Stable $\mathbb{A}^1$-homotopy and $R$-equivalence

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Abstract

We prove that existence of a $k$-rational point can be detected by the stable $\mathbb{A}^1$-homotopy category of $S^1$-spectra, or even a “rationalized” variant of this category.

1 Introduction

Suppose $k$ is a field and $X$ a smooth proper $k$-variety. By the Lang-Nishimura lemma [Nis55], one knows that existence of a $k$-point is a $k$-birational invariant. By a remark of Morel and Voevodsky, one also knows that existence of a $k$-rational point is an unstable $\mathbb{A}^1$-homotopy invariant; see, e.g., [MV99, §3 Remark 2.5], where it is observed that this is a consequence of the fact that the Nisnevich topology is used in the construction of the unstable $\mathbb{A}^1$-homotopy category. The purpose of this note is to, in a sense, combine the two results above and to show that the ability to detect rational points persists in the Morel-Voevodsky stable $\mathbb{A}^1$-homotopy category of $S^1$-spectra as well as in Morel’s $\mathbb{A}^1$-derived category and even the $\mathbb{A}^1$-derived category with $\mathbb{Q}$-coefficients. Very loosely speaking, our results say that existence of rational points can be detected by purely cohomological means.

Write $\text{SH}^k_+$ for the Morel-Voevodsky stable $\mathbb{A}^1$-homotopy category of $S^1$-spectra (see [Mor05, Definition 4.1.1] for a precise definition). Let $\Sigma_s^\infty X_+$ denote the $\mathbb{A}^1$-localization of the simplicial suspension spectrum of $X$ with a disjoint basepoint attached. The 0-th $S^1$-stable $\mathbb{A}^1$-homotopy sheaf of $X$, denoted $\pi^s_0(X_+)$, is the Nisnevich sheaf associated with the presheaf on $Sm_k$

$$U \mapsto \text{Hom}_{\text{SH}^k_+}(\Sigma_s^\infty U_+, \Sigma_s^\infty X_+).$$

The structure morphism $X \to \text{Spec } k$ induces a morphism of sheaves $\pi^s_0(X_+) \to \pi^s_0\text{Spec } k_+$. The sheaf $\pi^s_0(X_+)$ is a birational invariant of smooth, proper $k$-varieties. If $X$ has a $k$-rational point, the map $\pi^s_0(X_+) \to \pi^s_0(\text{Spec } k_+)$ is a split epimorphism. We prove a converse to this statement.

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Theorem 1. If $X$ is a smooth proper $k$-variety, then the following conditions are equivalent:

i) $X$ has a $k$-rational point,

ii) there is a stable $\mathbb{A}^1$-homotopy class of maps $\Sigma_\infty^s \text{Spec } k_+ \to \Sigma_\infty^s X_+$ splitting the structure map $\Sigma_\infty^s X_+ \to \Sigma_\infty^s \text{Spec } k_+$, and

iii) the morphism of sheaves $\pi^s_0(X_+) \to \pi^s_0(\text{Spec } k_+)$ is a split epimorphism.

That $(i) \implies (ii) \implies (iii)$ is clear, and the work goes into showing $(iii) \implies (i)$. Proposition 2.1 allows us to show that $\pi^s_0(\text{Spec } k_+) = \mathbb{Z}$, so $(iii)$ can be made more explicit. For a corresponding statement with $\mathbb{Q}$-coefficients see Remark 2.10. To put this result in context, we observe how these results combined with those of [AH10] give a framework for comparing rational points and 0-cycles of degree 1.

Remark 2. Let $\Sigma_{p_1}$ denote the operation of smashing with the simplicial suspension spectrum of $(\mathbb{P}^1, \infty)$, and let $\Omega_{p_1}$ be the adjoint looping functor. If $E$ is any $S^1$-spectrum, there is a map $E \to \Omega_{p_1} \Sigma_{p_1} E$. We can iterate this functor to obtain a tower

$$E \to \Omega_{p_1} \Sigma_{p_1} E \to \Omega_{p_1}^2 \Sigma_{p_1}^2 E \cdots .$$

The 0-th $\mathbb{P}^1$-stable $\mathbb{A}^1$-homotopy sheaf of $E$, denoted $\pi^s_{\mathbb{A}^1}^0(E)$, can be computed by means of the formula

$$\pi^s_{\mathbb{A}^1}^0(E) = \text{colim}_n \pi^s_0(\Omega_{p_1}^n \Sigma_{p_1}^n E).$$

