Homotopy theory of smooth affine quadrics revisited

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Abstract

We produce explicit affine quadric hypersurfaces, smooth over Spec \( \mathbb{Z} \), that are geometric models of motivic spheres, i.e., spheres from the standpoint of \( \mathbb{A}^1 \)-homotopy theory. To do this, we construct new \( \mathbb{A}^1 \)-contractible smooth schemes that cannot be realized as quotients of affine space by free actions of unipotent groups. This last result answers negatively a question about the structure of \( \mathbb{A}^1 \)-contractible smooth schemes. We then prove that our quadric hypersurfaces, exhibiting a feature analogous to one of classical spheres, can be covered by two \( \mathbb{A}^1 \)-contractible smooth schemes whose intersection is (\( \mathbb{A}^1 \)-weakly equivalent to) another motivic sphere. We use these results to compute various standard invariants for such quadrics (e.g., Chow groups, K-theory, or Hermitian K-theory).

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1 Introduction

Perhaps the most basic representation of an \( n \)-dimensional sphere \( S^n \) is as the set of real solutions to the equation \( \sum_{i=1}^{n+1} x_i^2 = 1 \), i.e., as the set of real points of an affine algebraic variety defined over the integers. If \( -\infty \) denotes the point with coordinates \((0,0,\ldots,-1)\) and \( \infty \) denotes the point with coordinates \((0,\ldots,0,1)\), then stereographic projection shows that both \( S^n \setminus \{\infty\} \) and \( S^n \setminus \{-\infty\} \) are diffeomorphic to \( \mathbb{R}^n \); we distinguish the two copies by calling them \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n_- \). Furthermore the intersection \( S^n \setminus \{\infty,-\infty\} = \mathbb{R}^n_+ \cap \mathbb{R}^n_- \) is diffeomorphic to \( \mathbb{R}^n \setminus 0 \). Now, \( \mathbb{R}^n \) is contractible, and radial projection determines a homotopy equivalence between the “equatorial” copy of \( S^{n-1} \), defined by the additional equation \( x_{n+1} = 0 \), and \( \mathbb{R}^n \setminus 0 \). Choose a base-point

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for $\mathbb{R}^n \setminus 0$ that lies in this copy of $S^{n-1}$. This geometric presentation provides an explicit link between $S^n$ and a sphere of one lower dimension which provides a tool to study inductively (co)homology, say via Mayer-Vietoris sequences, or the fundamental group, via the van Kampen theorem.

The above geometric observations are perhaps most compactly stated using homotopy theoretic terminology: the sphere $S^n$ is the homotopy pushout of the diagram

$$
* \leftarrow S^{n-1} \rightarrow *.
$$

Let $I = [0, 1]$ pointed by 1 denote the unit interval. To “compute” this homotopy pushout, we can replace the $*$ on the right hand side by the cone $C(S^{n-1}) = S^{n-1} \wedge I$ over $S^{n-1}$, which is a contractible topological space. If we include $S^{n-1}$ in $C(S^{n-1})$ as $S^{n-1} \times I$, then the push-out of the resulting diagram is the quotient $C(S^{n-1})/S^{n-1}$, which is the presentation of the unreduced suspension $S(S^{n-1})$ as a double cone. Thus, the homotopy push-out presentation above records the fact that $S^n$ is the suspension of $S^{n-1}$.

In a similar vein, note that the projection map for the tangent bundle $TS^n \rightarrow S^n$ is a homotopy equivalence. The tangent bundle $TS^n \rightarrow S^n$ trivializes on the open cover of $S^n$ specified in the previous paragraph since topological vector bundles over contractible spaces are necessarily trivial. We can identify $TS^n|_{\mathbb{R}^n \setminus 0}$ with $\mathbb{R}^{2n} \setminus E_n$ where $E_n$ is a codimension $n$ linear subspace. Thus, we can show that $TS^n$ is, up to homotopy equivalence, the suspension of $\mathbb{R}^{2n} \setminus E_n$.

In this paper, we use the last observation to inspect the theory of smooth affine quadric hypersurfaces from the standpoint of the $A^1$-homotopy theory (a.k.a. motivic homotopy theory) of Morel and Voevodsky [MV99]. Roughly speaking, they introduce a homotopy theory for smooth algebraic varieties where homotopies are parameterized by the affine line. The analog of “homotopy equivalence” in the resulting homotopy category will be called “$A^1$-weak equivalence” (references to precise definitions will be given later). To construct such a homotopy theory, one must first enlarge the category of smooth schemes to a category of spaces where various categorical constructions (e.g., arbitrary quotients) may be performed. Indeed, one works in a category of pointed spaces where constructions such as wedge sums, smash products and more generally, all limits and colimits indexed by a small category may be performed. The $A^1$-homotopy category is obtained by formally inverting all the projection maps $X \times A^1 \rightarrow X$.

There are two natural analogues of the circle in $A^1$-homotopy theory. The Tate circle is just $G_m$ (pointed by 1), and the simplicial circle, usually denoted $S^1_s$, is $A^1/\{0, 1\}$ (pointed by the image of $\{0, 1\}$ in the quotient). General spheres are then smash products of the above circles of the form and denoted $S^i_s \wedge G_m^j$. Understanding the geometry of these spaces is of fundamental interest in $A^1$-homotopy theory.

We consider the smooth affine quadric hypersurfaces defined by the equations

$$
Q_{2m-1} = \{ \sum_{i} x_i x_{m+i} = 1 \}, \text{ and }
Q_{2m} = \{ \sum_{i} x_i x_{m+i} = x_{2m+1}(1 + x_{2m+1}) \}.
$$
Our main result provides an explicit description of the $\mathbb{A}^1$-homotopy type of these quadrics in terms of motivic spheres.

**Theorem 1.1** (See Theorem 3.4). For any $n \geq 0$, we have explicit $\mathbb{A}^1$-weak equivalences

$$Q_n \sim_{\mathbb{A}^1} \begin{cases} S^{m-1}_s \wedge \mathbb{G}_m \wedge m & \text{if } n = 2m - 1 \\ S^m_s \wedge \mathbb{G}_m \wedge m & \text{if } n = 2m \end{cases}$$

of spaces over $\text{Spec } \mathbb{Z}$.

For the odd-dimensional quadrics $Q_{2m-1}$, it was classically known that projection onto $x_1, \ldots, x_m$ defines a Zariski locally trivial morphism $Q_{2m-1} \to \mathbb{A}^m \setminus \{0\}$ with fibers isomorphic to affine spaces of dimension $m - 1$ (and thus an $\mathbb{A}^1$-weak equivalence). Indeed, we can identify $\mathbb{A}^m \setminus \{0\}$ as a homogeneous space for $SL_{m+1}$ (the stabilizer of a point is an extension of a unipotent group of dimension $m$ by $SL_m$). The inclusion $SL_m \to H$ induces an $SL_{m+1}$-equivariant map $SL_{m+1}/SL_m \to SL_{m+1}/H$ that can be identified with the given map $Q_{2m-1} \to \mathbb{A}^m \setminus \{0\}$. Furthermore, $\mathbb{A}^m \setminus \{0\}$ is known to be a motivic sphere (up to $\mathbb{A}^1$-weak equivalence) by work of Morel and Voevodsky. Let us remark that the morphism above induces a morphism of the underlying complex analytic spaces $Q_{2m-1}(\mathbb{C}) \to \mathbb{C}^m \setminus \{0\}$. While this latter morphism is diffeomorphic to the projection for a vector bundle, as long as $m > 1$ it does not admit an algebraic (or holomorphic) section and thus is not an algebraic (or holomorphic) vector bundle!

In the even-dimensional case, we will cover $Q_{2m}$ by two open sets $X^+_m$ and $X^-_m$ both of which are $\mathbb{A}^1$-contractible (i.e., equivalent to a point in the $\mathbb{A}^1$-homotopy category), and whose intersection is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^m \setminus \{0\}$. We will thus see that, in analogy with our observations about the homotopy type of the tangent bundle above, $Q_{2m}$ is $\mathbb{A}^1$-homotopy equivalent to the reduced suspension of $\mathbb{A}^m \setminus \{0\}$.

While the discussion preceding the theorem statement is geometrically very appealing, it is not at all clear that one should expect any simple description of the $\mathbb{A}^1$-homotopy type of a quadric. The motivic spheres to which the smooth schemes $Q_{2m}$ are $\mathbb{A}^1$-homotopy equivalent are not a priori smooth schemes. Thus, this result can be thought of as providing the first non-trivial examples of spaces that are not smooth schemes, yet have the $\mathbb{A}^1$-homotopy type of smooth (affine) schemes. After this paper was written, we learned from Isaksen that he and Dugger (unpublished) had obtained a stable description of quadrics related to those in Theorem 3.4 in characteristic unequal to 2 stably, i.e., after a single simplicial suspension.

