\mathbb{A}^1$-invariant if and only if $G^{\text{ss}}$ is $\mathbb{A}^1$-invariant, so we can assume $G = G^{\text{ss}}$.

If $G$ is a (non-trivial) semi-simple group, then it splits over a finite separable extension $L/k$. Passing to such an extension, we obtain non-trivial morphisms from $\mathbb{A}^1_L$ (any root subgroup provides such a morphism), and thus $G$ is not $\mathbb{A}^1$-invariant.

Proposition 4.24. Suppose $k$ is a perfect field, and assume $G$ is a smooth $k$-group scheme. The sheaf $G$ is $\mathbb{A}^1$-invariant if and only if $G^0$ is an extension of an abelian variety by a $k$-torus.

Proof. By Chevalley’s theorem ([Con02]), there is a canonical extension of the form

$$1 \rightarrow G^{\text{aff}} \rightarrow G \rightarrow A \rightarrow 1,$$

where $G^{\text{aff}}$ is a normal, smooth closed affine algebraic group, and $A$ is an abelian variety. Since $A$ is $\mathbb{A}^1$-rigid (cf. Example [2.4]), and the underlying Nisnevich sheaf is flasque, we conclude
that $A$ is strongly $\mathbb{A}^1$-invariant. Thus, proving the result for $G$ is equivalent to proving the result for $G^{aff}$. Since $G^{aff}$ is a smooth affine algebraic $k$-group, we apply Lemma 4.23 to finish the proof.

**Proposition 4.25.** Assume $k$ is a field having characteristic 0, and suppose $G$ is a smooth $k$-group scheme. The étale sheaf $G$ is strongly $\mathbb{A}^1$-invariant in the étale topology if and only if $G^0$ is an extension of an abelian variety by a $k$-torus. If one of these equivalent conditions holds, then $G$ is strongly $\mathbb{A}^1$-invariant in the Nisnevich topology as well.

**Proof.** By Proposition 4.24 we know that $G$ is $\mathbb{A}^1$-invariant if and only if $G^0$ is an extension of an abelian variety by a $k$-torus. In this case, applying Lemma 4.26 we conclude that $G^0$ is an étale sheaf with transfers in the sense of [MVW06, §6 p. 39]. Then, since $k$ has characteristic 0, we may apply Lemma 4.17 to conclude that $G^0$ is in fact strongly $\mathbb{A}^1$-invariant in the étale topology. Also, since $k$ has characteristic 0, we know that finite groups are strongly $\mathbb{A}^1$-invariant in the étale topology. The last statement follows from the equivalences by applying Corollary 4.7.

**Lemma 4.26** ([Org04] Lemme 3.1.2). If $S$ is a smooth commutative $k$-group scheme, then the étale sheaf underlying $S$ can be equipped canonically with transfers (in the sense of [MVW06, §6 p. 39]).

**Example 4.27.** For fields having positive characteristic, the sheaf $\mathbb{G}_m$ is not strictly $\mathbb{A}^1$-invariant in the étale topology. Thus, $\mathbb{G}_m$ is an étale sheaf whose underlying Nisnevich sheaf is strictly $\mathbb{A}^1$-invariant, but which is not strictly $\mathbb{A}^1$-invariant in the étale topology in general. In other words, the converse to Corollary 4.15 is false. However, let us note that $\mathbb{G}_m$ is strongly $\mathbb{A}^1$-invariant in the étale topology because of Hilbert’s theorem 90, i.e., $H^1_{Nis}(X, \mathbb{G}_m) = H^1(\mathfrak{et}, X, \mathbb{G}_m)$ for any $X \in Sm_k$.

With more work, one can construct counter-examples to Corollary 4.17 even for fields having characteristic 0. If $C$ is a smooth curve of genus $g > 0$, then one can consider the free strongly invariant sheaf of abelian groups generated by $C$, often denoted $\mathbb{Z}_{\mathbb{A}^1}(C)$ (cf. [Mor06b, p.104]), in a manner similar to Definition 4.10. This sheaf of groups is actually an étale sheaf of groups, that is not strongly $\mathbb{A}^1$-invariant in the étale topology.

**Remark 4.28.** Strong $\mathbb{A}^1$-invariance (or its failure) for $GL_n$ has been studied in great detail. On the one hand, [Mor07] proves that if $X$ is a smooth affine scheme, that $[X, BGL_n]_{\mathbb{A}^1}$ is in canonical bijection with the set of isomorphism classes of rank $n$ vector bundles on $X$ whenever $n \neq 2$. On the other hand if $X$ is not affine, the examples of [AD08] show there is essentially no “lower bound” on how badly this identification can fail.

**Automorphism groups of smooth proper varieties**

The automorphism groups of smooth proper $k$-varieties form a quite restricted class. Combining the next result with Proposition 4.25, we obtain an essentially complete understanding of $\mathbb{A}^1$-$h$-cobordisms constructed by means of Proposition 3.4.
Proposition 4.29. Suppose $k$ is a field having characteristic 0. If $X \in Sm_k$ is also proper, then $\text{Aut}(X)$ is a smooth $k$-group scheme.

Proof. An automorphism of a scheme $X$ is a morphism $f : X \to X$. Such a morphism defines a graph $\Gamma_f \subset X \times X$. By this construction, we can identify the functor defining $\text{Aut}(X)$ as a sub-functor of an appropriate Hilbert functor. In the case $X$ is projective, representability of this functor follows from [Ko96, Theorem I.1.10 (cf. Exercise I.1.10.2)]. If $X$ is only a proper scheme, then the sheaf $\text{Aut}(X)$ is represented by an algebraic space by [Ols06, Lemma 5.1]. By [Knu71, II.6.7], any algebraic space has a dense open affine subscheme and one can use the group action to construct a Zariski open cover of $\text{Aut}(X)$ by such schemes. Thus, $\text{Aut}(X)$ is always a $k$-group scheme under the hypotheses. Finally, over fields having characteristic 0, one can show that the group scheme $\text{Aut}(X)$ is actually smooth by explicit computation of its tangent space and application of Cartier’s theorem, [DG70, Chapter II §6.1.1].

5 Computing the $\mathbb{A}^1$-fundamental group

This section is the spiritual center of the paper. Given an $\mathbb{A}^1$-connected space $X$, it is natural to study its higher $\mathbb{A}^1$-homotopy invariants. If $k$ is a field, Theorem 2.21 shows that $k$-rational smooth proper surfaces are $\mathbb{A}^1$-connected, and if furthermore $k$ has characteristic 0, then retract $k$-rational varieties are $\mathbb{A}^1$-connected. See Appendix A for a summary of results relating $\mathbb{A}^1$-connectivity and rationality properties. Thus, let $X$ be an $\mathbb{A}^1$-connected smooth variety, and fix a basepoint $x \in X(k)$. We focus now on computing the next $\mathbb{A}^1$-homotopy invariant of such varieties: the $\mathbb{A}^1$-fundamental sheaf of groups, denoted $\pi_{1, \mathbb{A}^1}(X,x)$.

To facilitate topological intuition, throughout this section we refer to $\pi_{1, \mathbb{A}^1}(X,x)$ as simply the $\mathbb{A}^1$-fundamental group. To partially justify this abuse of terminology, we begin by proving or recalling a collection of results that are analogous to corresponding topological statements. Theorem 5.1 establishes a version of the van Kampen theorem, and Proposition 5.2 indicates a relationship between the covering space theory associated with the $\mathbb{A}^1$-fundamental group and geometry. We also discuss in great detail the structure of the $\mathbb{A}^1$-fundamental group of $\mathbb{P}^1$, which is a fundamental computation in unstable $\mathbb{A}^1$-homotopy theory. Unlike its topological counterpart, Proposition 5.4 shows that the $\mathbb{A}^1$-fundamental group of a smooth proper $\mathbb{A}^1$-connected variety is always non-trivial; Extension 5.5 explains a corresponding result for étale-$\mathbb{A}^1$-connected varieties.

The ultimate goal of this section, accomplished in Corollary 5.14, is to show that if $X$ is a $k$-rational smooth proper surface over an algebraically closed field $k$, the $\mathbb{A}^1$-homotopy type of $X$ is determined by its $\mathbb{A}^1$-fundamental group. To establish this, we will simply compute the $\mathbb{A}^1$-fundamental groups of all $k$-rational smooth proper surfaces. Theorem 3.8 shows that we need only perform the computation for Hirzebruch surfaces, and for certain blow-ups of points. The first case is addressed by Proposition 5.8 and Proposition 5.9 addresses the second case by establishing a general “reduction theorem” for blow-ups of points on smooth schemes that are covered by affine spaces in the sense of Definition 2.15.
Suppose $X \in S_{m_1}$, choose a basepoint $x \in X(k)$, and consider the $\mathbb{A}^1$-fundamental group $\pi_1^{\mathbb{A}^1}(X, x)$. If furthermore $X$ is $\mathbb{A}^1$-connected, and we pick another base-point $x' \in X(k)$, the $\mathbb{A}^1$-fundamental group $\pi_1^{\mathbb{A}^1}(X, x')$ is conjugate to $\pi_1^{\mathbb{A}^1}(X, x)$. For this reason, we fix and (occasionally) suppress basepoints in all our subsequent discussion. In Definition 4.2 we recalled the notion of a strongly $\mathbb{A}^1$-invariant sheaf of groups. We also noted [Mor06b Theorem 3.1] shows that $\pi_1^{\mathbb{A}^1}(X, x)$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups. Here is a version of the classical van Kampen theorem (more general versions are known).

**Theorem 5.1** ($\mathbb{A}^1$-van Kampen theorem [Mor06b Theorem 4.12]). Suppose $X$ is a smooth $\mathbb{A}^1$-connected $k$-variety covered by $\mathbb{A}^1$-connected open subsets $U, V$ such that $U \cap V$ is $\mathbb{A}^1$-connected. Then we have a canonical isomorphism

$$
\pi_1^{\mathbb{A}^1}(U, x) \ast_{\pi_1^{\mathbb{A}^1}(U \cap V)} \pi_1^{\mathbb{A}^1}(V, x) \sim \pi_1^{\mathbb{A}^1}(X, x),
$$

where the operation $\ast_{\pi_1^{\mathbb{A}^1}}$ is given by Definition 4.9.