The structure map $X \to \text{Spec } k$ induces a morphism $\pi^s_{\mathbb{A}^1}^0(X_+) \to \pi^s_{\mathbb{A}^1}^0(\text{Spec } k_+)$. One says that $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy if the latter map is a split epimorphism. By [AH10] Theorem 1, if $k$ is an infinite perfect field having characteristic unequal to 2, we know that a smooth proper $k$-scheme $X$ has a 0-cycle of degree 1 if and only if it has a rational point up to stable $\mathbb{A}^1$-homotopy. Thus, under the stated hypotheses on $k$, the difference between a 0-cycle of degree 1 and $k$-rational point is measured by the difference between $S^1$-stable and $\mathbb{P}^1$-stable $\mathbb{A}^1$-homotopy theory. The existence of such a connection between rational points and 0-cycles of degree 1 was suggested in [Lev10] p. 395-6.

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2 Proof of Theorem

Let us introduce some notation for the rest of the note. Throughout, suppose $k$ is a field. Let $Sm_k$ denote the category of schemes that are separated, smooth, and have finite type over $\text{Spec } k$. Write $\text{Spec}_k$ for the category of simplicial Nisnevich sheaves of sets on $Sm_k$; objects
of this category will be called spaces. We identify Nisnevich sheaves with the corresponding simplicial objects.

Write \( \mathcal{H}(k) \) for the Morel-Voevodsky unstable \( \mathbb{A}^1 \)-homotopy category. This category is constructed from the category \( \text{Sp} \text{c}_{k} \) by localizing at the class of \( \mathbb{A}^1 \)-weak equivalences (see [MV99 §3.2]). We write \( \text{SH}^k \) for the stable \( \mathbb{A}^1 \)-homotopy category of \( S^1 \)-spectra, e.g., as defined in [Mor05, §5]. Loosely speaking, this category is obtained from \( \mathcal{H}(k) \) by formally inverting the simplicial suspension operation. We write \( \text{SH}^k \) for the stable \( \mathbb{A}^1 \)-homotopy category of \( \mathbb{P}^1 \)-spectra, e.g., as defined in [Jar00]; this category is obtained from \( \text{SH}^k \) by further inverting the operation of smashing with the suspension spectrum of \( \mathbb{G}_m \).

Recall that a presheaf of sets \( \mathcal{F} \) on \( \mathcal{S}m_k \) is called \( \mathbb{A}^1 \)-invariant, if for any smooth scheme \( U \) the map \( \mathcal{F}(U) \to \mathcal{F}(U \times \mathbb{A}^1) \) induced by pullback along the projection \( U \times \mathbb{A}^1 \to U \) is a bijection. If \( \mathcal{X} \) is any space, we write \( \mathbb{Z}(\mathcal{X}) \) for the simplicial sheaf of abelian groups freely generated by the simplices of \( \mathcal{X} \). The normalized chain complex of \( \mathbb{Z}(\mathcal{X}) \), for which we will write \( \mathcal{C}_*(\mathbb{Z}(\mathcal{X})) \), is a chain complex of sheaves of abelian groups.

Write \( D(\text{Ab}_{\text{Nis}}(k)) \) for the (unbounded) derived category of Nisnevich sheaves of abelian groups on \( \mathcal{S}m_k \). A complex of sheaves of abelian groups \( A \) on \( \mathcal{S}m_k \) is called \( \mathbb{A}^1 \)-local if for any complex \( B \) the map

\[
\text{Hom}_{D(\text{Ab}_{\text{Nis}}(k))}(B, A) \to \text{Hom}_{D(\text{Ab}_{\text{Nis}}(k))}(B \otimes \mathbb{Z}(\mathbb{A}^1), A)
\]

is a bijection. A sheaf \( \mathcal{F} \) of abelian groups is said to be strictly \( \mathbb{A}^1 \)-invariant if it is \( \mathbb{A}^1 \)-invariant viewed as a complex of sheaves situated in degree 0. Consider the full subcategory of \( D(\text{Ab}_{\text{Nis}}(k)) \) consisting of \( \mathbb{A}^1 \)-local complexes; the inclusion of this subcategory into \( D(\text{Ab}_{\text{Nis}}(k)) \) admits a left adjoint \( L_{\mathbb{A}^1} \) called the functor of \( \mathbb{A}^1 \)-localization [CD09 Proposition 4.3]. Morel’s \( \mathbb{A}^1 \)-derived category \( D_{\mathbb{A}^1}(k) \) is (equivalent to) the full subcategory of the derived category of Nisnevich sheaves of abelian groups consisting of \( \mathbb{A}^1 \)-local complexes.