The technically complicated part of the proof is to construct and prove $\mathbb{A}^1$-contractibility of the varieties $X^\pm_{2m}$; this is the content of Theorem 2.10. The examples in [AD07b] (together with much tinkering) indicate the method of construction. Indeed, we will define $X^\pm_{2m}$ in such a way that $X^\pm_{2m} \cong Q_{2m} \setminus E_m$, where $E_m$ is the closed subscheme of codimension $m$, isomorphic to $\mathbb{A}^m$, defined by $x_1 = \ldots = x_m = 0, x_{2m+1} = -1$. One might expect (as we initially did) that $\mathbb{A}^1$-contractibility of $X_{2m}$ could then be proved using exactly the techniques developed in [AD07b]. In fact, we have on several occasions (see, e.g., [AD07d]) indicated the hope that every $\mathbb{A}^1$-contractible smooth scheme could be realized as a quotient of $\mathbb{A}^n$ by the free action of a unipotent group. Unfortunately, the following result shows that this (in retrospect extremely naïve) expectation is completely unreasonable.
Theorem 1.2 (See Theorem 2.10 and Corollary 2.12). No smooth quasi-affine scheme that can be realized as an open subscheme of a smooth affine scheme with complement of codimension $d \geq 3$ can be realized as a quotient of a smooth affine scheme by the free action of a unipotent group. In particular, the varieties $X^\pm_{2m}$ cannot be realized as unipotent quotients of affine space.

The proof of this result uses the techniques of the Cousin complex to study cohomology with coefficients in a unipotent group, and, in particular, to give conditions ensuring that all unipotent group torsors over a scheme are trivial. Thus, the proof of $\mathbb{A}^1$-contractibility of $X_{2m}$ must proceed by means other than those developed in [AD07b]. Instead, we use an explicit form of the so-called Jouanolou-Thomason homotopy lemma (together with a generous helping of brute force) to construct an explicit morphism $\mathbb{A}^{3m} \to Q_{2m} \setminus E_m$ that is Zariski locally trivial with fibers isomorphic to affine space. Thus, while $Q_{2m} \setminus E_m$ is not a quotient by a unipotent group action, it is a sort of “generalized” unipotent quotient, i.e., the base of a torsor under a vector bundle. We close with a precise conjecture describing the structure of non-affine $\mathbb{A}^1$-contractible smooth schemes (see Conjecture 2.15). Along the way, we will produce many “highly $\mathbb{A}^1$-connected” smooth affine hypersurfaces. The quadrics above will only be one example that we can study in a particularly explicit manner.

An immediate upshot of our discussion is that “computation” of various classical algebro-geometric invariants, e.g., higher Chow groups, algebraic K-theory, and Hermitian K-theory, together with some more modern invariants, e.g., oriented higher Chow groups, or, indeed, any “$\mathbb{A}^1$-representable” theory becomes essentially formal. Indeed, $\mathbb{A}^1$-representable (co)homology theories are bigraded with indices corresponding to the two circles mentioned above. Thus, the “reduced” version of any such theory (obtained, as usual, by splitting off the (co)homology of a point) evaluated on one of our smooth quadrics can be obtained from (co)homology of a point by just shifting indices as soon as the circles are invertible (i.e., stably)! For precise results, we refer the reader to Lemmas 3.12, 3.14, and 3.17; these results also make very explicit the kind of “Bott periodicity” present in these algebro-geometric cohomology theories. Morel’s deep results on the first non-vanishing $\mathbb{A}^1$-homotopy groups of spheres (from [Mor06]) can be used in conjunction with Theorem 3.4 to provide a study of $\mathbb{A}^1$-homotopy classes of maps between smooth schemes.

Let us note that there are a host of applications of the ideas present here to provide constructions analogous to classical homotopy theory. For example, using our computations, one can provide motivic analogues of the Hopf construction, Hopf maps, the Hopf invariant, the Adams $e$-invariant, etc. We have limited the scope of our discussion here to preliminary geometric results (together with a few computations) to streamline the presentation. All the applications have been consolidated and will appear in the sequel [Aso08b].

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Conventions, notations and recollections on \(A^1\)-homotopy theory

Throughout this paper, \(k\) will denote a commutative unital ring. Often, \(k\) will be a field, but sometimes \(k\) will be \(\mathbb{Z}\) or perhaps an open-subscheme there-of. Henceforth the word *scheme* will be synonymous with separated scheme of finite type over \(k\). The word *variety* will mean reduced scheme, i.e., varieties are not assumed to be irreducible. With these terminological conventions, we let \(\mathcal{S}_m_k\) denote the category of smooth schemes over \(k\).

Following the usual conventions in \(A^1\)-homotopy theory, we let \(\mathcal{S}_{pc_k}\) stand for the category of *spaces over \(k\)*, i.e., the category of Nisnevich sheaves of sets on \(\mathcal{S}_m_k\). The word *sheaf* will mean sheaf in the Nisnevich topology, unless otherwise indicated. The Nisnevich topology is, by construction, finer than the Zariski topology, but coarser than the étale topology. For a precise definition and various equivalent conditions, we refer the reader to [MV99] §3.1.

We let \(\Delta^0\mathcal{S}_{pc_k}\) denote the category of simplicial spaces, \(\mathcal{H}_s((\mathcal{S}_m/k)_{Nis})\) denote the *simplicial homotopy category*, and \(\mathcal{H}(k)\) denote the \(A^1\)-homotopy category. The pointed versions of these categories will be denoted \(\mathcal{S}_{pc_k}^e\), \(\Delta^0\mathcal{S}_{pc_k}^e\), \(\mathcal{H}_s((\mathcal{S}_m/k)_{Nis})^e\) and \(\mathcal{H}(k)^e\). We will denote by \([\cdot,\cdot]_{A^1}\) morphisms in either the category \(\mathcal{H}(k)\) or \(\mathcal{H}(k)^e\); to distinguish the latter case from the former, we will explicitly specify the base-points. We will designate spaces by upper case calligraphic letters (e.g., \(\mathcal{X}, \mathcal{Y}, \ldots\)) and smooth schemes by upper case roman letters (e.g., \(X, Y, \ldots\)).

We refer the reader to the work [MV99] for detailed foundational results in \(A^1\)-homotopy theory. In particular [MV99] §3.2.1 defines the notion of \(A^1\)-weak equivalence. For us, it suffices to know that a Zariski locally trivial smooth morphism of smooth schemes with fibers isomorphic to affine spaces is an \(A^1\)-weak equivalence (cf. [MV99] §3 Example 2.3). We will also occasionally use the functor \(E_{\Delta^1}\) and its properties, in particular the fact that it commutes with formation of finite limits; for an overview of these kinds of facts, we refer the reader to [AD07a].

For the purposes of this paper, it is important to know that \(\mathcal{S}_{pc_k}^e\) has all small limits and colimits. If \(k\) is clear from context, to emphasize the analogy with topology, we denote by \(\ast\) the final (and initial) object of the category \(\mathcal{S}_{pc_k}^e\) rather than using \(\text{Spec} \, k\). In this direction, given two pointed spaces \((\mathcal{X}, x)\) and \((\mathcal{Y}, y)\) the wedge sum \(\mathcal{X} \vee \mathcal{Y}\), which can be thought of as “one-point union,” is the push-out of the diagram \((\mathcal{X}, x) \leftarrow \ast \rightarrow (\mathcal{Y}, y)\). Also, the smash product \(\mathcal{X} \wedge \mathcal{Y}\) of two pointed spaces is obtained by collapsing (i.e., forming a quotient) \(\mathcal{X} \vee \mathcal{Y} \subseteq \mathcal{X} \times \mathcal{Y}\) to a point.

As we noted in the introduction, the \(A^1\)-homotopy category possesses two circles: the *Tate circle*, sometimes denoted \(S^1_t\), is just the space \(G_m\) (pointed by 1) and the *simplicial circle*, denoted \(S^1_s\) is the space \(A^1/\{0,1\}\) (cf. [MV99] §3.2.2). We let \(\Omega_i^e\) denote the \(i\)-fold \(G_m\)-looping functor, though the operation of smashing with \(G_m\), i.e., *Tate suspension*, will be denoted as usual by \(G_m \wedge \cdot\). We let \(S_i^e\) denote the \(i\)-fold simplicial sphere, and with this in mind, we will write \(\Sigma_i^e\) for the \(i\)-fold simplicial suspension functor, i.e., the functor \(S_i^e \wedge \cdot\), and \(\Omega_i^e\) for the \(i\)-fold simplicial looping functor.