A version of covering space theory for the $\mathbb{A}^1$-fundamental group dubbed $\mathbb{A}^1$-covering space theory has been developed by the second author (see [Mor06b §4.1]). For our purposes, the following result will suffice.

**Proposition 5.2** (cf. [AD07a Corollary 5.3]). Suppose $\tilde{X}$ and $X$ are two smooth $\mathbb{A}^1$-connected $k$-varieties. If $f : \tilde{X} \to X$ is a $\mathbb{G}_m \times^r$-torsor over $X$, then the morphism $f$ is an $\mathbb{A}^1$-fibration, and one has a short exact sequence of the form

$$
1 \longrightarrow \pi_1^{\mathbb{A}^1}(\tilde{X}) \longrightarrow \pi_1^{\mathbb{A}^1}(X) \longrightarrow \mathbb{G}_m \times^r \longrightarrow 1,
$$

and isomorphisms $\pi_i^{\mathbb{A}^1}(\tilde{X}) \sim \pi_i^{\mathbb{A}^1}(X)$ for every $i > 1$.

**Remark 5.3.** Slightly more generally, one can show that Zariski locally trivial torsors with $\mathbb{A}^1$-rigid fibers are always $\mathbb{A}^1$-fibrations. Torsors under split tori over smooth schemes are examples of $\mathbb{A}^1$-covering spaces in the sense of [Mor06b §4.1] by ibid. Lemma 4.5. This fact has been used in [AD07a] and [Wen07] to describe the $\mathbb{A}^1$-fundamental group of a smooth proper toric variety as an extension of a torus by a strongly $\mathbb{A}^1$-invariant sheaf of groups of arithmetic nature.

The main problem with Proposition 5.2 is that it does not provide an explicit identification of the extension or the group structure on the $\mathbb{A}^1$-fundamental group. The problem of identifying this additional data, which Proposition 5.8 shows to be very subtle, will occupy us in what follows. Nevertheless, we can use the geometry behind Proposition 5.2 to establish that smooth proper $\mathbb{A}^1$-connected schemes necessarily have non-trivial $\mathbb{A}^1$-fundamental groups.

**Proposition 5.4.** Suppose $X \in S_{m_1}$ is $\mathbb{A}^1$-connected and $x \in X(k)$. We have a canonical isomorphism

$$
\text{Hom}_{\mathbb{G}_m^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(X, x), \mathbb{G}_m) \sim \text{Pic}(X).
$$

In particular, if $X$ is a strictly positive dimensional, $\mathbb{A}^1$-connected, smooth proper $k$-variety then $\text{Pic}(X)$ is non-trivial and thus $\pi_1^{\mathbb{A}^1}(X)$ is non-trivial.
Proof. We know that the canonical map \([(X, x), (BG_m, *)]_{\mathbb{A}^1} \to \text{Hom}_{\mathbb{G}_m^A}(\pi_{1, \mathbb{A}^1}(X, x), G_m)\) (induced by the Postnikov tower) is an isomorphism (cf. Mor06b Remark 4.11 or AD07a Theorem 3.31). Since \(G_m\) is abelian, we know that the canonical map from base-pointed to base-point free maps is an isomorphism. Thus, we see that \([(X, x), (BG_m, *)]\sim [X, BG_m]_{\mathbb{A}^1}\sim Pic(X)\). If \(Pic(X)\) is non-trivial, it follows that \(\text{Hom}_{\mathbb{G}_m^A}(\pi_{1, \mathbb{A}^1}(X, x), G_m)\) is non-trivial and thus, by the Yoneda lemma, that \(\pi_{1, \mathbb{A}^1}(X, x)\) is itself non-trivial.

Now, if \(X\) is a strictly positive dimensional smooth proper variety, we claim \(Pic(X)\) is non-trivial. Indeed, since \(X\) is smooth scheme over a field, it is, by our assumptions and conventions, separated, regular and Noetherian, and so admits an ample family of line bundles. Since \(X\) is strictly positive dimensional and proper, it is not affine, and thus one of these line bundles must be non-trivial. Since \(X\) is \(\mathbb{A}^1\)-connected, we know \(X(k)\) is non-empty. Upon choice of a base-point \(x \in X(k)\), we can appeal to the first part of the statement to finish the proof. \(\square\)

Extension 5.5. If \(X \in Sm_k\) is proper and \(\mathbb{A}^1\)-connected, and \(X(k)\) is non-empty, the \(\mathbb{A}^1\)-fundamental group (defined using \(\mathcal{H}_{\text{et}}^1(k)\); see the beginning of §4] is always non-trivial as well. Using the Postnikov tower and the fact that \(G_m\) is strongly \(\mathbb{A}^1\)-invariant in the \(\mathbb{A}^1\)-fundamental group, one can interpret \(H^1_{\text{et}}(X, G_m)\) as the set of homomorphisms (of sheaves of groups) from the \(\mathbb{A}^1\)-fundamental group of \(X\) to \(G_m\). Then, one need only observe that \(H^1_{\text{et}}(X, G_m) = Pic(X)\) by Hilbert’s theorem 90. More generally, it seems reasonable to expect that, generalizing Mor06b Theorem 3.1, the \(\mathbb{A}^1\)-fundamental group is always strongly \(\mathbb{A}^1\)-invariant in the \(\mathbb{A}^1\)-fundamental group, and a proof formally identical to the one above may be used as well.

Extension 5.6. Using the discussion of Example 2.32 we leave the reader the (easy) exercise of applying the results of Extension 5.5 to prove an \(\mathbb{A}^1\)-h-cobordism theorem, i.e., that any \(\mathbb{A}^1\)-h-cobordism between \(\mathbb{A}^1\)-connected and \(\mathbb{A}^1\)-simply connected smooth proper varieties over a field is always trivial.

The \(\mathbb{A}^1\)-fundamental group of \(\mathbb{P}^1\)

We now discuss the computation of the \(\mathbb{A}^1\)-fundamental group of \(\mathbb{P}^1\); this example, which is the simplest non-trivial case, is studied in great detail in Mor06b §4.3. To begin, let us first describe the \(\mathbb{A}^1\)-homotopy type of \(\mathbb{P}^1\) (cf. MV99 §3 Corollary 2.18)). The usual open cover of \(\mathbb{P}^1\) by two copies of the affine line with intersection \(G_m\) presents \(\mathbb{P}^1\) as a push-out of the following diagram

\[
\mathbb{A}^1 \leftarrow G_m \rightarrow \mathbb{A}^1.
\]

The push-out of this diagram can also be computed in the \(\mathbb{A}^1\)-homotopy category, where up to \(\mathbb{A}^1\)-weak equivalence, it can be replaced by the diagram

\[
* \leftarrow G_m \rightarrow C(G_m).
\]

Here, \(C(G_m) = G_m \wedge \Delta^1\) is the cone over \(G_m\) (where the simplicial interval \(\Delta^1\) is pointed by 1). The canonical map from the homotopy colimit to the colimit gives a morphism \(\Sigma^* G_m \to \mathbb{P}^1\).
that is an $A^1$-weak equivalence (since either morphism $G_m \hookrightarrow A^1$ is a cofibration). Now, consider Definition 4.10.

**Notation 5.7.** Set $F_{A^1}(1) := F_{A^1}(G_m) = \pi^{A^1}_1(\mathbb{P}^1)$ where $G_m$ is pointed by 1.

The defining property of free strongly $A^1$-invariant sheaves of groups gives rise to a canonical morphism $\theta : G_m \to F_{A^1}(G_m)$. On the other hand, Proposition 5.2 shows that the standard $G_m$-torsor $A^2 \setminus 0 \to \mathbb{P}^1$ induces a short exact sequence of $A^1$-homotopy groups

\[
(5.1) \quad 1 \longrightarrow \pi^{A^1}_1(A^2 \setminus 0) \longrightarrow F_{A^1}(1) \longrightarrow G_m \longrightarrow 1
\]

that is *split* by $\theta$.

The $A^1$-fundamental group of $A^2 \setminus 0$ can be studied using an explicit description of its $A^1$-homotopy type. The open cover of $A^2 \setminus 0$ by $A^1 \times G_m$ and $G_m \times A^1$ with intersection $G_m \times G_m$ shows that $A^2 \setminus 0$ is the pushout of the diagram

\[
G_m \leftarrow G_m \times G_m \longrightarrow G_m
\]

computed in the $A^1$-homotopy category. This realizes $A^2 \setminus 0$ as the *join* of $G_m$ with itself. One can show that this join is always $A^1$-weakly equivalent to $\Sigma^4_1 G_m \wedge G_m$. According to our above definitions, $\pi^{A^1}_1(A^2 \setminus 0)$ is the free strongly $A^1$-invariant sheaf of groups generated by $G_m \wedge G_m$; this group is sometimes denoted $F_{A^1}(2)$, but we will describe it more explicitly momentarily.

There is a projection morphism $SL_2 \to A^2 \setminus 0$ that is an $A^1$-weak equivalence (being Zariski locally trivial with affine space fibers). In classical topology, one knows that the fundamental group of a topological group is abelian, and this proof can be adapted to show that the $A^1$-fundamental group of a group space is necessarily a sheaf of abelian groups. This sheaf of abelian groups, called the sheaf of second Milnor-Witt K-theory groups, will be denoted $K_2^{MW}$. It is closely related to both Milnor K-theory and Witt groups as explained in [Mor06b §2], where a completely explicit presentation via “symbols” (generators and relations) is given.

In any case, $F_{A^1}(1)$ fits into a split short exact sequence of the form

\[
1 \longrightarrow K_2^{MW} \longrightarrow F_{A^1}(1) \longrightarrow G_m \longrightarrow 1.
\]

Theorem 4.29 of [Mor06b] demonstrates that this short exact sequence is in fact a *central extension*. As a sheaf of sets $F_{A^1}(1)$ is a product $K_2^{MW} \times G_m$ and we can be extremely explicit about the group structure on this sheaf of sets.