Set \( \mathcal{C}^{\mathbb{A}^1}_s(\mathcal{X}) := L_{\mathbb{A}^1} \mathcal{C}_s(\mathbb{Z}(\mathcal{X})) \); this complex is called the \( \mathbb{A}^1 \)-chain complex of \( \mathcal{X} \). The 0-th \( \mathbb{A}^1 \)-homology sheaf of \( \mathcal{X} \), denoted \( \mathcal{H}^{\mathbb{A}^1}_0(\mathcal{X}) \), is just the 0-th homology sheaf of \( \mathcal{C}^{\mathbb{A}^1}_s(\mathcal{X}) \). The functor \( \mathcal{X} \mapsto \mathcal{C}^{\mathbb{A}^1}_s(\mathcal{X}) \) induces a functor \( \mathcal{H}(k) \to D_{\mathbb{A}^1}(k) \). The suspension isomorphism for homology shows that this functor factors through a functor \( \text{SH}^k \to D_{\mathbb{A}^1}(k) \) that we will call abelianization. For recollections about the \( \mathbb{A}^1 \)-derived category, see [Mor06, §3.2].

**The Hurewicz homomorphism**

The abelianization functor induces a Hurewicz morphism \( \pi^0_0(\mathcal{X}_+) \to \mathcal{H}^{\mathbb{A}^1}_0(\mathcal{X}) \) (note: the definition of \( \pi^0_0(-) \) given in the introduction makes sense for any \( \mathcal{X} \in \text{Sp} \text{c}_{k} \)). The following result is a consequence of the stable \( \mathbb{A}^1 \)-connectivity theorem [Mor05 Theorem 6.1.8], which states that \(-1\)-connected spectra or complexes are preserved by \( \mathbb{A}^1 \)-localization.

**Proposition 2.1.** If \( \mathcal{X} \) is a space, the canonical morphism \( \pi^0_0(\mathcal{X}_+) \to \mathcal{H}^{\mathbb{A}^1}_0(\mathcal{X}) \) is an isomorphism of strictly \( \mathbb{A}^1 \)-invariant sheaves.

Because of this proposition, we can (and will) replace the 0-th stable \( \mathbb{A}^1 \)-homotopy sheaf by the 0-th \( \mathbb{A}^1 \)-homology sheaf of a space in the sequel. The next result follows immediately from Proposition 2.1 and, e.g., [Aso10 Theorem 2.2.9].
Corollary 2.2. If $k$ is an infinite field, the sheaf $\pi_0^0(X_+)$ is a birational invariant of smooth and proper $k$-varieties.

2.1 Strict $\mathbb{A}^1$-invariance and birationality

Definition 2.3. Suppose $\mathcal{F}$ is a presheaf of sets on $Sm_k$. We say $\mathcal{F}$ is birational if for any open dense immersion $U \to U'$ in $Sm_k$, the map $\mathcal{F}(U') \to \mathcal{F}(U)$ is an isomorphism.

In the following lemma, we summarize some technical properties of birational presheaves. This result is “well known to the experts” and we include it for the convenience of the reader; results along these lines can also be found in, e.g., [Lev10, §2].

Lemma 2.4. If $\mathcal{F}$ is a birational presheaf, the free presheaf of abelian groups $\mathbb{Z}(\mathcal{F})$ is also birational, and both $\mathcal{F}$ and $\mathbb{Z}(\mathcal{F})$ are Nisnevich sheaves. If $\mathcal{F}$ is furthermore $\mathbb{A}^1$-invariant, then $\mathbb{Z}(\mathcal{F})$ is Nisnevich flasque, and $\mathbb{Z}(\mathcal{F})$ is strictly $\mathbb{A}^1$-invariant.

Proof. To show that $\mathcal{F}$ is a Nisnevich sheaf is, we just have to check that $\mathcal{F}$ takes an elementary distinguished square

$$
\begin{array}{ccc}
V' & \longrightarrow & V \\
\downarrow & & \downarrow \\
U & \longrightarrow & X,
\end{array}
$$

(where $\psi$ is étale, $U \to X$ is an open immersion, $X \setminus U$ is given the usual reduced scheme structure, and the map $\psi^{-1}(X \setminus U) \to X \setminus U$ is an isomorphism) to a cartesian square. Since $\mathcal{F}$ is birational, both the bottom and top maps are isomorphisms and so the diagram is cartesian. Now, if $\mathcal{F}$ is birational, then by definition $\mathbb{Z}(\mathcal{F})$ is also birational, and by what we just showed $\mathbb{Z}(\mathcal{F})$ is also a Nisnevich sheaf.