We also highlight some analogues of various pieces of terminology from classical homotopy theory. The \(A^1\)-homotopy groups of a pointed space \((\mathcal{X}, x)\), denoted \(\pi_i^{A^1}(\mathcal{X}, x)\), are the sheaves
associated with the presheaves
\[ U \mapsto [S^n_i \wedge U_+, (\mathcal{X}, x)]_{\mathbb{A}^1}, \]
and the sheaf \( \pi_0^{\mathbb{A}^1}(\mathcal{X}) \) of \( \mathbb{A}^1 \)-connected components is the sheaf associated with the presheaf
\[ U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1}. \]
A space is called \( \mathbb{A}^1 \)-connected if \( \pi_0^{\mathbb{A}^1}(X) \) is the trivial sheaf. A space is called \( n \)-\( \mathbb{A}^1 \)-connected (for \( n \geq 1 \)) if \( \pi_i^{\mathbb{A}^1}(\mathcal{X}, x) \) is the trivial sheaf of groups whenever \( i \leq n \). A space will be called \( \mathbb{A}^1 \)-contractible if the structure morphism \( \mathcal{X} \to * \) is an \( \mathbb{A}^1 \)-weak equivalence; by the \( \mathbb{A}^1 \)-Whitehead theorem (see [MV99 §Proposition 2.13]) this is equivalent to the statement that \( \mathcal{X} \) is \( n \)-\( \mathbb{A}^1 \)-connected for every \( n \geq 0 \). For a very brief survey of structural properties of the \( \mathbb{A}^1 \)-homotopy groups, statements of results like the \( \mathbb{A}^1 \)-Hurewicz theorem, \( \mathbb{A}^1 \)-homology together with its basic properties (e.g., Mayer-Vietoris) we refer the reader to [AD07a]. For a detailed treatment of this circle of results including all proofs, we refer the reader to the other main foundational work in unstable \( \mathbb{A}^1 \)-homotopy theory: [Mor06].

2 Homotopy equivalence by affine vector bundle torsor

In this section, we revisit the so-called Jouanolou-Thomason homotopy lemma. In our language, the Jouanolou-Thomason homotopy lemma states that any regular scheme is the base of a Zariski locally trivial fibration with total space a regular affine scheme and fibers isomorphic to affine spaces. In some cases, the total space appearing in the Jouanolou-Thomason lemma can be made quite explicit and explicit hypersurface presentations of the total spaces can be given; Proposition 2.4 gives a representative special case of this kind of statement.

The specific kind of affine space fiber bundles constructed below can be viewed as a generalization of the unipotent quotients studied in [AD07b]. We will use the techniques introduced in the proof of the Jouanolou-Thomason homotopy lemma to prove the existence of some \( \mathbb{A}^1 \)-contractible smooth schemes not isomorphic to affine space (see Theorem 2.10). Furthermore, the sense in which affine space fiber bundles are different from principal bundles under unipotent groups is made precise in Theorem 2.11 and its important Corollary 2.12. We close the section with a construction of some highly \( \mathbb{A}^1 \)-connected smooth affine hypersurfaces.

Affine vector bundle torsors

Given a locally free sheaf \( \mathcal{F} \) on a scheme \( X \), the associated geometric vector bundle \( \text{Spec Sym}(\mathcal{F}^\vee) = \mathbb{V}(\mathcal{F}) \to X \) (the symmetric algebra of the dual of \( \mathcal{F} \)) is naturally equipped with a (commutative) multiplication \( \mathbb{V}(\mathcal{F}) \times_X \mathbb{V}(\mathcal{F}) \) induced by the diagonal morphism \( \mathcal{F} \to \mathcal{F} \oplus \mathcal{F} \) of \( \mathcal{O}_X \)-modules, we refer to the resulting action as the “translation” action of \( \mathbb{V}(\mathcal{F}) \) on itself. Of course, if \( X = \text{Spec} k \) with \( k \) a field, this corresponds to the usual addition for \( k \)-vector spaces. By abuse of notation, we will sometimes denote locally free sheaves and their associated geometric vector bundles using the same symbols.
Definition 2.1. Let $X$ be a scheme, $\mathcal{F}$ a locally free sheaf on $X$, and $\pi : \mathbb{V}(\mathcal{F}) \to X$ a geometric vector bundle over $X$. An affine vector bundle torsor over $X$ (and under $\mathbb{V}(\mathcal{F})$) consists of a Zariski locally trivial smooth affine morphism $f : Y \to X$, together with an action morphism $\mathbb{V}(\mathcal{F}) \times_X Y \to Y$, such that i) $Y$ is an affine scheme, ii) the action of $\mathbb{V}(\mathcal{F})$ on $Y$ is Zariski locally isomorphic to $\mathbb{V}(\mathcal{F})$ acting on itself by translations.

One can show that Zariski locally trivial torsors under the vector bundle $\mathbb{V}(\mathcal{F})$ over $X$ are classified by elements of the group $H^1_{\text{Zar}}(X, \mathcal{F})$ (which can be interpreted as a Čech cohomology group) (cf. [SGA03] Exposé XI 4.7). In the special case where $\mathcal{F} = \mathcal{O}_X$, and $\mathbb{V}(\mathcal{F}) = \mathbb{A}^1_X$, affine vector bundle torsors are the same as everywhere stable actions of $\mathbb{A}^1_X$ on schemes studied in [AD07b] Theorem 3.10.

If $X = \text{Spec} A$ is an affine scheme, Serre’s theorem on vanishing of coherent cohomology on an affine scheme ([Gro61b Théorème 1.3.1]) shows that affine vector bundle torsors over $X$ are trivial, i.e., isomorphic to actual vector bundles. Any geometric vector bundle admits a zero section, though, as we shall see, general torsors under vector bundles need not admit sections.

It is not a priori clear that one can construct affine vector bundle torsors over a non-affine scheme. The following result, first proved by Jouanolou for quasi-projective varieties, and extended by Thomason to varieties admitting an ample family of line bundles guarantees existence of affine vector bundle torsors in good situations. Before we state the general result, let us discuss the motivating example.

Example 2.2. Let $X = \mathbb{P}^n$. Consider the product $\mathbb{P}^n \times \mathbb{P}^n$ where the second factor is viewed as the dual projective space to the first factor. There is an incidence hyperplane $H \subset \mathbb{P}^n \times \mathbb{P}^n$, and one can check that $\mathbb{P}^n = \mathbb{P}^n \times \mathbb{P}^n \setminus H$ is an affine variety (e.g., because $H$ is an ample divisor). The projection onto the first factor $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ induces a morphism $\mathbb{P}^n \to \mathbb{P}^n$ that is Zariski locally trivial with affine space fibers. The space $\mathbb{P}^n$ can, upon choice of a base-point, also be identified as a homogeneous space of the form $SL_{n+1}/H$, where $SL_{n+1}$ acts diagonally on the product. The stabilizer group $H$ is conjugate to the sub-group of $\mathbb{G}_m \times GL_n$ obtained by forming the kernel of the homomorphism $\mathbb{G}_m \times GL_n \to \mathbb{G}_m$ defined by $(\lambda, X) \mapsto \lambda \cdot \det(X)$.

Proposition 2.3 (Jouanolou-Thomason homotopy lemma ([Wei89 Proposition 4.3])). Suppose $X$ is a regular $k$-scheme (by convention, schemes are separated and finite type!). There exists a locally free sheaf $\mathcal{F}$ on $X$, and an affine vector bundle torsor $f : Y \to X$ under $\mathbb{V}(\mathcal{F})$.

Proposition 2.4. Let $X$ be an affine scheme. Let $d \geq 2$ be an integer. Suppose $Z \subset X$ is a codimension $d$ closed subscheme defined by the vanishing of functions $g_1, \ldots, g_m \in k[X]$. Choose coordinates $x_1, \ldots, x_m$ on $\mathbb{A}^m$ and define

$$Y \subset X \times \mathbb{A}^m := \{ \sum_i x_i g_i = 1 \}.$$ 

The composite morphism $f := Y \leftarrow X \times \mathbb{A}^m \to X$ induces a surjective morphism $f : Y \to X \setminus Z$ that is an affine vector bundle torsor.

Proof. The functions $g_i$ induce a morphism

$$g = (g_1, \ldots, g_m) : \mathcal{O}_X \setminus Z \to \mathcal{O}^{\oplus m}_{X \setminus Z}.$$
Since \( g \) is a split monomorphism on each open affine where some \( g_i \) is non-vanishing, it follows that the cokernel of \( g \) is, essentially by construction, a locally free sheaf on \( X \setminus Z \) that we will denote by \( \mathcal{F} \). For simplicity of notation, we let \( \mathcal{E} = \mathcal{O}_{X \setminus Z}^m \). Consider the projective space bundle \( \mathbb{P}(\mathcal{E}) = \text{Proj} \text{Sym}(\mathcal{E}^\vee) \). Since we have a surjective map \( \mathcal{E} \twoheadrightarrow \mathcal{F} \), we get a closed immersion of projective space bundles \( \mathbb{P}_{X \setminus Z}(\mathcal{F}) \hookrightarrow \mathbb{P}_{X \setminus Z}(\mathcal{E}) \). We set

\[
Y = \mathbb{P}_{X \setminus Z}(\mathcal{E}) \setminus \mathbb{P}_{X \setminus Z}(\mathcal{F}).
\]

Let \( \pi \) denote the projection morphism \( \mathbb{P}_{X \setminus Z}(\mathcal{E}) \twoheadrightarrow X \setminus Z \), and let \( f : Y \to X \setminus Z \) the morphism induced by composition of inclusion and projection. We will now show that \( Y \) has the form described in the statement of the result.