We will need a few pieces of notation about the sheaf $K_2^{MW}$. There is a canonical *symbol* morphism

\[
\Phi : G_m \times G_m \longrightarrow K_2^{MW}
\]

obtained via composition of the projection $G_m \times G_m \to G_m \wedge G_m$ and the canonical morphism $G_m \wedge G_m \to \pi^{A^1}_1(\Sigma^4_1 G_m \wedge G_m)$ described above. Given a Henselian local scheme $S$, and sections $a, b \in G_m(S)$, we write $[a][b]$ for the image in $K_2^{MW}(S)$. The symbol morphism is, up to an explicit automorphism of $K_2^{MW}$, related to the morphism $\theta$ by the following formula ([Mor06b Theorem 4.29 and Remark 4.30]):

\[
[a][b] = \Phi(a, b) = \theta(a)\theta(b)\theta(ab)^{-1}.
\]
This formula provides an explicit description of the multiplication on $K_2^{MW} \times \mathbb{G}_m$ giving $F_{\mathbb{A}^1}(1)$ its group structure.

**The $\mathbb{A}^1$-fundamental group of a Hirzebruch surface**

The Hirzebruch surface $F_a$ is isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-a))$ and comes equipped with a structure morphism $F_a \to \mathbb{P}^1$ admitting a section. This morphism induces (split) group homomorphisms

$$\pi_{\mathbb{A}^1}^1(F_a) \to F_{\mathbb{A}^1}(1)$$

for any integer $a$. Pulling back the structure morphism along the $\mathbb{G}_m$-torsor $\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$ produces a trivial bundle of the form $\mathbb{A}^2 \setminus 0 \times \mathbb{P}^1$. Let $V$ be the 2-dimensional representation of $\mathbb{G}_m$ defined by the action $v \cdot (x_1, x_2) = (v^0 x_1, v^a x_2)$. This action induces an action of $\mathbb{G}_m$ on $\mathbb{P}^1 = \mathbb{P}(V)$. Furthermore, this $\mathbb{G}_m$-action preserves the point with homogeneous coordinates $[1, 0]$, which we refer to as $\infty$. The induced map $\mathbb{G}_m \to \text{Aut}(\mathbb{P}^1)$ gives a morphism of sheaves

$$\mathbb{G}_m \to \text{Aut}(F_{\mathbb{A}^1}(1)).$$

Note that $\mathbb{G}_m$ also acts on $\mathbb{A}^2 \setminus 0$, in a manner inducing the central extension of Equation 5.1. The inclusion of a fiber $\mathbb{P}^1$ (say over the image of a chosen base-point) then gives a morphism $F_{\mathbb{A}^1}(1) \to \pi_{\mathbb{A}^1}^1(F_a)$, and the aforementioned discussion shows that one has a split short exact sequence of groups

$$1 \to F_{\mathbb{A}^1}(1) \to \pi_{\mathbb{A}^1}^1(F_a) \to F_{\mathbb{A}^1}(1) \to 1.$$

Our discussion of the actions shows that the action of $F_{\mathbb{A}^1}(1)$ acts on itself factors through the quotient map $F_{\mathbb{A}^1}(1) \to \mathbb{G}_m \to \text{Aut}(F_{\mathbb{A}^1}(1))$, and this last map is completely determined by the integer $a$. We write

$$\pi_{\mathbb{A}^1}^1(F_a) := F_{\mathbb{A}^1}(1) \times^a F_{\mathbb{A}^1}(1).$$

With this notation in place, we can state the first computation.

**Proposition 5.8.** We have isomorphisms of sheaves of groups

$$\pi_{\mathbb{A}^1}^1(F_a) \cong \begin{cases} F_{\mathbb{A}^1}(1) \times F_{\mathbb{A}^1}(1) & \text{if } a \text{ is even, and} \\ F_{\mathbb{A}^1}(1) \times^1 F_{\mathbb{A}^1}(1) & \text{if } a \text{ is odd.} \end{cases}$$

Furthermore, the sheaves of groups $F_{\mathbb{A}^1}(1) \times F_{\mathbb{A}^1}(1)$ and $F_{\mathbb{A}^1}(1) \times^1 F_{\mathbb{A}^1}(1)$ are not isomorphic.

**Proof.** The first statement follows immediately from the proof of Lemma [3.11] the isomorphisms of $\mathbb{A}^1$-fundamental groups are induced by inclusions in the appropriate $\mathbb{A}^1$-h-cobordisms.

To establish the second statement, we study the morphism of sheaves $\mathbb{G}_m \to \text{Aut}(F_{\mathbb{A}^1}(1))$ in more detail. For any finitely generated separable field extension $L/k$ and any element $u \in L^*$, consider the map $\mathbb{P}^1 \to \mathbb{P}^1$ defined on homogeneous coordinates by $[1, u^a]$. This preserves the point with homogeneous coordinates $[1, 0]$, which we called $\infty$, and induces the map $F_{\mathbb{A}^1}(1)(L) \to F_{\mathbb{A}^1}(1)(L)$ that we’d like to study. If $a = 0$, this map is the trivial map.
For $a \neq 0$, we use the identification of [Mor06b, Corollary 4.34]. Indeed, $\text{Aut}(F_{A^1}(1))$ can be identified with the sheaf of units in $\mathbb{Z} \oplus K_{MW}^1$. Now, for any finitely generated separable extension $L/k$, and any $u \in L^*$, the map $L^* = G_m(L) \to \mathbb{Z} \oplus K_{MW}^1(L)$, which is not a morphism of sheaves of groups, is given by the formula $u \mapsto (1, [u])$. We thus want to compute the action of the element $(1, [u^n])$ by conjugation on an element of $F_{A^1}(1)$. In case $a = 1$, this is exactly the action mentioned in Remark 4.31 of [Mor06b], and, in particular, not the trivial action.

A^1-fundamental groups of blow-ups of points

The computation of the $A^1$-fundamental group of the blow-up of a finite collection of distinct points of $\mathbb{P}^2$ is in some ways more straightforward, though less explicit, than the computation of the $A^1$-fundamental group of a Hirzebruch surface. We study the $A^1$-fundamental groups of blow-ups of smooth schemes that are covered by affine spaces. Before we proceed, we recall that for any integer $n > 1$ and arbitrary choices of base-point, [Mor06b, Theorem 3.40] shows $\pi_{A^1}^1(A^n \setminus 0) = 1$; using Proposition 5.2 we deduce that $\pi_{A^1}^1(\mathbb{P}^n) = G_m$.

Proposition 5.9. Suppose $X$ is a smooth $k$-variety of dimension $n \geq 2$ that is covered by affine spaces and $x \in X(k)$ is a $k$-point. In case $n = 2$, we have an exact sequence of the form

$$1 \to K_{MW}^2 \to \pi_{A^1}^1(X \setminus x) \to \pi_{A^1}^1(X) \to 1.$$ 

If $n > 2$, then for any choice of base-point, the open immersion $X \setminus x \hookrightarrow X$ induces an isomorphism $\pi_{A^1}^1(X \setminus x) \to \pi_{A^1}^1(X)$. Furthermore, we have isomorphisms:

$$\pi_{A^1}^1(\text{Bl}_x(X)) \cong \begin{cases} \pi_{A^1}^1(X \setminus x) \star_{A^1} K_{MW}^1 F_{A^1}(1) & \text{if } n = 2, \\ \pi_{A^1}^1(X) \star_{A^1} G_m & \text{if } n > 2. \end{cases}$$

Proof. Suppose $X$ is a smooth $k$-variety of dimension $n$ covered by affine spaces, where $n \geq 2$. Either $X \cong A^n$, or we can cover $X$ by two open sets, the first isomorphic to $A^n$, the second isomorphic to $X \setminus x$, and having intersection $A^n \setminus 0$ (after applying an automorphism of affine space if necessary). Note that $A^n \setminus 0$ and $A^n$ are $A^1$-connected (e.g., they are both $A^1$-chain connected). Using this, the space $X \setminus x$ is $A^1$-connected since it admits an open cover by $A^1$-chain connected open sets (namely open sets isomorphic to $A^n \setminus 0$ and $A^n$).

We know that $\pi_{A^1}^1(A^n \setminus 0)$ is trivial if $n > 2$ and isomorphic to $K_{MW}^2$ (for any choice of base-point) for $n = 2$; this establishes the first part of the proposition in case $X = A^n$. In general, we can write $X$ as the push-out of the diagram

$$\begin{array}{ccc} A^n \setminus 0 & \to & A^n \\ \downarrow & & \downarrow \\ X \setminus x & \to & X \end{array}$$
Now, since $\mathbb{A}^n$ has trivial $\mathbb{A}^1$-fundamental group, the $\mathbb{A}^1$-van Kampen theorem (5.1) gives us an exact sequence of the form

$$1 \rightarrow \pi_1^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \rightarrow \pi_1^{\mathbb{A}^1}(X \setminus x) \rightarrow \pi_1^{\mathbb{A}^1}(X) \rightarrow 1.$$ 

We deduce that, if $n > 2$, then the open immersion $X \setminus x \rightarrow X$ induces an isomorphism $\pi_1^{\mathbb{A}^1}(X \setminus x) \sim \pi_1^{\mathbb{A}^1}(X)$. On the other hand, if $n = 2$, we get an exact sequence of the form

$$1 \rightarrow \mathbf{K}_2^{MW} \rightarrow \pi_1^{\mathbb{A}^1}(X \setminus x) \rightarrow \pi_1^{\mathbb{A}^1}(X) \rightarrow 1,$$

which establishes the first part of our statement.

Now, consider $\text{Bl}_x(X)$. If $X \cong \mathbb{A}^n$, one knows that $\text{Bl}_0(\mathbb{A}^n)$ is isomorphic to the total space of the line bundle associated with the locally free sheaf $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ over $\mathbb{P}^{n-1}$. Thus $\text{Bl}_0(\mathbb{A}^n)$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^{n-1}$, and we deduce that $\pi_1^{\mathbb{A}^1}(\text{Bl}_0(\mathbb{A}^n))$ is $\mathbb{G}_m$ if $n > 2$ and $F_{\mathbb{A}^1}(1)$ if $n = 2$.