If $\mathcal{F}$ is also $\mathbb{A}^1$-invariant, it follows immediately that $\mathbb{Z}(\mathcal{F})$ is also $\mathbb{A}^1$-invariant. We will now show that $\mathbb{Z}(\mathcal{F})$ is Nisnevich flasque. To see this, recall that the Nisnevich cohomology can be computed by means of Čech cochains: [MV99, p. 95] mentions this without proof, but the proof is essentially identical to the corresponding statement in the étale topology; one uses Čech-derived functor spectral sequence and the fact [Nis89, Lemma 1.18.1] that the higher cohomology sheaves of a Nisnevich sheaf of abelian groups vanish. Therefore, suppose $X$ is an irreducible smooth scheme, $u : U \to X$ is a Nisnevich cover of $X$. By lifting the generic point $\eta$ of $X$, we can find a component of $U$ that is birational to $X$. Since each map $U^{\times n+1} \to U^{\times n}$ is also a Nisnevich cover, it follows that $\mathbb{Z}(\mathcal{F})(U^{\times n}) \to \mathbb{Z}(\mathcal{F})(U^{\times n+1})$ is injective and thus all higher Nisnevich cohomology of $\mathbb{Z}(\mathcal{F})$ vanishes.

Corollary 2.5. If $\mathcal{F}$ is a birational and $\mathbb{A}^1$-invariant sheaf of sets, the canonical map $\mathcal{F} \to \mathbb{Z}(\mathcal{F})$ induces an isomorphism $H_0^{\mathbb{A}^1}(\mathcal{F}) \to \mathbb{Z}(\mathcal{F})$.

Proof. By definition $H_0^{\mathbb{A}^1}(\mathcal{F}) = H_0(L_{\mathbb{A}^1}\mathbb{Z}(\mathcal{F}))$. However, since $\mathbb{Z}(\mathcal{F})$ is Nisnevich flasque, it follows that $\mathbb{Z}(\mathcal{F})$ is $\mathbb{A}^1$-local, i.e., the canonical map $L_{\mathbb{A}^1}(\mathbb{Z}(\mathcal{F})) \to \mathbb{Z}(\mathcal{F})$ is an isomorphism.
Example 2.6. Suppose \( X \) is an \( \mathbb{A}^1\)-rigid smooth proper \( k \)-scheme (see [MV99, §3 Example 2.4]). Given an open dense immersion \( U \to U' \), the map \( X(U') \to X(U) \) is an isomorphism; indeed any such map is uniquely determined by where it sends the generic point of each component. As a consequence \( Z(X) \) is a strictly \( \mathbb{A}^1 \)-invariant sheaf. Because \( Z(X) \) is \( \mathbb{A}^1 \)-local, we see that the canonical map \( C^A_{\ast}(X) = L_{\mathbb{A}^1}Z(X) \to Z(X) \) is an isomorphism, and thus that \( H^A_0(X) = Z(X) \). Thus, \( X, Z(X), \) and \( H^A_0(X) \) are all birational sheaves and by Lemma 2.4 all these sheaves are all strictly \( \mathbb{A}^1 \)-invariant. As a consequence of Corollary 2.5 we deduce that if \( k \) is infinite and \( X' \) is any smooth proper variety that is stably \( k \)-birationally equivalent to a smooth proper \( \mathbb{A}^1 \)-rigid variety \( X \), then \( H^A_0(X') = H^A_0(X) \).

2.2 Birational connected components and the main result

Suppose \( X \) is a smooth proper variety over a field \( k \). If \( L/k \) is a separable, finitely generated extension, recall that two \( L \)-points in \( X \) are \( R \)-equivalent if they can be connected by the images of a chain of morphisms from \( \mathbb{P}^1_L \) to \( X \) (over \( k \)) [Man86]. There is a birational sheaf related to \( R \)-equivalence classes of points in \( X \).

Theorem 2.7. If \( X \) is a smooth proper \( k \)-variety, there is a birational and \( \mathbb{A}^1 \)-invariant sheaf \( \pi^0\mathbb{A}^1(X) \) together with a canonical map \( X \to \pi^0\mathbb{A}^1(X) \) functorial for morphisms of proper varieties such that for any separable finitely generated extension \( L/k \) the induced map \( X(L) \to \pi^0\mathbb{A}^1(X)(L) \) factors through a bijection \( X(L)/R \to \pi^0\mathbb{A}^1(X)(L) \).