The morphism \( g \) is a section of \( \mathcal{E} \), which can be identified with a global section of \( \mathcal{O}_{\mathbb{P}_{X \setminus Z}(\mathcal{E})}(1) \) over \( \mathbb{P}_{X \setminus Z}(\mathcal{E}) \). We can then identify \( Y \) with \( \text{Spec} \text{Sym}(\mathcal{E})/(g-1) \). One can check that \( Y \) is a torsor under the geometric vector bundle \( \mathcal{F} \), which can be identified with \( \text{Spec} \text{Sym}(\mathcal{O}_{X \setminus Z}^m)/(g-0) \). In particular, the morphism \( f : Y \to X \setminus Z \) is affine.

Now, let us check that \( Y \) can be described as a hypersurface as claimed. We have already shown that \( Y \) is defined by a hypersurface in \( X \setminus Z \times \mathbb{A}^m \cong \text{Spec} \text{Sym}(\mathcal{O}_{X \setminus Z}^m) \). To see that \( Y \) is affine, note that each function \( g_i \) induces an element \( f_i \in \Gamma(Y, \mathcal{O}_Y) \) such that \( \sum_i f_i = 1 \). Explicitly, we can identify \( f_i \) with \( x_i g_i \). Furthermore, each \( Y_{f_i} = f_i^{-1}(X \setminus Z_{g_i}) \) is affine, being an open affine subscheme of \( \pi^{-1}(X_\alpha) \). It follows from the local criterion of affineness (see \cite[Théorème 5.2.1(a) and (b)]{Gro61a}) that \( Y \) is affine. It follows from the defining equation that \( Y \) has the required form, since if the \( g_i \) all simultaneously vanish, the hypersurface equation cannot be satisfied.

\[ \square \]

Remark 2.5. The Jouanolou-Thomason homotopy lemma is often used as a “black-box” to reduce problems about general smooth schemes to problems about affine schemes. It is important to note that the construction given above is not functorial for general maps. However, it is functorial for affine morphisms, as affine morphisms are stable under base change (\cite[Proposition 1.5.1]{Gro61a}). We will use this fact repeatedly in the sequel.

Remark 2.6. The hypersurfaces studied in the previous proposition are not “generic” among hypersurfaces of a given degree; see \cite[§1.1.2]{Kol98} for a discussion of this point. It is a very interesting question to determine the \( \mathbb{A}^1 \)-connectivity of a generic hypersurface in affine space of a given degree.

Example 2.7. Take \( \mathbb{A}^m \setminus 0 \) where 0 is defined by the vanishing of \( x_1, \ldots, x_m \). Then Proposition 2.4 shows that the hypersurface \( Q_{2m-1} \subset \mathbb{A}^{2m} \) defined by the equation \( \sum_i x_i x_{m+i} = 1 \) admits a morphism \( \varphi \) to \( \mathbb{A}^m \setminus 0 \) that is Zariski locally trivial with fibers isomorphic to affine space.

Remark 2.8. Example 2.7 is universal among Zariski locally trivial fiber spaces with affine space fibers in the following sense. Suppose \( X \) is an affine scheme and \( Z \subset X \) is a closed subscheme whose ideal is generated by \( f_1, \ldots, f_d \) as in the proof of Proposition 2.4. The functions \( f_1, \ldots, f_d \) define an affine morphism \( f : X \to \mathbb{A}^d \). By construction \( f^{-1}(0) \cong Z \) and thus there is an induced affine morphism \( f : X \setminus Z \to \mathbb{A}^d \setminus 0 \).
If we pull back the torsor $Q_{2d-1} \to \mathbb{A}^d \setminus 0$ by $f$, we get a cartesian square of the form

$$
\begin{array}{ccc}
X \setminus Z \times_{\mathbb{A}^d \setminus 0} Q_{2d-1} & \to & Q_{2d-1} \\
\downarrow \varphi' & & \downarrow \varphi \\
X \setminus Z & \to & \mathbb{A}^d \setminus 0.
\end{array}
$$

Since $f$ and $\varphi$ are affine morphisms, it follows that so are $f'$ and $\varphi'$. It is easy to check that $\varphi'$ is a Zariski locally trivial fibration with affine space fibers (in fact a torsor under the pull-back of the vector bundle under which $Q_{2d-1}$ is an affine vector bundle torsor). Furthermore, since $Q_{2d-1}$ is affine, it follows that the fiber product $X \setminus Z \times_{\mathbb{A}^d \setminus 0} Q_{2d-1}$ is an affine scheme as well.

Let us also observe that changing the generators of $Z$ will change the presentation of this fiber product. Without too much additional effort, the ideas sketched here can be turned into a proof of Proposition 2.4. The knowledgeable reader will observe that this discussion provides nothing more than a (partial) description of the functor of points of $\mathbb{A}^d \setminus 0$.

**Some new $\mathbb{A}^1$-contractible smooth schemes**

Recall that an *exotic $\mathbb{A}^1$-contractible smooth scheme* is an $\mathbb{A}^1$-contractible smooth scheme not isomorphic to affine space. In [AD07b] we constructed $\mathbb{A}^1$-contractible smooth schemes as quotients of affine space by free actions of the additive group $\mathbb{G}_a$ (or, more generally, free actions of unipotent groups on affine spaces). All of the examples constructed in [AD07b] could be realized as open subschemes of affine schemes with complement of codimension $\leq 2$. We will now produce examples of $\mathbb{A}^1$-contractible smooth schemes that have complement of codimension $d \geq 3$ in a smooth affine scheme.

Choose coordinates $x_1, \ldots, x_n$ on $\mathbb{A}^n$. If $n = 2m + 1$, let $Q_{2m}$ be the hypersurface defined by the equation

$$
\sum_{i} x_i x_{m+i} = x_{2m+1}(1 + x_{2m+1}).
$$

Let $E_m$ be the closed subvariety of $Q_{2m}$ defined by the equations $x_1 = \cdots = x_m = 0, x_{2m+1} = -1$, and set

$$
X_{2m} = Q_{2m} \setminus E_m.
$$

We will see in Lemma 3.1 that the quadrics $Q_{2m}$ are smooth over $\text{Spec} \mathbb{Z}$, and it follows immediately (via [SGA03] Proposition II.1.3) that $X_{2m}$ is smooth over $\text{Spec} \mathbb{Z}$ as well.

**Remark 2.9.** If $k = \mathbb{C}$, one can check that $X_{2m}(\mathbb{C})$ is a contractible complex manifold. We will see in the next section that over $\mathbb{C}$, the variety $Q_{2m}$ is isomorphic to the variety defined by $\sum_{i=1}^{2m+1} x_i^2 = 1$. Wood explains ([Woo93] §2) how to identify the last variety with the tangent bundle of a sphere. Under this isomorphism, the variety $E_m$ can be identified with the tangent space at a point. The complement is, by the method indicated in the introduction, then a contractible topological space diffeomorphic to $\mathbb{R}^{4m}$. The following result implies that $X_{2m}(\mathbb{R})$ is in fact a contractible manifold as well (via realization functors, see [Asa08a] Lemma 2.5).
Theorem 2.10. For each integer $m \geq 0$, the variety $X_{2m}$ is $\mathbb{A}^1$-contractible over $\text{Spec} \mathbb{Z}$.

Proof. If $m = 0$ or 1, the variety in question is easily seen to be isomorphic to affine space of dimension $2m$. If $m = 2$, the variety in question was studied in [AD07b] (see p. 3 and Remark 5.2). Suppose $m > 2$. By Proposition 2.4 we know that $X_{2m}$ is $\mathbb{A}^1$-weakly equivalent to the hypersurface $Y$ in $Q_{2m} \times \mathbb{A}^{m+1}$ defined by the quadric $\sum_{i=1}^{m} y_i x_i + y_{m+1}(x_{2m+1} + 1) = 1$, i.e., to an intersection of two quadrics in $\mathbb{A}^{3m+2}$. We claim that $Y$ is in fact isomorphic to $\mathbb{A}^{3m}$ and the argument we give below lends credence to this assertion. However, and most unfortunately, it seems quite difficult to check this statement.