If $X$ is covered by more than one copy of affine space, using the open cover above, together with the fact that blowing up is Zariski local, we get a Mayer-Vietoris diagram of the form

$$\begin{array}{c}
\mathbb{A}^n \setminus 0 \longrightarrow \text{Bl}_0\mathbb{A}^n \\
\downarrow \quad \downarrow \\
X \setminus x \longrightarrow \text{Bl}_xX.
\end{array}$$

**Case $n > 2$.** If $n > 2$, then we know that $\pi_1^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$ is trivial. The $\mathbb{A}^1$-van Kampen theorem then provides an isomorphism

$$\pi_1^{\mathbb{A}^1}(\text{Bl}_0(\mathbb{A}^n)) \ast^{\mathbb{A}^1} \pi_1^{\mathbb{A}^1}(X \setminus x) \sim \pi_1^{\mathbb{A}^1}(\text{Bl}_x(X)).$$

Thus, by the discussion in the case of $\mathbb{A}^n$, it follows that $\pi_1^{\mathbb{A}^1}(\text{Bl}_x(X))$ is isomorphic to the amalgamated sum $\pi_1^{\mathbb{A}^1}(X \setminus x) \ast^{\mathbb{A}^1} \mathbb{G}_m$. Finally, the first part of the proposition allows us to conclude that $\pi_1^{\mathbb{A}^1}(X \setminus x) \sim \pi_1^{\mathbb{A}^1}(X)$.

**Case $n = 2$.** For $n = 2$, we have $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) = \mathbf{K}_2^{MW}$, which we recall is abelian. Since $\text{Bl}_0(\mathbb{A}^2)$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1$, we know $\pi_1^{\mathbb{A}^1}(\text{Bl}_0(\mathbb{A}^2)) \cong F_{\mathbb{A}^1}(1)$.

Much more can be said if we use aspects of $\mathbb{A}^1$-homology developed in [Mor06b, §3.2]. We claim the canonical morphism $H_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \rightarrow H_1^{\mathbb{A}^1}(X \setminus x)$ is an isomorphism. Indeed, by assumption, $X$ can be covered by open sets isomorphic to affine space, and we can use the Mayer-Vietoris sequence (see [AD07a]) together with a straightforward induction argument to deduce this fact. Since $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ is abelian, the $\mathbb{A}^1$-Hurewicz theorem ([Mor06b, Theorem 3.57]) shows that the morphism

$$\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \rightarrow \pi_1^{\mathbb{A}^1}(X \setminus x)$$

factors through the abelianization of $\pi_1^{\mathbb{A}^1}(X \setminus x)$. Since $\text{Bl}_0(\mathbb{A}^2)$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1$, the upper horizontal map of the diagram is, up to $\mathbb{A}^1$-weak equivalence, a morphism $\mathbb{A}^2 \setminus 0 \rightarrow \mathbb{P}^1$. The induced map on $\mathbb{A}^1$-fundamental groups is thus a map $\mathbf{K}_2^{MW} \rightarrow F_{\mathbb{A}^1}(1)$. □
Corollary 5.10. Suppose $m$ is an integer $\geq 3$. For distinct points $x_1, \ldots, x_n \in \mathbb{P}^m(k)$, we have an isomorphism of strongly $\mathbb{A}^1$-invariant sheaves of groups:
\[
\mathbb{G}_m \ast \mathbb{A}^1 \ast \cdots \ast \mathbb{A}^1 \mathbb{G}_m \xrightarrow{\cong} \pi_1^\mathbb{A} \left( \text{Bl}_{x_1, \ldots, x_n}(\mathbb{P}^m) \right),
\]
where the amalgamated sum on the left hand side has $n$-factors of $\mathbb{G}_m$.

Proof. This follows immediately by induction from Proposition 5.9. \qed

Example 5.11. We can be somewhat more explicit about the structure of some of the above amalgamated products. For example, for any point $x \in \mathbb{P}^m(k)$, there is a $\mathbb{G}_m \times \mathbb{A}^2$-torsor $\mathbb{A}_m \times \mathbb{A}_2 \to \text{Bl}_x(\mathbb{P}^m)$. In particular, if $m > 2$, Proposition 5.2 shows that this $\mathbb{G}_m \times \mathbb{A}^2$-torsor gives rise to an exact sequence of the form
\[
1 \longrightarrow \mathbb{K}^\text{MW}_2 \longrightarrow \mathbb{G}_m \ast \mathbb{A}^1 \mathbb{G}_m \longrightarrow \mathbb{G}_m \times \mathbb{G}_m \longrightarrow 1.
\]
More generally, for any integer $n \geq 2$, $\mathbb{A}^1$-covering space theory of [Mor06b, §4.1] can be used to show that one always has a surjective homomorphism:
\[
\mathbb{G}_m \ast \mathbb{A}^1 \ast \cdots \ast \mathbb{A}^1 \mathbb{G}_m \longrightarrow \mathbb{G}_m \times \cdots \times \mathbb{G}_m,
\]
where both sides contain $n$ copies of $\mathbb{G}_m$. Precisely, one can check that the blow-up of $n$-points of $\mathbb{P}^m$ has an $\mathbb{A}^1$-covering space corresponding to a torsor under the torus dual to the Picard group, and the above map is just the abelianization map. The factor on the right can be thought of as having motivic weight 1, and the kernel of this homomorphism, can be thought of as having motivic weight 2. A similar filtration should exist on the $\mathbb{A}^1$-fundamental group of any smooth $\mathbb{A}^1$-connected $k$-variety.

Corollary 5.12. Let $n$ be an integer $\geq 1$. For distinct points $x_1, \ldots, x_n \in \mathbb{P}^2(k)$, there is a surjective homomorphism of strongly $\mathbb{A}^1$-invariant sheaves of groups
\[
F_\mathbb{A}^1(1) \ast \mathbb{A}^1 \ast \cdots \ast \mathbb{A}^1 F_\mathbb{A}^1(1) \longrightarrow \pi_1^\mathbb{A} \left( \text{Bl}_{x_1, \ldots, x_n}(\mathbb{P}^2) \right),
\]
where the amalgamated sum on the left hand side has $n$ factors.

Remark 5.13. Note that Proposition 5.9 gives an inductive description of the $\mathbb{A}^1$-fundamental group of the blow-up of finitely many distinct points on $\mathbb{P}^2$. In particular, it follows immediately from the proof that the number of points being blown up is an invariant of the $\mathbb{A}^1$-fundamental group. Furthermore, the proof shows that, in some sense, the $\mathbb{A}^1$-fundamental group “sees” the fundamental group of the real points (the connected sum of some number of copies of $\mathbb{R} \mathbb{P}^2$). However, it is unclear how to give a simple closed form expression for these sheaves of groups in a manner similar to Corollary 5.10.

Finally, we can deduce the remaining theorem statement (Theorem 1.13) from §1. Indeed, combining Theorem 3.8, Proposition 5.8 and Proposition 5.9 we obtain the following result.

Corollary 5.14. If $X$ and $Y$ are two rational smooth proper surfaces over an algebraically closed field, the following conditions are equivalent.
i) The varieties $X$ and $Y$ are $\mathbb{A}^1$-h-cobordant.

ii) The varieties $X$ and $Y$ are $\mathbb{A}^1$-weakly equivalent.

iii) The varieties $X$ and $Y$ have isomorphic $\mathbb{A}^1$-fundamental groups.

Furthermore the set $\mathcal{S}_{\mathbb{A}^1}(X)$ consists of exactly 1 element.

**Remark 5.15.** Assuming Conjecture 2.13, Corollary 5.14 provides the classification of $\mathbb{A}^1$-connected surfaces over an algebraically closed field. Combining Corollaries 2.41 and 5.14 we obtain the solution to the $\mathbb{A}^1$-surgery problem (1.10) for smooth proper $\mathbb{A}^1$-connected surfaces over fields $k$ having characteristic 0. If $k$ is not algebraically closed, Corollary 5.14 will not provide the classification of smooth proper $\mathbb{A}^1$-connected surfaces due to the examples of stably $k$-rational, non rational surfaces (see Example 2.24). Thus, minimality is presumably not preserved by field extension. Nevertheless, we expect that smooth proper $\mathbb{A}^1$-connected surfaces over an arbitrary field are classified up to $\mathbb{A}^1$-h-cobordism by their $\mathbb{A}^1$-fundamental group.

**Extension: torsion of an $\mathbb{A}^1$-weak equivalence**

We use the notation of §1. Barden, Mazur and Stallings reconsidered the $h$-cobordism theorem in the non-simply connected case using J.H.C. Whitehead’s notion of torsion of a homotopy equivalence. The $s$-cobordism theorem states that an $h$-cobordism $(W, M, M')$ of manifolds of dimension $\geq 5$ such that the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are simple homotopy equivalences (i.e., the torsion vanishes) is diffeomorphic to a product. Furthermore, $h$-cobordisms $(W, M, M')$ with $\dim M \geq 5$ are parameterized by the Whitehead group of $M$, which is a certain quotient of the algebraic $K_1$ of the group algebra $\mathbb{Z}[\pi_1(M)]$. (Note: in low dimensions, $h$-cobordisms between simply connected manifolds can fail to be products.)

We know that $\mathbb{A}^1$-h-cobordisms of strictly positive dimensional smooth proper $\mathbb{A}^1$-connected varieties are always non-trivial by Proposition 5.3. The $s$-cobordism theorem suggests an explanation for non-triviality of such $\mathbb{A}^1$-h-cobordisms in terms of the $\mathbb{A}^1$-fundamental group. One can formulate a notion of $\mathbb{A}^1$-Whitehead torsion of an $\mathbb{A}^1$-weak equivalence. Suppose given an $\mathbb{A}^1$-h-cobordism $(W, f)$ between smooth proper $\mathbb{A}^1$-connected $k$-varieties $X$ to $X'$. The inclusion $X \hookrightarrow W$ induces a morphism of $\mathbb{A}^1$-singular chain complexes (see [Mor06b] §3.2 for a definition), and the cone of this morphism is an $\mathbb{A}^1$-contractible chain complex of (sheaves of) modules over the (sheaf of) group algebra(s) $\mathbb{Z}[\pi_1^{\mathbb{A}^1}(X)]$. When the $\mathbb{A}^1$-singular chain complexes are sufficiently well understood, one can associate with this complex a computable $\mathbb{A}^1$-Whitehead torsion. Optimistically, one can hope for an $\mathbb{A}^1$-s-cobordism theorem stating that $\mathbb{A}^1$-h-cobordisms can be parameterized by an appropriately defined Whitehead group of the $\mathbb{A}^1$-fundamental group.