Proof. Everything except the statement of functoriality is included in [AM09 Theorem 6.2.1]. Since \( \pi^0\mathbb{A}^1(X) \) is a birational and \( \mathbb{A}^1 \)-invariant sheaf, to construct a morphism \( \pi^0\mathbb{A}^1(Y) \to \pi^0\mathbb{A}^1(X) \), it suffices to observe that by the definition of \( R \)-equivalence a morphism \( f : X \to Y \) induces morphisms \( X(L)/R \to Y(L)/R \) for every finitely generated separable extension \( L/k \).

If \( X \) is a smooth proper variety, we can consider the sheaf \( Z(\pi^0\mathbb{A}^1(X)) \). By Lemma 2.4 it follows that \( Z(\pi^0\mathbb{A}^1(X)) \) is a strictly \( \mathbb{A}^1 \)-invariant sheaf, and Corollary 2.5 gives rise to a canonical identification \( H^A_0(\pi^0\mathbb{A}^1(X)) \to Z(\pi^0\mathbb{A}^1(X)) \). As a consequence of Theorem 2.7 we deduce the existence of a canonical morphism

\[ \phi_X : H^A_0(X) \to Z(\pi^0\mathbb{A}^1(X)). \]

Because \( Z(\pi^0\mathbb{A}^1(X)) \) is a strictly \( \mathbb{A}^1 \)-invariant sheaf, existence of this morphism also follows immediately from [Aso10 Lemma 2.2.3], which states that \( H^A_0(X) \) is initial among strictly \( \mathbb{A}^1 \)-invariant sheaves \( M \) admitting a morphism of sheaves \( X \to M \).

Remark 2.8. It seems reasonable to expect that the morphism \( \phi_X : H^A_0(X) \to Z(\pi^0\mathbb{A}^1(X)) \) is an isomorphism. Since our goal is to get as quickly as possible to the connection with rational points we did not pursue this further.

Corollary 2.9. If \( X \) is a smooth proper \( k \)-variety, then the set \( X(k) \) is non-empty if and only if the map \( H^A_0(X) \to Z \) induced by the structure map is a split surjection.
Proof. If \( X(k) \) is non-empty, then we get a morphism \( Z = H_{0}^{A_{1}}(\text{Spec } k) \rightarrow H_{0}^{A_{1}}(X) \) that splits the map induced by the structure morphism. Conversely, note that the map \( H_{0}^{A_{1}}(X) \rightarrow \mathbb{Z}(\pi_{0}^{b_{1}}(X)) \) is functorial in \( X \), and thus the morphism \( H_{0}^{A_{1}}(X) \rightarrow \mathbb{Z} \) factors through the morphism \( \phi_{X} \). A splitting \( Z \rightarrow H_{0}^{A_{1}}(X) \) therefore gives rise to a non-trivial morphism \( Z \rightarrow H_{0}^{A_{1}}(X) \), i.e., an element of \( \mathbb{Z}(\pi_{0}^{b_{1}}(X))(k) \). The group \( \mathbb{Z}(\pi_{0}^{b_{1}}(X))(k) \) is by Theorem 2.7 the free abelian group on the set \( X(k)/R \). Since the group \( \mathbb{Z}(\pi_{0}^{b_{1}}(X))(k) \) is a non-trivial free abelian group, we deduce that \( X(k)/R \) has at least 1 element, and therefore \( X(k) \) is non-empty.

\[ \square \]

Proof of Theorem 2.10. The rationalized \( A_{1} \)-derived category is obtained by following the construction of the \( A_{1} \)-derived category sketched above and replacing abelian groups by \( \mathbb{Q} \)-vector spaces throughout. Replacing \( \mathbb{Z} \) by \( \mathbb{Q} \) in all of the above allows one to deduce that existence of \( k \)-rational point is detected by the rationalized \( A_{1} \)-derived category. To be precise, if \( X \) is a smooth proper \( k \)-scheme, then \( X \) has a \( k \)-rational point if and only if the canonical map \( H_{0}^{A_{1}}(X, \mathbb{Q}) \rightarrow \mathbb{Q} \) induced by the structure morphism \( X \rightarrow \text{Spec } k \) is a split epimorphism. This statement also implies a statement about an appropriate “rational” version of the stable \( A_{1} \)-homotopy category of \( S^{1} \)-spectra, but we leave this to the reader.

References