Instead, we'll construct an explicit surjective morphism $w : \mathbb{A}^{3m} \to Q_{2m} \setminus E_m$, which is Zariski locally trivial with fibers isomorphic to affine spaces. To do this, we'll just describe the coordinate functions of the morphism. Choose coordinates $x_1, \ldots, x_{2m}, y_1, \ldots, y_m$ on $\mathbb{A}^{3m}$. Let $w_1, \ldots, w_{2m+1}$ be coordinates on $\mathbb{A}^{3m+1}$. Let $Q_x = \sum_i x_i x_{m+i}$. Consider the morphism $w = (w_1, \ldots, w_{2m+1})$ defined by the functions

\begin{equation}
    w_i = \begin{cases}
        x_i & \text{for } i = 1, \ldots, n \\
        x_i (1 + Q_x) + f_{i-m} & \text{for } i = m + 1, \ldots, 2m \\
        Q_x & \text{for } i = 2m + 1,
    \end{cases}
\end{equation}

where the $f_i$ are defined by the following formulae:

\begin{align*}
    f_1 &= 0 + x_2 y_2 + x_3 y_3 + x_4 y_4 + \cdots + x_m y_m, \\
    f_2 &= -x_1 y_2 + 0 + x_3 y_4 + x_4 y_5 + \cdots + x_m y_1, \\
    f_3 &= -x_1 y_3 - x_2 y_4 + 0 + x_4 y_1 + \cdots + x_m y_2, \\
    f_4 &= -x_1 y_4 - x_2 y_5 - x_3 y_6 + 0 + \cdots + x_m y_3, \\
    \vdots &= \vdots \vdots \vdots \vdots \vdots \vdots \vdots \\
    f_m &= -x_1 y_m - x_2 y_1 - x_3 y_2 - x_4 y_3 + \cdots \cdots + 0.
\end{align*}

By construction one has $\sum_i x_i f_i = 0$ and so the functions $w_i$ satisfy the equation

$$
\sum_i w_i w_{m+i} = w_{2m+1}(1 + w_{2m+1}).
$$

If $w_1 = \cdots = w_m = 0$, then this implies $x_1 = \cdots = x_m = 0$ and hence $w_{2m+1} = Q_x = 0$. Thus, the image of $w$ is contained in $Q_{2m} \setminus E_m$. (One may check by (extremely tedious) explicit computation of $dw$, that $w$ has the correct rank (i.e., $2m$); this computation will be unnecessary by the argument below.)

We will check that $w$ is Zariski locally trivial with fibers isomorphic to affine spaces. Assuming this, we first show that $w$ is in fact a smooth morphism. To check $w$ is smooth, suppose we can find an open cover $U_i$ of $X_{2m}$ over which $w$ trivializes. Let $u : \prod_i U_i = U \to X_{2m}$ denote the corresponding covering morphism. We may use [SGA03] Corollaire II.4.13 to observe that $w$ is smooth if and only if the its base-change along $u$

$$
w' : \mathbb{A}^{3m} \times_{X_{2m}} U \to U
$$
is smooth. Now, by assumption $\mathbb{A}^{3m} \times X_{2m} \xrightarrow{\sim} \mathbb{A}^m \times U$. Under this isomorphism $w'$ is transformed into projection onto the second factor. The scheme $U$ is smooth (being an open subscheme of a smooth scheme) and affine space is smooth so it follows that $w'$ is smooth as well.

Now, we construct the open cover of $X_{2m}$ over which $w$ will trivialize. We will show that over each open affine subscheme of the form $w_i \neq 0$ ($i = 1, \ldots, m$) and $w_{2m+1} \neq -1$, $w$ admits an explicit trivializing section. Note that assuming $w_i \neq 0$ implies $x_i \neq 0$, and assuming $w_{2m+1} \neq -1$ implies that $Q_X \neq -1$. The inclusion $k[y_1, \ldots, y_m] \hookrightarrow k[x_1, \ldots, x_{2m}, y_1, \ldots, y_m]$ induces a projection morphism $\psi : \mathbb{A}^{3m} \hookrightarrow \mathbb{A}^m$. We thus have a product morphism

$$w \times \psi : \mathbb{A}^{3m} \longrightarrow Q_{2m} \setminus E_m \times \mathbb{A}^m.$$ 

We claim that there are explicit sections $s_i : Q_{2m} \setminus \{w_i = 0\} \longrightarrow \mathbb{A}^{3m} \setminus \{x_i = 0\}$ ($i = 1, \ldots, m$) and $s_{2m+1} : Q_{2m} \setminus \{w_{2m+1} = -1\} \longrightarrow \mathbb{A}^{3m} \setminus \{Q_X = -1\}$ that split this projection morphism.

Constructing the sections $s_i$ is equivalent to writing down morphisms

$$k[x_1, \ldots, x_{2m}, y_1, \ldots, y_m, x_i^{-1}] \longrightarrow k[w_1, \ldots, w_{2m+1}, w_i^{-1}]/(\sum_j w_j w_{m+j} = w_{2m+1}(1 + w_{2m+1})).$$

To do this, send $x_1, \ldots, x_{2m}$ to $w_1, \ldots, w_m, w_{m+1}, \ldots, w_m, w_{2m+1}$ in order (where the hat indicates that one elides the corresponding variable). Since under the assumption that $w_i \neq 0$ we can solve for $w_{m+i}$ in terms of the other variables, one can check that the above map determines an isomorphism $Q_{2m} \setminus \{w_i = 0\} \xrightarrow{\sim} \mathbb{A}^{2m-1} \times \mathbb{G}_m$ (the $\mathbb{G}_m$ factor corresponds to the variable $w_i$). In total, we get a product decomposition $\mathbb{A}^{3m} \setminus \{x_i = 0\} \xrightarrow{\sim} \mathbb{A}^{2m-1} \times \mathbb{G}_m \times \mathbb{A}^m$ which gives the trivializing section.

Similarly, we construct $s_{2m+1}$ as follows. If $w_{2m+1} \neq -1$, then we have $w_{2m+1} = \frac{1}{w_{2m+1} + 1} \sum_i w_i w_{m+i}$. The change of coordinates $w_i' = \frac{w_i}{w_{2m+1} + 1}$ is invertible. Note that $w_{2m+1} \neq -1$ is then equivalent to $\sum_i w_i' w_{m+i} = -1$. The map sending $x_1, \ldots, x_{2m}$ to $w_1, \ldots, w_m$ in order then determines an isomorphism $Q_{2m} \setminus \{w_{2m+1} = -1\} \xrightarrow{\sim} \mathbb{A}^{2m} \setminus \{Q_X = -1\}$. Thus, we get a product decomposition $\mathbb{A}^{3m} \setminus \{Q_X = -1\} \xrightarrow{\sim} \mathbb{A}^{2m} \setminus \{Q_X = -1\} \times \mathbb{A}^m$. Taken together, the above constructions provide the required trivialization of $w$.

\[\square\]

### Some counter-examples

In [AD07b] Question 1.1 and [AD07c] Remark 1.4, we asked whether every $\mathbb{A}^1$-contractible smooth scheme can be realized as a quotient of an affine space by the free action of a unipotent group. We will now see that for $m \geq 3$, the varieties $X_{2m}$ studied in Theorem 2.10 cannot be realized as quotients of affine space by free actions of unipotent groups $U$. We begin by proving a general “excision” style result (Theorem 2.11) for (Zariski) cohomology with coefficients in a unipotent group; the main result then follows from this excision result when applied in the special case of degree 1 cohomology (Corollary 2.12). Indeed, the degree 1 cohomology group in question can be identified as the (pointed) set of $U$-torsors on the scheme $X$. We warn the reader that, contrary to “understood” prior conventions, the letter $U$ in this section stands for a unipotent group, as opposed to an open subscheme; we refer the reader to [Bor91] §15 (Definition 15.1) for general information on split unipotent groups.
Theorem 2.11 (Excision for unipotent groups). Let $d \geq 2$ be an integer. Suppose $X$ is a regular scheme over a field and $W \subset X$ is an open subscheme whose closed complement has codimension $\geq d$. If $U$ is any split unipotent group, then for $i \leq 1$ the induced restriction map

$$H^i_{\text{Zar}}(X,U) \longrightarrow H^i_{\text{Zar}}(W,U)$$

is an epimorphism for $i \leq d - 1$, and an isomorphism for $i \leq d - 2$. If furthermore $U$ is a commutative unipotent group, then the same result holds without the first restriction on $i$.

Proof. Since $U$ is split, by definition it admits an increasing filtration by normal subgroups with subquotients isomorphic to $\mathbb{G}_a$. Given an exact sequence of the form

$$1 \longrightarrow U' \longrightarrow U \longrightarrow \mathbb{G}_a$$

we get long exact sequences in cohomology of the form

$$\cdots \longrightarrow H^i(X,U') \longrightarrow H^i(X,U) \longrightarrow H^i(X,\mathbb{G}_a) \longrightarrow \cdots$$

where the exact sequence does not continue to the right for $i > 1$ if $U'$ is non-commutative. It follows that, by induction on the dimension of $U$, together with the 5-lemma, to prove the result in general, it suffices to prove the result for $U = \mathbb{G}_a$. In this case, one knows that by definition $H^i(X,\mathbb{G}_a) = H^i(X,\mathcal{O}_X)$. Thus, our goal is to show that the map

$$H^i(X,\mathcal{O}_X) \longrightarrow H^i(W,\mathcal{O}_W)$$

induced by pull-back via the inclusion $W \hookrightarrow X$ is an epimorphism for $i \leq d - 1$ and an isomorphism for $i \leq d - 2$.

To prove the last fact, we use the Cousin complex for $\mathcal{O}_X$ as introduced in e.g., [Har66] IV.2. Under the assumption that $X$ is regular, the Cousin complex provides an injective resolution of $\mathcal{O}_X$ (see especially ibid. p. 239). Let $X^{(p)}$ denote the set of codimension $p$ points in $X$ (recall this means $\dim \mathcal{O}_{X,x} = p$). The $p$-th term is

$$\prod_{x \in X^{(p)}} i_x(H^p\mathcal{O}_X),$$

and in particular $H^i(X,\mathcal{O}_X)$ only depends on points of codimension $d - 1$ in $X$. The construction of the Cousin complex is furthermore contravariantly functorial for morphisms of schemes. Since the inclusion $W \hookrightarrow X$ is, by definition, an isomorphism on points of codimension $d - 1$, the result follows from the fact that cokernel of the induced morphism of cousin complexes for $\mathcal{O}_X$ and $\mathcal{O}_W$ only depends on points of codimension $\geq d$.