**Remark 5.16.** The computations above suggest that the $\mathbb{A}^1$-Whitehead torsion will likely be quite complicated in general. According to Corollary 5.14, non-minimal $\mathbb{A}^1$-homotopy types can, in general, contain moduli of non-isomorphic varieties. On the other hand, the minimal $\mathbb{A}^1$-homotopy types for surfaces over an algebraically closed field are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, and
each of these $A^1$-homotopy types only contains a discrete set of isomorphism classes of smooth proper varieties.

Extension: determining $\mathcal{S}_{A^1}(X)$

Given a finite CW complex $X$, we now recall some aspects of the surgery problem and determination of the structure set $\mathcal{S}(X)$. If $X$ is homotopy equivalent to a manifold then (a) the cohomology of $X$ satisfies Poincaré duality, and (b) $X$ has a tangent bundle or, by Atiyah duality, $X$ has a stable normal bundle. Amazingly, these two pieces of data turn out to be essentially sufficient to identify manifolds among CW complexes, provided certain compatibility conditions are satisfied.

A finite CW complex satisfying Poincaré duality is called a geometric Poincaré complex. Any geometric Poincaré complex $X$ admits a Spivak normal fibration, which is a homotopy theoretic substitute for the stable normal bundle. The Spivak normal fibration is a homotopy sphere bundle and is “classified” by a map $\nu_X : X \to BG$ where $BG$ is the colimit of the classifying spaces of the monoids of homotopy self-equivalences of the sphere of dimension $n$ for a natural sequence of inclusions. If $X$ is homotopy equivalent to a manifold, the Spivak normal fibration will admit a vector bundle reduction (classifying the stable normal bundle). If $BO$ denotes the classifying space for the stable orthogonal group, there is a map $i : BO \to BG$ that induces the Hopf-Whitehead $J$-homomorphism. A vector bundle reduction of $\nu_X$ is a lift along $i$. The homotopy cofiber of $i$ can be identified with $B(G/O)$ and reductions exist if and only if the induced map $X \to B(G/O)$ is homotopically trivial. A primary obstruction to $X$ admitting a manifold structure is the homotopic triviality of this map, and if this obstruction vanishes, lifts are classified by the set of homotopy classes of maps $[X, G/O]$.

The secondary surgery obstruction provides a map from $[X, G/O]$ to a group $L_n(\mathbb{Z}(\pi_1(X)))$ defined in terms of complexes of $\mathbb{Z}(\pi_1(X))$-modules with duality (where $n$ is the “formal” dimension of $X$). Finally, the structure set $\mathcal{S}(X)$ fits into an exact sequence of sets of the form:

$$L_{n+1}(\mathbb{Z}(\pi_1(X))) \to \mathcal{S}(X) \to [X, G/O] \to L_n(\mathbb{Z}(\pi_1(X)))$$

called the surgery exact sequence. Our notation signifies that group $L_{n+1}(\mathbb{Z}(\pi_1(X)))$ acts on the set $\mathcal{S}(X)$, and the last map on the right hand side is not in general a group homomorphism (even though both its source and target are groups)!

For an appropriate analog in $A^1$-homotopy theory, note that Algebraic K-theory is representable in the $A^1$-homotopy category (see [MV99 §4 Theorem 3.13]). Smooth schemes have tangent bundles, which in some situations are classified by maps to an infinite Grassmannian ([Mor07]), and the statement of $A^1$-Atiyah duality for smooth projective schemes (see [Hu05 Theorem A.1] or [Rio05 Théorème 2.2]) tells us how to define the notion of an $A^1$-Poincaré complex.

One may define the natural analog of “$G$” as the $\pm 1$-components of the $\mathbb{P}^1$-infinite loop space $Q_{\mathbb{P}^1}S^0_k$ corresponding to the $\mathbb{P}^1$-sphere spectrum $S^0_k$. Except at the “zeroth” level, the $A^1$-homotopy groups of this space coincide with the stable motivic homotopy groups of spheres. One needs to prove existence of analogs of Spivak normal fibrations for $A^1$-Poincaré complexes.
To develop the primary K-theory obstruction for Problem 1.10, one needs to study the “(sheaf theoretic) motivic $J$-homomorphism,” and the “$\mathbb{P}^1$-loop space recognition problem” as mentioned by Voevodsky (to show that spaces like “$G/GL$” are $\mathbb{P}^1$-infinite loop spaces). Adopting this point of view, computations of stable $\mathbb{A}^1$-homotopy (sheaves of) groups (e.g., [Mor04b]) have bearing on the geometry and arithmetic of algebraic varieties. Analogs of the secondary surgery obstruction theory (even conjectural) involving the $\mathbb{A}^1$-fundamental group are still mysterious.

6 Birational sheaves and $\mathbb{A}^1$-chain connectedness

The goal of this section is to establish Theorem 2.38. We have deferred the proof of this result here because, while it is in a sense elementary, the techniques used in the proof differ substantially from those (explicitly) used in previous sections. We present the proof in outline here.

Proof of Theorem 2.38. We will construct a new sheaf $\pi^{b\mathbb{A}^1}_0(X)$ (see Definition 6.14) that is explicitly $\mathbb{A}^1$-invariant in the sense of Definition 4.16 and has an additional birational property. The existence of the sheaf $\pi^{b\mathbb{A}^1}_0(X)$, together with the proof of its $\mathbb{A}^1$-invariance, follows from Theorem 6.10; this point is both technically and notationally the most complicated part of the proof. We will show in Proposition 6.15 that there is a factorization

$$\pi^{ch}_0(X) \longrightarrow \pi^{b\mathbb{A}^1}_0(X) \longrightarrow \pi^{b\mathbb{A}^1}_0(X)$$

inducing a bijection between sections of the first and last sheaves over finitely generated separable extensions $L/k$ that establishes the required result.

Remark 6.1. We believe that techniques analogous to those developed in this section can be used to show that, at least for $X$ proper, étale $\mathbb{A}^1$-chain connectedness (in the sense of Remark 2.35) is equivalent to étale $\mathbb{A}^1$-connectedness (in the sense of Definition 2.31).

Birational and $\mathbb{A}^1$-invariant sheaves

To establish the propositions referenced in the proof of Theorem 2.38 above, we need to introduce some terminology. For $X \in Sm_k$, we write $X^{(p)}$ for the set of codimension $p$ points of $X$. We introduce a notion of birational and $\mathbb{A}^1$-invariant sheaves along the lines of the axiomatic framework developed in [Mor06b, §1.1]; the superscript “$b\mathbb{A}^1$” in $\pi^{b\mathbb{A}^1}_0$ indicates these properties. We refer the reader to [CT95, §2] and [Mor06b, §1.1] for some discussion of ideas leading up to the next definition.

Definition 6.2. Suppose $\mathcal{S}$ is a presheaf of sets on $Sm_k$. We will say that $\mathcal{S}$ is birational if it satisfies the following two properties.
(i) For any $X \in Sm_k$ having irreducible components $X_\eta$ ($\eta \in X^{(0)}$), the map

$$S(X) \longrightarrow \prod_{\eta \in X^{(0)}} S(X_\eta)$$

is a bijection.

(ii) For any $X \in Sm_k$, and any open dense subscheme $U \subset X$, the restriction map $S(X) \to S(U)$ is a bijection.

A sheaf $S$ of sets on $Sm_k$ is called birational if the underlying presheaf of sets on $Sm_k$ is birational.

**Lemma 6.3.** If $S$ is a birational presheaf on $Sm_k$, then $S$ is automatically a (Nisnevich) sheaf.

**Proof.** Using [MV99, §3 Proposition 1.4], it suffices to show that given any Nisnevich distinguished square (see loc. cit. Definition 1.3), the induced square of sets obtained by applying $S$ is Cartesian. This property follows immediately from the definition of birationality. \(\square\)

**Notation 6.4.** Let $Shv^{bh1}_k$ denote the full subcategory of $Shv_k$ consisting of sheaves that are both birational in the sense of Definition 6.2 and $A^1$-invariant in the sense of Definition 4.16.

**An equivalence of categories**

Let $F_k$ denote the category whose objects are finitely generated separable extension fields $L$ of $k$ and whose morphisms are field extensions.

**Notation 6.5.** We write $F_k - Set$ for the category of covariant functors from $F_k$ to the category of sets.

Suppose that $(L, \nu)$ is a pair consisting of a finitely generated separable field extension $L/k$ and a discrete valuation $\nu$ on $L$; write $O_\nu$ for the corresponding discrete valuation ring and $\kappa_\nu$ for associated residue field. Given an object $S \in Shv^{bh1}_k$, birationality of $S$ implies that the map $S(Spec O_\nu) \to S(Spec L)$ is a bijection. The morphism $O_\nu \to \kappa_\nu$ induces a map $S(Spec O_\nu) \to S(Spec \kappa_\nu)$ that when composed with the inverse to the aforementioned bijection induces a specialization map $S(L) \to S(\kappa_\nu)$.

**Definition 6.6.** The category $F^e_k - Set$ has as objects elements $S \in F_k - Set$ together with the following additional structure:

(R) for each pair $(L, \nu)$ consisting of $L \in F_k$ and a discrete valuation $\nu$ on $L$ with residue field $\kappa_\nu$ separable over $k$, a specialization (or residue) morphism $s_\nu : S(L) \to S(\kappa_\nu)$.

Morphisms in the category $F^e_k - Set$ are those natural transformations of functors preserving the data (R).
With these definitions, evaluation on sections determines a functor

\[
\text{Shv}_{k}^{hA^1} \longrightarrow \mathcal{F}_{k}^{r} - \text{Set}
\]

that we refer to simply as restriction. We will see momentarily that restriction is fully-faithful; let us first study its essential image.

**Definition 6.7.** An object \( S \in \mathcal{F}_{k}^{r} - \text{Set} \) is called sheaflike if it has the following three properties.