Corollary 2.12. Suppose $X$ is a regular affine scheme and $W \subset X$ is an open subscheme whose complement has codimension $d \geq 3$. Every torsor under a split unipotent group over $W$ is trivial. In particular, the varieties $X^m_{2m}$ cannot be realized as quotients of affine space by free actions of split unipotent groups for $m \geq 3$. 

\[ \Box \]
Proof. By the identification of \(U\)-torsors on a scheme \(X\) with the group \(H^1_{\text{zar}}(X, U)\), it suffices to prove that \(H^1(W, U)\) is trivial for any \(U\)-torsor on \(X\). Under the assumptions, \(H^1(W, U) \cong H^1(X, U)\) by Theorem 2.11. Since the group \(H^1(X, U)\) is the cohomology of a coherent sheaf on an affine scheme, its vanishing follows from Serre’s theorem about vanishing of coherent cohomology over an affine scheme ([Gro61b] Théorème 1.3.1).

Now any \(U\)-torsor on \(W\) is trivial. Thus, it suffices to show that \(W \times U\) can never be affine. This last statement follows easily from the fact that \(W \times U\) is an open subscheme of \(X \times U\) (which is affine) with complement of codimension \(d\). The last statement comes from the observation that \(X_{2m}\) has complement of codimension 3 in a smooth affine variety as long as \(m \geq 3\).

\[\square\]

Remark 2.13. It follows from Corollary 2.12 that, using the notation of Proposition 2.4, whenever \(Z \subset \mathbb{A}^n\) has codimension \(\geq 3\), the torsor under a vector bundle constructed by the Jouanolou-Thomason homotopy lemma is not a torsor under a unipotent group. The above corollary admits the following partial converse.

Lemma 2.14. If \(X\) is a regular affine scheme, \(W \subset X\) is an open subscheme whose closed complement is of codimension 2 and whose ideal is generated by exactly 2 elements, then \(W\) can be realized as the quotient of an affine scheme by the free action of a unipotent group.

Proof. The proof of this statement is essentially identical to that sketched in Remark 2.8. Observe that \(\mathbb{A}^2 \setminus 0\) can be realized as a quotient of \(SL_2\) by a free action of \(\mathbb{G}_a\). Now, if \(W \subset X\) has complement \(Z\) of codimension 2 then under the hypothesis that \(Z\) is generated by exactly two elements \(f_1, f_2\) we can proceed in a manner analogous to the one sketched previously. \[\square\]

The structure of non-affine exotic \(\mathbb{A}^1\)-contractibles

We can now give a precise conjecture regarding the structure of quasi-affine but not affine \(\mathbb{A}^1\)-contractible smooth schemes.

Conjecture 2.15. Suppose \(X\) is a quasi-affine yet not affine \(\mathbb{A}^1\)-contractible smooth scheme that can be realized as the complement of codimension \(d\) subscheme \(Z\) of an affine scheme. There exist an integer \(n\) and a Zariski locally trivial smooth morphism with affine space fibers \(f: \mathbb{A}^n \rightarrow X\).

Remark 2.16. Ideally, we would like to take \(n = \dim X + d\) where \(d\) is the codimension of \(Z\) in \(X\). The naïve obstruction to doing this is provided by the fact that \(Z\) need not be a set-theoretic complete intersection. In general, suppose the ideal of \(Z\) is generated by at least \(d'\) elements. Any choice of generators \(f_1, \ldots, f_{d'}\) of the ideal of \(Z\) induces a Zariski locally trivial smooth morphism with affine space fibers \(\mathbb{A}^n \rightarrow X\) \((n = \dim X + d')\) by pulling back the morphism \(Q_{2m-1} \rightarrow \mathbb{A}^{d'} \setminus 0\) as in Remark 2.8 and we could conjecture that the total space of the fiber product is isomorphic to affine space (it is obviously \(\mathbb{A}^1\)-contractible smooth and affine).

\(\mathbb{A}^1\)-homotopy types of some hypersurfaces

Combining the results of the Proposition 2.4 with some excision results proved in [AD07a] and [As08a], we can obtain some bounds on \(\mathbb{A}^1\)-connectivity of hypersurfaces. Throughout the rest
of this subsection, we assume that \( k \) is an infinite field as this is a necessary condition for the theorem statements that follow.

**Theorem 2.17** ([AD07a] Theorem 4.1). Let \( d \geq 2 \) be an integer, and \( m = \max(d - 3, 0) \). Suppose \( X \) is an \( m \)-\( \mathbb{A}^1 \)-connected smooth scheme. Suppose \( U \subset X \) is an \( \mathbb{A}^1 \)-connected open subscheme whose closed complement has codimension \( d \). For any choice of base-point \( x \in U(k) \), the induced morphism

\[
\pi_i^{\mathbb{A}^1}(U, x) \longrightarrow \pi_i^{\mathbb{A}^1}(X, x)
\]

is an epimorphism for \( i \leq d - 1 \) and an isomorphism for \( i \leq d - 2 \).

**Proposition 2.18** ([Aso08a] Proposition 2.1). Suppose \( X \) is an \( \mathbb{A}^1 \)-contractible smooth scheme. If \( U \subset X \) be an open subscheme whose closed complement has codimension \( d \geq 2 \), then \( U \) is at least \((d - 2)\)-\( \mathbb{A}^1 \)-connected.

**Corollary 2.19** (Hypersurface connectivity). Let \( g_1, \ldots, g_m \in k[t_1, \ldots, t_n] \), and the dimension of the subscheme \( Z \) of \( \mathbb{A}^n \) whose ideal is generated by \( g_1, \ldots, g_m \) is \( d \geq 2 \). If \( x_1, \ldots, x_m \) are coordinates on \( \mathbb{A}^m \), the hypersurface \( Y \) in \( \mathbb{A}^{n+m} \) defined by the equation \( \sum_i x_i g_i = 1 \) is at least \((d - 2)\)-\( \mathbb{A}^1 \)-connected.

**Proof.** Since affine space \( \mathbb{A}^m \) is \( \mathbb{A}^1 \)-contractible, we can conclude that \( \mathbb{A}^m \setminus Z \) is at least \((d - 2)\)-\( \mathbb{A}^1 \)-connected by combining Theorem 2.17 and Proposition 2.18. Next, by Proposition 2.4 we know that the morphism \( f : Y \longrightarrow \mathbb{A}^m \setminus Z \) is an affine vector bundle torsor. Since affine vector bundle torsors are Zariski locally trivial smooth morphisms with fibers isomorphic to affine spaces, it follows from, e.g., [MV99] §3.2.4 that \( f \) is in fact an \( \mathbb{A}^1 \)-weak equivalence. \( \square \)

We can use these results to deduce existence of “many” highly \( \mathbb{A}^1 \)-connected” smooth affine schemes.

**Corollary 2.20.** There exist at least countably many \((n - 2)\)-\( \mathbb{A}^1 \)-connected smooth affine varieties of dimension \( 2n - 1 \). There exist arbitrary dimensional moduli of \((n - 2)\)-\( \mathbb{A}^1 \)-connected smooth affine varieties of dimension \( 2n \).

**Proof.** The space \( \mathbb{A}^n \setminus \{x_1, \ldots, x_m\} \) is \( \mathbb{A}^1 \)-weakly equivalent to a smooth affine scheme of dimension \( 2n - 1 \) and is \((n - 2)\)-\( \mathbb{A}^1 \)-connected by excision.

Start with a codimension \((n - 1)\)-subvariety \( Z \) of \( \mathbb{A}^n \), and assume that \( n > 3 \). We can assume the subvariety is generated by exactly \( n - 1 \) equations. By the excision results quoted above \( \mathbb{A}^n \setminus Z \) is \((n - 3)\)-\( \mathbb{A}^1 \)-connected. By the Jouanolou-Thomason homotopy lemma \( \mathbb{A}^n \setminus Z \) has the \( \mathbb{A}^1 \)-homotopy type of a smooth affine scheme of dimension \( n + n - 2 = 2(n - 1) \). \( \square \)

**Remark 2.21.** Corollary 2.19 begs the question of actually computing \( \pi_i^{\mathbb{A}^1}(\mathbb{A}^m \setminus Z, x) \) for any choice of base-point \( x \in \mathbb{A}^m \setminus Z(k) \). For \( Z \) a finite collection of points, we do this next.
Some further examples

Recall that the free strongly $\mathbb{A}^1$-invariant sheaf of groups generated by a pointed space $(\mathcal{S}, s)$ is by definition $\pi_1^A(\Sigma_1^s \mathcal{S})$. Likewise, the free strictly $\mathbb{A}^1$-invariant sheaf of groups generated by $(\mathcal{S}, s)$ is the sheaf $H_0^A(\mathcal{S})$.