**(A1)** Given pairs \((L, \nu)\) and \((L', \nu')\) consisting of finitely generated separable extensions \( k \subset L \subset L' \) such that \( \nu' \) restricts to a discrete valuation \( \nu \) on \( L \) with ramification index 1, and both \( \kappa_\nu \) and \( \kappa_{\nu'} \) are separable over \( k \), the following diagram commutes:

\[
\begin{array}{ccc}
S(L) & \longrightarrow & S(L') \\
\downarrow & & \downarrow \\
S(\kappa_\nu) & \longrightarrow & S(\kappa_{\nu'}). \\
\end{array}
\]

**(A2)** Given pairs \((L, \nu)\) and \((L', \nu')\) consisting of finitely generated separable extensions \( k \subset L \subset L' \) such that \( \nu' \) restricts to 0 on \( L \), and \( \kappa_{\nu'} \) is separable over \( k \), the composite map \( S(L) \to S(L') \to S(\kappa_{\nu'}) \) and the map \( S(L) \to S(\kappa_\nu) \) are equal.

**(A3)** For any \( X \in \text{Sm}_k \) irreducible with function field \( F \), for any point \( z \in X^{(2)} \) of codimension 2 with residue field \( \kappa(z) \) separable over \( k \), and for any point \( y_0 \in X^{(1)} \) with residue field \( \kappa(y_0) \), such that \( z \in y_0 \) and such that \( y_0 \in \text{Sm}_k \), the composition

\[
S(F) \longrightarrow S(\kappa(y_0)) \longrightarrow S(\kappa(z))
\]

is independent of the choice of \( y_0 \).

If furthermore, \( S \) has the following property, we will say that it is a sheaflike and \( A^1 \)-invariant \( \mathcal{F}_{k}^{r} - \text{Set} \).

**(A4)** For any \( L \in \mathcal{F}_k \), the map \( S(L) \to S(L(t)) \) is a bijection.

**Theorem 6.8.** The restriction functor

\[
\text{Shv}_{k}^{hA^1} \longrightarrow \mathcal{F}_{k}^{r} - \text{Set}
\]

of Equation 6.1 is fully-faithful and has essential image the full subcategory of \( \mathcal{F}_{k}^{r} - \text{Set} \) spanned by objects that are sheaflike and \( A^1 \)-invariant.

**Proof.** The full-faithfulness follows easily from birationality. It is also straightforward to check that the essential image of restriction is contained in the full subcategory of \( \mathcal{F}_{k}^{r} - \text{Set} \) spanned by objects that are sheaflike and \( A^1 \)-invariant. Indeed, (A1) follows immediately from the
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property of being a Nisnevich sheaf (via the distinguished square characterization), both (A2) and (A3) follow by choosing explicit smooth models for appropriate closed immersions. We will thus show how to construct an explicit quasi-inverse functor. Suppose given an object $\mathcal{S} \in \mathcal{F} \rightarrow \text{Set}$ that is sheaflike and $\mathbb{A}^1$-invariant. We define a presheaf $\tilde{\mathcal{S}}$ as on $\mathcal{S}m_k$ as follows.

**Step 1.** To each irreducible $U \in \mathcal{S}m_k$ assign the set $\mathcal{S}(k(U))$, and extend this assignment in the unique way to make Property (i) of Definition 6.2 hold.

**Step 2.** Given a morphism $f: Y \rightarrow X$, we define a morphism $\tilde{\mathcal{S}}(f): \tilde{\mathcal{S}}(X) \rightarrow \tilde{\mathcal{S}}(Y)$ as follows. If $f$ is dominant, we define $\tilde{\mathcal{S}}(f)$ to be the induced morphism $\mathcal{S}(f): \mathcal{S}(k(X)) \rightarrow \mathcal{S}(k(Y))$. In the case $f: Y \rightarrow X$ is a closed immersion of smooth schemes, we proceed as follows. Let $\mathcal{N}_f$ denote the normal bundle to the immersion. Consider the diagram

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{N}_f) & \xrightarrow{\iota} & \text{Bl}_Z(X) \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Y & \xrightarrow{f} & X
\end{array}
$$

The top horizontal morphism is a codimension 1 closed immersion, the left vertical arrow is a Zariski locally trivial morphism with projective space fibers, and the right vertical morphism is a proper birational morphism (in particular dominant). Using Zariski local triviality of $\pi'$, together with the fact that $\mathcal{S}$ is $\mathbb{A}^1$-invariant and $\tilde{\mathcal{S}}$ only depends on the function field of its input, we observe that $\tilde{\mathcal{S}}(\mathbb{P}(\mathcal{N}_f)) = \tilde{\mathcal{S}}(Y)$. Again using the fact that $\tilde{\mathcal{S}}$ only depends on the function field of its input, we observe that $\tilde{\mathcal{S}}(X_Z) = \tilde{\mathcal{S}}(X)$. The top vertical morphism is a codimension 1 closed immersion, so we can define $\tilde{\mathcal{S}}(X_Z) \rightarrow \tilde{\mathcal{S}}(\mathbb{P}(\mathcal{N}_f))$ using the specialization morphism for the associated morphism of function fields.

**Step 3.** A general morphism $f: Y \rightarrow X$ can be factored as a closed immersion $Y \rightarrow Y \times X$ (the graph) followed by a projection (which is dominant). We can then define $\tilde{\mathcal{S}}(f)$ as the composite of these two morphisms. We now need to check that the above constructions are actually compatible and define a functor. These facts are checked in Lemma 6.9.

Assuming these compatibilities, note that $\tilde{\mathcal{S}}$ is by construction a birational and $\mathbb{A}^1$-invariant presheaf. Thus, Lemma 6.3 shows $\tilde{\mathcal{S}}$ actually defines a birational and $\mathbb{A}^1$-invariant sheaf. Furthermore, it is straightforward to check that this construction provides an inverse to restriction.

**Lemma 6.9.** Continuing with notation as in Theorem 6.8 (and its proof), we have the following two facts.

- **Given a Cartesian square of the form**

$$
\begin{array}{ccc}
Y' & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \rightarrow & X
\end{array}
$$

where the vertical morphisms are smooth and the horizontal morphisms are closed immersions, the diagram of sets obtained by applying $\tilde{\mathcal{S}}$ commutes. 

\]
Given a sequence of closed immersions \( Z \hookrightarrow Y \hookrightarrow X \), the triangle (of sets) obtained by applying \( \tilde{S} \) commutes.

**Proof.** The first point follows immediately from the functorial properties of blow-ups together with \( \mathbb{A}^1 \)-invariance and the fact that \( S \) is an \( Fk \)-Set.

For the second point, we proceed as follows. Let \( \mathcal{N} \) denote the normal bundle to the closed immersion \( Y \hookrightarrow X \). Consider the morphism \( \text{Bl}_Y(X) \rightarrow X \). Pulling back this morphism along the closed immersion \( Z \hookrightarrow Y \) induces the projective bundle \( \mathbb{P}(\mathcal{N}|_Z) \rightarrow Z \). Again, using the fact that \( \tilde{S} \) only depends on the function field of its argument together with \( \mathbb{A}^1 \)-invariance, we conclude that \( \tilde{S}(Z) = \tilde{S}(\mathbb{P}(\mathcal{N}|_Z)) \) and \( \tilde{S}(Y) = \tilde{S}(\mathbb{P}(\mathcal{N})) \). This observation reduces us to proving the property when \( Y \) is a codimension 1 closed subscheme of \( X \). Repeating this process for \( Z \), we can reduce to the case where \( Z \subset Y \subset X \) is a sequence of codimension 1 closed immersions. In this case, using the fact that \( S \) is sheaflike, we can apply Property \((A3)\) to finish.

Birational connected components associated with proper schemes

Via Theorem 6.8, to define a birational and \( \mathbb{A}^1 \)-invariant sheaf, it suffices to specify an object of \( F_k - \text{Set} \) and check the properties listed in Definition 6.7. The main goal of this section is to prove the following result.

**Theorem 6.10.** Suppose \( X \) is proper scheme having finite type over a field \( k \). Denote, by abuse of notation, the Nisnevich sheaf defined by (the functor of points of) \( X \). There exists a birational and \( \mathbb{A}^1 \)-invariant sheaf \( \pi_{\mathbb{A}^1}^{\text{bir}}(X) \) together with a morphism of sheaves \( X \rightarrow \pi_{\mathbb{A}^1}^{\text{bir}}(X) \) such that, for any \( L \in F_k \), the induced map \( X(L) \rightarrow \pi_{\mathbb{A}^1}^{\text{bir}}(X)(L) \) factors through a bijection

\[
X(L)/\sim \cong \pi_{\mathbb{A}^1}^{\text{bir}}(X)(L)
\]

(recall Notation 6.8).

**Remark 6.11.** One can show that the canonical morphism \( X \rightarrow \pi_{\mathbb{A}^1}^{\text{bir}}(X) \) of the previous result is initial among morphisms of \( X \) to birational \( \mathbb{A}^1 \)-invariant sheaves, but we will not need this fact.

Suppose given a finitely generated separable extension \( L \) of \( k \), and a discrete valuation \( \nu \) on \( L \) with associated valuation ring \( O_\nu \) and residue field \( \kappa_\nu \). Assume \( \kappa_\nu \) is separable over \( k \). Consider the \( F_k - \text{Set} \) defined by

\[
L \mapsto X(L)/\sim.
\]

The valuative criterion of properness induces a canonical bijection \( X(O_\nu) = X(L) \). Furthermore, the morphism \( O_\nu \rightarrow \kappa_\nu \) induces a morphism \( X(O_\nu) \rightarrow X(\kappa_\nu) \).

**Lemma 6.12.** If \( X \) is a proper scheme having finite type over a field \( k \), the composite morphism \( X(L) = X(O_\nu) \rightarrow X(\kappa_\nu) \rightarrow X(\kappa_\nu)/\sim \) factors through a morphism

\[
s_\nu : X(L)/\sim \rightarrow X(\kappa_\nu)/\sim
\]

providing the data \((R)\) of Definition 6.6. Denote by \( \varpi_{\mathbb{A}^1}^{\text{bir}}(X) \) the \( F_k - \text{Set} \) that associates with \( L \in F_k \) the set \( X(L)/\sim \).
Proposition 6.13. The object $\pi_0^{bh^1}(X)$ of $\mathcal{F}_k$—Set from Lemma 6.12 is both sheaflike and $\mathbb{A}^1$-invariant in the sense of Definition 6.7.