Also, recall that the join of two pointed spaces $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$, denoted $\mathcal{X} \ast \mathcal{Y}$, is the homotopy push-out of the diagram

$$
\mathcal{X} \leftarrow \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}
$$

computed in $\mathcal{H}(k)$. Under the hypotheses, one can show that $\mathcal{X} \ast \mathcal{Y} \cong \Sigma_1^s \mathcal{X} \land \mathcal{Y}$.

**Proposition 2.22.** Suppose $p_1, \ldots, p_m$ are a sequence of (distinct) closed points in $\mathbb{A}^n$, $n \geq 2$, $m \geq 1$. The complement $\mathbb{A}^n \setminus \{p_1, \ldots, p_m\}$ is exactly $(n-2)$-$\mathbb{A}^1$-connected. For any choice of base-point, we have:

$$
\pi_{n-1}^A(\mathbb{A}^n \setminus \{p_1, \ldots, p_m\}) \cong \begin{cases} 
F^A_1(m) & \text{if } n = 2 \\
K_{MW}^n & \text{if } n \geq 3,
\end{cases}
$$

where $F^A_1(m)$ is the free strongly $\mathbb{A}^1$-invariant sheaf of groups generated $\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \land \mathbb{G}_m$, for any collection of distinct closed points $x_1, \ldots, x_m \subset \mathbb{A}^1$.

**Proof.** The automorphism group of affine space $\mathbb{A}^n$ ($n \geq 2$) over any field is $d$-transitive for any $d \geq 2$. (For a proof in the case $k = \mathbb{C}$ of a slightly stronger statement, we refer the reader to [Win90].) Choose coordinates $y_1, \ldots, y_n$ on $\mathbb{A}^n$. Thus, after applying an appropriate automorphism of affine space, we can assume that the points $p_1, \ldots, p_m$ have $y_i = 0$ for $i > 1$ and $y_1 = a(p_i)$ for some $a(p_i) \in \mathbb{A}^1$. Let $x_i = a(p_i)$.

We can cover $\mathbb{A}^n \setminus \{p_1, \ldots, p_m\}$ by two open sets isomorphic to $\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \times \mathbb{A}^{n-1}$ and $\mathbb{A}^1 \times \mathbb{A}^{n-1} \setminus 0$ with intersection $\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \times \mathbb{A}^{n-1} \setminus 0$. In other words, we have a cocartesian diagram of the form

$$
\begin{array}{ccc}
\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \times \mathbb{A}^{n-1} \setminus 0 & \longrightarrow & \mathbb{A}^1 \times \mathbb{A}^{n-1} \setminus 0 \\
\downarrow & & \downarrow \\
\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \times \mathbb{A}^{n-1} & \longrightarrow & \mathbb{A}^n \setminus \{p_1, \ldots, p_m\}.
\end{array}
$$

Since $\mathbb{A}^{n-1}$ and $\mathbb{A}^1$ are $\mathbb{A}^1$-contractible, we see that $\mathbb{A}^n \setminus \{p_1, \ldots, p_m\}$ is in fact $\mathbb{A}^1$-weakly equivalent to the homotopy colimit of the diagram

$$
\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \longrightarrow \mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \times \mathbb{A}^{n-1} \setminus 0 \longrightarrow \mathbb{A}^{n-1} \setminus 0
$$

This homotopy colimit is the join $\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \ast \mathbb{A}^{n-1} \setminus 0$, which as we noted above is $\mathbb{A}^1$-weakly equivalent to $\Sigma_1^s(\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \land \mathbb{A}^{n-1} \setminus 0)$. An inductive computation using the same kind of open cover (with one point) shows that $\mathbb{A}^{n-1} \setminus 0$ is $\mathbb{A}^1$-weakly equivalent to $\Sigma_s^{n-2} \mathbb{G}_m^\wedge n-1$. Thus, in total, we have shown that one has an isomorphism

$$
\mathbb{A}^n \setminus \{p_1, \ldots, p_m\} \sim_{\mathbb{A}^1} \Sigma_s^{n-1}(\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \land \mathbb{G}_m^\wedge n-1),
$$
which of course depends on the choice of the \(x_1, \ldots, x_m\). However, the \(\mathbb{A}^1\)-homotopy type of the resulting space is independent of this choice.

The rest of the proof consists of applying standard tools from homotopy theory and homology theory, though in the \(\mathbb{A}^1\)-context. For example, it follows from Morel’s unstable \(\mathbb{A}^1\)-connectivity theorem ([Mor06] Theorem 3.38) that \(\mathbb{A}^n \setminus \{p_1, \ldots, p_m\}\) is always at least \((n-2)\)-\(\mathbb{A}^1\)-connected. For \(n = 2\), we have

\[
\pi_1^1(\mathbb{A}^2 \setminus \{p_1, \ldots, p_m\}) \cong \pi_1^1(\Sigma_s(\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \wedge \mathbb{G}_m))
\]

and the result follows from the definition of the free strongly \(\mathbb{A}^1\)-invariant sheaf generated by a sheaf.

For \(n > 2\), the \(\mathbb{A}^1\)-Hurewicz theorem (see [Mor06] Theorem 3.57) identifies \(\pi_{n-1}^1(\Sigma_s^{-1}(\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \wedge \mathbb{G}_m))\) with \(H_{n-1}^1(\Sigma_s^{-1}(\mathbb{A}^1 \setminus \{x_1, \ldots, x_m\} \wedge \mathbb{G}_m))\). The Mayer-Vietoris sequence (see [AD07a] Proposition 3.16) and induction can be used to compute the \(\mathbb{A}^1\)-homotopy of \(\mathbb{A}^n \setminus \{p_1, \ldots, p_m\}\), as long as one knows that \(H_{n-1}^1(\mathbb{A}^n \setminus 0) \cong KMW^n\).

\section{3 Homotopy types of some smooth affine quadrics}

We now study the \(\mathbb{A}^1\)-homotopy types of the smooth affine quadric hypersurfaces mentioned in the introduction. We emphasize here that all of our theorem statements are formulated over \(\text{Spec} \, \mathbb{Z}\), i.e., it is not necessary to invert 2 anywhere! We begin by fixing conventions regarding affine quadric hypersurfaces and quadratic forms. The main theorem is Theorem 3.4. Afterward, we use Theorem 3.4 to study cohomological properties of smooth affine quadrics and we mention some facts about vector bundles on such varieties.

\textbf{Definitions and conventions regarding quadrics}

We begin by fixing notations and definitions regarding quadrics. Choose coordinates \(x_1, \ldots, x_n\) on \(\mathbb{A}^n\). The two main objects of consideration will be the odd-dimensional quadric hypersurface \(Q_{2m-1} \subset \mathbb{A}^{2m}\) defined by the equation

\[
\sum_i x_i x_{m+i} = 1,
\]

and the even-dimensional quadric hypersurface \(Q_{2m} \subset \mathbb{A}^{2m+1}\) defined by the equation

\[
\sum_i x_i x_{m+i} = x_{2m+1}(1 + x_{2m+1}).
\]

The main justification for using these models of quadrics comes from the following fact.

\textbf{Lemma 3.1.} For any integer \(m \geq 0\), the quadrics \(Q_n\) are smooth over \(\text{Spec} \, \mathbb{Z}\).

\textbf{Proof.} Consider the structure morphism \(Q_n \longrightarrow \text{Spec} \, \mathbb{Z}\). We apply [SGA03] Théorème II.2.1, the local criterion for flatness and a direct computation to show that \(Q_n\) is regular when reduced modulo any prime number \(p\). \(\square\)
Let $q_{2m-1}$ be the quadratic form $\sum_i x_i x_{m+i}$, and let $q_{2m}$ be the quadratic form $\sum_i x_i x_{m+i} + x_{2m+1}^2$. We will also consider the affine quadric hypersurfaces defined by the equations $q_n = 1$; these are the “usual” split or hyperbolic quadric hypersurfaces. For a concise view of the geometric importance of these quadrics, we refer the reader to [SGA73] Expose VII Proposition 1.2. The variety defined by the equation $q_{2m} = 1$ will also be denoted $Q_{m,m}$.

Note that, if 2 is invertible in $k$, the morphism defined by the assignment

$$x_i \mapsto \begin{cases} 
-\frac{x_i}{2} & \text{if } 1 \leq i \leq m \\
\frac{x_i}{2} & \text{if } m + 1 \leq i \leq 2m \\
\frac{1}{2}x_i - \frac{1}{2} & \text{if } i = 2m + 1
\end{cases}$$

determines an isomorphism from $Q_{2m}$ to the “usual” split even-dimensional quadric hypersurface $Q_{m,m}$. We will use this isomorphism repeatedly (and without mention, as long as the appropriate hypotheses are in place) in the future. While defined over Spec $\mathbb{Z}$, the “usual” split even-dimensional quadric has the unfortunate failing of being singular when reduced modulo 2. Indeed, the resulting variety over the finite field $\mathbb{F}_2$ is singular exactly at the point whose coordinates are given by $x_1 = \ldots = x_{2m} = 0$, and $x_{2m+1} = 1$.