The proofs of these results require some technical results about surfaces that we recall below. Assuming for the moment the truth of this lemma and proposition, let us finish the proofs of the other results stated above.

Definition 6.14. Suppose $X$ is proper scheme having finite type over a field $k$. Write $\pi_0^{bh^1}(X)$ for the birational and $\mathbb{A}^1$-invariant sheaf associated with the the object $\pi_0^{bh^1}(X)$ of $\mathcal{F}_k$—Set via the equivalence of categories of Theorem 6.8. We refer to $\pi_0^{bh^1}(X)$ as the sheaf of birational $\mathbb{A}^1$-connected components of $X$.

Proof of Theorem 6.10. After Lemma 6.12 and Proposition 6.13 it remains to construct the morphism of sheaves $X \to \pi_0^{bh^1}(X)$. Let us construct a morphism of the underlying presheaves. For any smooth $k$-scheme $U$, there is a well-defined map

$$X(U) \to X(U)/\sim \to \pi_0^{bh^1}(X)(U)$$

induced by functoriality of the construction $U \mapsto X(U)/\sim$ (“shrink $U$”). The associated morphism of sheaves is the required morphism. Observe that the induced morphism $X \to \pi_0^{bh^1}(X)$ factors through the canonical epimorphism $X \to \pi_0^{ch^1}(X)$ of Remark 2.11.

Proposition 6.15. If $X$ is a proper scheme having finite type over a field $k$, there is a canonical map $\pi_0^{ch^1}(X) \to \pi_0^{bh^1}(X)$ such that the composite map $\pi_0^{ch^1}(X) \to \pi_0^{bh^1}(X) \to \pi_0^{bh^1}(X)$ (the first map is the canonical epimorphism of Remark 2.11) induces a bijection on sections over finitely generated separable extension fields $L/k$.

Proof. By Theorem 6.10 we know that $\pi_0^{bh^1}(X)$ is a birational and $\mathbb{A}^1$-invariant sheaf. This fact implies the space $\pi_0^{bh^1}(X)$ is $\mathbb{A}^1$-local by the equivalent conditions of [MV99, §2 Proposition 3.19]. By the universal property of $\mathbb{A}^1$-localization, the canonical map $X \to \pi_0^{bh^1}(X)$ factors uniquely through the $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(X)$ of $X$ (see [MV99, §2 Theorem 3.2] for this notation). Thus, we obtain a factorization

$$X \to L_{\mathbb{A}^1}(X) \to \pi_0^{bh^1}(X).$$

The second morphism induces for any $U \in Sm_k$ a morphism $[U, L_{\mathbb{A}^1}(X)]_s \to [U, \pi_0^{bh^1}(X)]_s$ functorial in $U$. The set $[U, \pi_0^{bh^1}(X)]_s$ coincides with $\pi_0^{bh^1}(X)(U)$ by [MV99, §2 Remark 1.14]. Sheafifying for the Nisnevich topology produces a morphism of sheaves

$$\pi_0^{ch^1}(X) \to \pi_0^{bh^1}(X).$$

Finally, note that the morphism $X \to L_{\mathbb{A}^1}(X)$ induces the morphism of sheaves $X \to \pi_0^{ch^1}(X)$, and this morphism factors through $\pi_0^{ch^1}(X)$.

Conjecture 6.16. For any proper scheme $X$ having finite type over a field $k$, the canonical morphism $\pi_0^{ch^1}(X) \to \pi_0^{bh^1}(X)$ is an isomorphism, i.e., $\pi_0^{ch^1}(X)$ is birational and $\mathbb{A}^1$-invariant.
Some technical results about surfaces

We will recall a pair of results that will be used in the Proofs of Lemma 6.12 and Proposition 6.13. Throughout, $X$ denotes a proper scheme having finite type over a field $k$.

Lemma 6.17. Let $S$ be an irreducible essentially smooth $k$-scheme of dimension 2 with function field $F$, and suppose $\alpha \in X(F)$. There exist finitely many closed points $z_1, \ldots, z_r$ in $S$, a proper birational morphism $f : S_\alpha \to S$, and $\beta \in X(S_\alpha)$ such that $S_\alpha$ is regular, $f$ is an isomorphism over the complement of $z_1, \ldots, z_r$ in $S$, and $\beta$ restricts to $\alpha$ under the induced map $X(S_\alpha) \to X(F)$.

Proof. Recall that $S$ is essentially smooth if it can be written as limit of smooth schemes having finite type over $k$ with smooth affine bonding morphisms. In any case, this result follows from a form of resolution of indeterminacy. As $X$ is proper, $\alpha$ extends to a morphism $\Omega \to X$ where $\Omega \subset S$ is an open subscheme whose complement consists of finitely many closed points $z_1, \ldots, z_r$. Denote by $S'$ the closure of the graph of the induced morphism $\Omega \to X$. The projection $S' \to S$ is then proper and an isomorphism over $\Omega$. By [Lip78, p. 101 Theorem and p. 155 (B)], there exists a morphism $S_\alpha \to S'$ with $S_\alpha$ regular that is proper, birational and isomorphism on $\Omega$. The composite map $S_\alpha \to X$ is the required morphism $\beta$.

Lemma 6.18. Given a proper birational morphism $S' \to S$ between regular $k$-schemes of dimension 2, for any point $s$ of $S$, the scheme-theoretic fiber $S'_{\kappa(s)} := S' \times_S \text{Spec} \kappa(s)$ is a $\kappa(s)$-scheme that is $\mathbb{A}^1$-chain connected.

Proof. This result follows immediately from a strong factorization style result. E.g., [Lic68, Section II Theorem 1.15] states that a proper birational morphism between regular surfaces is a composition of blow-ups at closed points. The fiber over any point $s$ in $S$ is a proper variety and one just needs to apply Proposition 2.23 (note: that result is stated under they hypothesis $k$ is perfect, but that assumption is used only to establish that regular varieties over $k$ are in fact smooth).

Proofs of Lemma 6.12 and Proposition 6.13

Proof of Lemma 6.12. Suppose $L \in \mathcal{F}_k$, and suppose $\nu$ is a discrete valuation on $L$ with associated valuation ring $\mathcal{O}_\nu$, and residue field $\kappa_\nu$ assumed to be separable over $k$. We want to prove that $X(L) \to X(\kappa_\nu)/\sim$ factors through $\mathbb{A}^1$-equivalence of points (see the discussion just prior to Notation 2.8). To do this, it suffices to prove that given a pair points $x_0, x_1 \in X(L)$, and a morphism $H : \mathbb{A}^1_L \to X$ restricting to $x_0$ and $x_1$, the image of $x_0$ and $x_1$ under the map $X(L) \cong X(\mathcal{O}_\nu) \to X(\kappa_\nu)$ are $\mathbb{A}^1$-chain equivalent.

Consider the open immersion $\mathbb{A}^1_L \to \mathbb{A}^1_{\mathcal{O}_\nu}$. Applying Lemma 6.17, we can assume that $H$ is defined on an open subscheme $U \subset \mathbb{A}^1_{\mathcal{O}_\nu}$ whose complement is a collection of finitely many closed points $z_1, \ldots, z_r \in \mathbb{A}^1_{\mathcal{O}_\nu}$, and extends to a morphism $\beta : S_\alpha \to X$, where $S_\alpha$ is smooth, and $S_\alpha \to \mathbb{A}^1_{\mathcal{O}_\nu}$ is a proper birational morphism that is an isomorphism over $U$.

Again using the valuative criterion of properness, the $L$-points $x_i$ of $X$ uniquely extend to $\mathcal{O}_\nu$-points of $X$ ($i \in \{0, 1\}$). We view the sections $\mathcal{O}_\nu \to \mathbb{A}^1_{\mathcal{O}_\nu}$ associated with these points
as regular closed subschemes $C_i$ in $\mathbb{A}^1_{\mathcal{O}_Y}$. The proper transforms $\tilde{C}_i$ of $C_i$ in $S_\alpha$ are closed subschemes that map properly and birationally onto $C_i$. This observation implies that the maps $\tilde{C}_i \to C_i$ are in fact isomorphisms. Thus, the closed points $z_i$ in $C_i$ lift uniquely to closed points $\tilde{z}_i$ in $\tilde{C}_i$.

Again for $i \in \{0, 1\}$, the image of $x_i$ under the map $X(L) \cong X(\mathcal{O}_\nu) \to X(\kappa_\nu)$ is the composition

$$\text{Spec } \kappa_\nu \xrightarrow{x_i} C_i \to X$$

where $C_i \to X$ is determined by $x_i \in X(L)$. Using the observations of the last paragraph, this composite map factors as

$$\text{Spec } \kappa_\nu \xrightarrow{\tilde{z}_i} \tilde{C}_i \to S_\alpha \xrightarrow{\beta} X;$$

the morphisms $\tilde{C}_i \to S_\alpha \xrightarrow{\beta} X$ and $C_i \to S$ are equal because they agree on $\text{Spec } L$.

The two points $z_i$ lie in $\mathbb{A}^1_{\mathcal{O}_Y}$ over the smooth curve $\mathbb{A}^1_{\mathcal{O}_\nu}$ whose proper transform $D$ in $S_\alpha$ is isomorphic to $\mathbb{A}^1_{\mathcal{O}_\nu}$. By Lemma 6.18, the lifts of $z_i$ in $\tilde{C}_i$ and in $D$ are $\mathbb{A}^1$-chain equivalent being in the same fiber of $S_\alpha \to \mathcal{O}_Y$. It follows that the two lifts $\tilde{z}_i$ are $\mathbb{A}^1$-chain equivalent in $S_\alpha$, which implies that the images of $x_0$ and $x_1$ through $X(L) \cong X(\mathcal{O}_\nu) \to X(\kappa_\nu)$ are $\mathbb{A}^1$-chain equivalent.

**Proof of Proposition 6.13.** We just have to check conditions (A1)-(A4) of Definition 6.7. For (A1), we observe that given an extension $L \subset L'$ in $\mathcal{F}_k$ together with discrete valuations $\nu$ and $\nu'$ satisfying the stated hypotheses, the diagram

$$
\begin{array}{ccc}
X(L) & \longrightarrow & X(L') \\
\downarrow & & \downarrow \\
X(\kappa_\nu) & \longrightarrow & X(\kappa_{\nu'})
\end{array}
$$

commutes. Lemma 6.12 shows that the induced maps on $\mathbb{A}^1$-equivalence classes of points also commute. A similar argument can be used to establish (A2).

For (A3), we proceed along the same lines as the proof of Lemma 6.12. Thus, let $z$ be a point of codimension 2 on a smooth scheme $S$ with residue field $\kappa(z)$ separable over $k$. Write $S_z$ for the corresponding local scheme. Assume $y_0$ and $y_1$ are two points of codimension 1 in $S_z$ whose closure $Y_i := \bar{y}_i \subset S_z$ is essentially smooth over $k$. Let $F$ be the fraction field of $S_z$, and consider a morphism $\alpha : \text{Spec } F \to X$. By Lemma 6.17, we can find $S_\alpha \to S_z$ a proper birational morphism that is an isomorphism over $S_z \setminus z$ with $S_\alpha$ a smooth $k$-scheme, and $\beta : S_\alpha \to X$ inducing $\alpha$.

Since the $Y_i$ are assumed smooth, the morphisms $\tilde{Y}_i \to Y_i$ from the proper transform $\tilde{Y}_i$ is an isomorphism. Let $\tilde{y}_i$ be the unique lifts of the closed point $z$ in $\tilde{Y}_i$. The composite maps $X(F) \to X(\kappa(y_i)) \to X(\kappa(z))$ are equal to the corresponding composite maps $X(F) \to X(\kappa(\tilde{y}_i)) \to X(\kappa(\tilde{z}_i)) = X(\kappa(z))$. Now, using Lemma 6.18 the points $\tilde{z}_i$ both lie in the fiber over $z$ and are hence $\mathbb{A}^1$-chain equivalent. This observation implies that the images of the composite maps in $X(\kappa(z))/\sim$ are equal, which is what we wanted to show.
induced by the inclusion \( L \subset L(t) \). Since \( X \) is proper, for any \( L \in \mathcal{F}_k \), the restriction map \( X(\mathbb{A}^1_L) \to X(L(t)) \) is bijective. Thus, the map \( X(\mathbb{A}^1_L) \to X(L(t))/\sim \) is surjective. Given \( H : \mathbb{A}^1_L \to X \), \( H \) is \( \mathbb{A}^1 \)-chain homotopic to \( H_0 : \mathbb{A}^1_L \to \text{Spec} L \to X \), where \( \text{Spec} L \to X \) is the restriction of \( H \) to 0. However, the product map \( \mathbb{A}^1_L \times \mathbb{A}^1_L \to \mathbb{A}^1_L \) induces a chain homotopy in \( X(\mathbb{A}^1_L) = X(L(t)) \) between \( H \) and \( H_0 \). This proves that the map \( X(L)/\sim \to X(L(t))/\sim \) is surjective.

To prove the required injectivity, we proceed as follows. It suffices to prove that given any field \( L \in \mathcal{F}_k \), and two points \( x_0, x_1 \in X(L) \), if the associated \( L(t) \)-points of \( X \), which we denote \( x'_0 \) and \( x'_1 \), are related by an elementary \( \mathbb{A}^1 \)-equivalence \( H : \mathbb{A}^1_{L(t)} \to X \), then \( x_0 \) and \( x_1 \) are themselves \( \mathbb{A}^1 \)-equivalent \( L \)-points.

There is an open dense subscheme \( U \subset \mathbb{A}^1_F \) such that \( H \) is induced by a \( k \)-morphism \( h : \mathbb{A}^1_U \to X \), and the composite maps \( U \overset{i}{\hookrightarrow} \mathbb{A}^1_U \to X \) and \( U \to \text{Spec}(L) \overset{\pi}{\to} X \) are equal. If \( U \) admits an \( L \)-rational point (e.g., if \( L \) is infinite), call it \( y_1 \), then composition with \( y_1 \) induces a morphism \( h_y : \mathbb{A}^1_U \to \mathbb{A}^1_U \to X \) providing an elementary \( \mathbb{A}^1 \)-equivalence between \( x_0 \) and \( x_1 \).

If \( U \) does not contain a rational point, then \( h \) is defined on an open subscheme \( \Omega \subset \mathbb{A}^1_{\mathbb{A}^1} \) whose complement is a finite collection of closed points, call them \( z_j \). Applying Lemma 6.17, we obtain a birational morphism \( S_h : \mathbb{A}^1_{\mathbb{A}^1} \to X \) that is an isomorphism on \( \Omega \), and a \( k \)-morphism \( \tilde{h} : S_h \to X \) extending \( h \). Consider the copy of the affine line \( D := \mathbb{A}^1_U \subset \mathbb{A}^1_{\mathbb{A}^1} = \mathbb{A}^1_L \times \mathbb{A}^1_L \) defined by \( id \times 0 \). The intersection of \( D \) with \( \Omega \) is a dense open subscheme of \( D \) (as \( \mathbb{A}^1_L \) has infinitely many closed points). The proper transform of \( D \) in \( S_h \) is a closed curve that we denote by \( \tilde{D} \). We observe that the induced projection \( \tilde{D} \to D \) is an isomorphism.

Now, the inclusion \( \tilde{D} \hookrightarrow S_h \) is an elementary \( \mathbb{A}^1 \)-equivalence between the two \( L \)-points obtained by composing with 0 and 1; call these points \( d_0 \) and \( d_1 \). Consider also the closed curves \( Y_i \) of \( \mathbb{A}^1_{\mathbb{A}^1} \) (also isomorphic to \( \mathbb{A}^1_L \)) defined by the inclusions \( 0 \times id \) and \( 1 \times id \). The proper transforms \( \tilde{Y}_i \subset S_h \) of \( Y_i \) are again isomorphic to \( \mathbb{A}^1_L \). The \( L \)-points of \( \tilde{Y}_i \) defined by inclusion at 0, call them \( \tilde{x}_i \), both lie in the same fiber of the morphism \( S_h \to \mathbb{A}^1_{\mathbb{A}^1} \) as the \( L \)-point \( d_i \). By Lemma 6.18, these fibers are \( \mathbb{A}^1 \)-chain connected, and thus the points \( d_i \) and \( \tilde{x}_i \) are \( \mathbb{A}^1 \)-equivalent \( L \)-points. Finally, the composite map \( \tilde{x}_i : \text{Spec} L \to S_h \to X \) is equal to \( x_i \) because the composite map \( U \hookrightarrow \mathbb{A}^1 = \tilde{Y}_i \to S_h \to X \) is by construction equal to \( x_i \); this provides the required \( \mathbb{A}^1 \)-equivalence. 

\( \square \)
A Notational postscript

We felt that introducing so many (somewhat subtilely) different notions of connectedness and rationality in the preceding sections warranted inclusion of a summary of the notation and various implications.

Notions of connectedness

- $\pi_0^{A^1}(\cdot)$ - the sheaf of $A^1$-connected components; see Definition 2.1
- $\pi_0^{ch}(\cdot)$ - the sheaf of $A^1$-chain connected components; see Definition 2.10.
- $\pi_0^{A^1,\text{ét}}(\cdot)$ - the sheaf of étale $A^1$-connected components; see Definition 2.31.
- $\pi_0^{\text{ét},ch}(\cdot)$ - the sheaf of étale $A^1$-chain connected components; see the proof of Theorem 2.33.
- $a_{\text{ét}} \pi_0^{A^1}(\cdot)$ - the étale sheafification of the functor $U \mapsto [U, X]_{A^1}$; see Equation 2.1.

Assume $X \in S_{mk}$. In sequence, Remark 2.11, the proof of Theorem 2.33, and Lemma 2.36 provided natural epimorphisms

\[
\begin{align*}
\pi_0^{ch}(X) & \twoheadrightarrow \pi_0^{A^1}(X), \\
\pi_0^{\text{ét},ch}(X) & \twoheadrightarrow \pi_0^{\text{ét},A^1}(X), \text{ and} \\
a_{\text{ét}} \pi_0^{A^1}(X) & \twoheadrightarrow \pi_0^{\text{ét},A^1}(X).
\end{align*}
\]

(A.1)

We also introduced two geometric notions of connectedness: $A^1$-chain connectedness (see Definition 2.9) and étale $A^1$-chain connectedness (see Remark 2.35), together with the collection of varieties covered by affine spaces (see Definition 2.15). The implications of various connectivity properties induced by the above epimorphisms, together with Lemma 2.16 are summarized in the following diagram.

(A.2) covered by affine spaces

\[
\begin{align*}
\text{étale } A^1\text{-connected } & \quad \text{étale } A^1\text{-chain connected } \\
A^1\text{-connected } & \quad A^1\text{-chain connected }
\end{align*}
\]

Notions of near rationality

The interrelationships between the various rationality properties (for smooth proper varieties over a field $k$) we considered are summarized in the following diagram.

(A.3) $k$-rational $\quad \Rightarrow \quad$ stably $k$-rational $\quad \Rightarrow \quad$ factor $k$-rational

separably rationally connected $\iff$ separably $k$-unirational $\iff$ retract $k$-rational
Definitions of the first four terms can be found in Definition 2.18; the last two terms are studied in [Ko96, IV.3.1-2]. For proofs of the first four implications, see Lemma 2.19; the last two implications follow easily from the definitions.

**Connections between $\mathbb{A}^1$-connectivity and rationality properties**

Finally, we can connect the two diagrams above. Again, let us restrict ourselves to considering only smooth proper varieties. If $k$ is a perfect field, Theorem 2.33 shows that

\[(A.4) \quad \text{separably rationally connected} \implies \text{\'{e}tale } \mathbb{A}^1\text{-chain connected.}\]

Theorem 2.38 (together with Proposition 2.12) shows that for arbitrary fields $k$,

\[(A.5) \quad \mathbb{A}^1\text{-chain connected} \iff \mathbb{A}^1\text{-connected.}\]

If furthermore, $k$ has characteristic 0, Theorem 2.21 shows that

\[(A.6) \quad \text{retract } k\text{-rational} \implies \mathbb{A}^1\text{-chain connected.}\]

**References**


REFERENCES