**Remark 3.2.** We also take a moment to emphasize some “low dimensional exceptions,” which support our conventions. We consider the even-dimensional quadric $Q_0$ as the closed subscheme of $\mathbb{A}^1$ defined by the equation $x_1(x_1 + 1) = 0$. This is a variety (i.e., reduced) with two points over Spec $k$. If 2 is invertible in $k$, then the isomorphism of the previous paragraph identifies $Q_0$ (as a scheme) with $\mu_2 = \text{Spec } k[x]/(x^2 - 1)$; the latter is a non-reduced group scheme if $k$ is a field of characteristic 2.

**Remark 3.3.** Over any field $k$ of characteristic unequal to 2, every (homogeneous) smooth affine quadric hypersurface is isomorphic to a hypersurface of the form $\sum a_i x_i^2$ where $a_i \in k$ ([Lam05] Corollary 2.4). If furthermore $-1$ is a square in $k$, then the quadrics $Q_n$ are isomorphic to the “diagonal” quadrics given by the hypersurface equations $\sum_{i=1}^{n+1} x_i^2 = 1$ (irrespective of whether $n$ is even or odd). If $k$ is furthermore algebraically closed, it follows that every smooth affine quadric hypersurface is isomorphic to either $Q_{2m}$ or $Q_{2m-1}$.

**Main theorem**

We can now state the main result; note that one need not impose any condition on $k$ and, in fact, the statement of the result is true over Spec $\mathbb{Z}$.

**Theorem 3.4.** For any $n \geq 0$, with $Q_n$ as above, we have unstable $\mathbb{A}^1$-weak equivalences

$$Q_n \sim_{\mathbb{A}^1} \begin{cases} 
S^m_s & \text{if } n = 2m \\
S^{m-1}_s & \text{if } n = 2m - 1,
\end{cases}$$

where, by convention $S^0_s = \text{Spec } k$. In particular, if $k$ is a field, the quadric $Q_{2m}$ is an $(m-1)$-$\mathbb{A}^1$-connected smooth affine 2m-fold and $Q_{2m-1}$ is an $(m-2)$-$\mathbb{A}^1$-connected smooth affine 2m-1-fold.
Proof. If $k$ is a field, the connectivity statements follow from the descriptions of unstable $A^1$-homotopy types using Morel’s unstable $A^1$-connectivity theorem (Theorem 3.38 of [Mor06]). Thus, let us prove that one has the given descriptions of unstable $A^1$-homotopy types.

If $n = 2m - 1$, this is essentially classical. Indeed, Proposition 2.4 shows that $Q_{2m-1}$ is $A^1$-weakly equivalent to $A^m \setminus 0$ (see Example 2.7). We then use an open cover of $A^m \setminus 0$ by the two open sets $A^1 \times A^{m-1} \setminus 0$ and $G_\mathbb{m} \times A^m$ with intersection $G_\mathbb{m} \times A^{m-1} \setminus 0$. In particular, this allows one to realize $A^m \setminus 0$ as $A^1$-weakly equivalent to the join $G_\mathbb{m} \ast A^{m-1} \setminus 0$, which is isomorphic in $\mathcal{H}(k)$ to $\Sigma^1(G_\mathbb{m} \wedge A^{m-1} \setminus 0)$ (see Proposition 2.22). The result then follows by induction (cf. [MV99] §3 Example 2.20).

If $n = 2m$, we have to do more work. If $n = 0$, this follows immediately from our conventions. Thus, let us suppose $n > 0$. In this case, we can cover $Q_{2m}$ by two open sets isomorphic to the $A^1$-contractible smooth scheme $X_{2m}$. Indeed, one open set is exactly $Q_{2m} \setminus E_m$ as defined immediately prior to Theorem 2.10; we will denote this open set by $X_{2m}^+$. Let $X_{2m}^-$ be the open set defined as the complement of the codimension $m$ closed subscheme $E_m'$ defined by $x_1 = \cdots = x_m = 0$, $x_{2m+1} = 1$.

Note that if 2 is invertible in $k$, the isomorphism $Q_{2m} \xrightarrow{\sim} Q_{m,m}$ identifies $X_{2m}^+$ with the open subschemes of $Q_{m,m}$ defined as the complements of $x_1 = \cdots = x_m = 0$, $x_{2m+1} = \pm 1$ (whence the terminology). For the sake of comparison, note the similarity with the discussion of spheres in the introduction. The automorphism $x_{2m+1} \mapsto -x_{2m+1}$ then defines the isomorphism $X_{2m}^+ \xrightarrow{\sim} X_{2m}^-$. If 2 is not invertible in $k$, one can modify the proof of Theorem 2.10 by redefining $w$ with the modification what $w_{2m+1} = 1 + Q_x$. With evident modifications (e.g., changing the open sets over which the resulting morphism trivializes), essentially the same proof establishes $A^1$-contractibility of $X_{2m}^-$.

In either case, we get a cocartesian (Mayer-Vietoris) square of the form

$$
\begin{array}{ccc}
X_{2m}^+ \setminus E_m' & \longrightarrow & X_{2m}^- \\
\downarrow & & \downarrow \\
X_{2m}^+ & \longrightarrow & Q_{2m}.
\end{array}
$$

Now, since both $X_{2m}^+$ and $X_{2m}^-$ are $A^1$-contractible, $Q_{2m}$ is $A^1$-weakly equivalent to the homotopy pushout of the diagram

$$
\ast \longleftarrow X_{2m}^+ \setminus E_m' \longrightarrow \ast,
$$

which is easily seen to be $A^1$-weakly equivalent to $\Sigma^1 X_{2m}^+ \setminus E_m'$. Thus, it suffices to study the $A^1$-homotopy type of $X_{2m}^+ \setminus E_m'$.

To describe $A^1$-homotopy type of $X_{2m}^+ \setminus E_m'$, we use the proof of Theorem 2.10. There, we constructed an explicit morphism $w : A^{3m} \longrightarrow X_{2m}^+$ that was Zariski locally trivial with fibers isomorphic to affine spaces. Since $E_m'$ is isomorphic to an affine space of dimension $m$, its pre-image under $w$ is an affine space of dimension $2m$ in $\mathbb{A}^{3m}$. We also know that $w$ determines an $A^1$-weak equivalence from $A^{3m} \setminus w^{-1}(E_m')$ to $X_{2m}^+ \setminus E_m'$. We claim that $A^{3m} \setminus w^{-1}(E_m')$ is $A^1$-weakly equivalent to $A^m \setminus 0$. Indeed, this can easily be checked via the explicit definition of the map $w$ in the proof of Theorem 2.10.

\[\square\]
Question 3.5. Which spheres $S_i^j \wedge \mathbb{G}_m^j$ have the $\mathbb{A}^1$-homotopy type of smooth (affine) schemes?

Remark 3.6. We expect that not all motivic spheres have the $\mathbb{A}^1$-homotopy type of smooth affine schemes. Furthermore, we know that one can construct infinitely many smooth affine schemes having the $\mathbb{A}^1$-homotopy type of certain motivic spheres. For example, for any strictly positive integer $n$, all the smooth affine hypersurfaces $D_n$ in $\mathbb{A}^3$ defined by the equations $x^ny = z(z+1)$ are $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1$. However, one may show that $D_n$ and $D_n'$ are not isomorphic if $n$ and $n'$ are not equal. Thus, a natural extension of the above question is to describe all the smooth affine schemes having the $\mathbb{A}^1$-homotopy type of $S_i^j \wedge \mathbb{G}_m^j$.

Some $\mathbb{A}^1$-homotopy classes of maps

For the rest of this section, we assume $k$ is a field. The first application of Theorem 3.4 is to, in conjunction with Morel’s results from [Mor06], determine the first non-vanishing $\mathbb{A}^1$-homotopy group of a split quadric.

Proposition 3.7. If $n = 1$, $Q_1$ is isomorphic to $\mathbb{G}_m$ and thus $\mathbb{A}^1$-rigid (and, in particular, not $\mathbb{A}^1$-connected). If $n = 2$, then $\pi^i_1(Q_2)$ is an extension of $\mathbb{G}_m$ by $K_{MW}^2$. Suppose $n \geq 3$. If $n = 2m$ is even, then one has isomorphisms of sheaves

$$\pi^i_1(Q_{2m}) \cong \begin{cases} 0 & \text{if } 0 \leq i \leq m - 1 \\ K_{MW}^m & \text{if } i = m \end{cases}$$

If $n = 2m - 1$ is odd, then one has isomorphisms of sheaves

$$\pi^i_1(Q_{2m-1}) \cong \begin{cases} 0 & \text{if } 0 \leq i \leq m - 2 \\ K_{MW}^m & \text{if } i = m - 1 \end{cases}$$

Proof. These statements follow immediately by combining Theorems 3.4 and Theorem 4.29 (2) of [Mor06] with Theorem 3.4 of this paper.

The above identification also gives us a “computation” of certain groups of $\mathbb{A}^1$-homotopy classes of maps in terms of $\mathbb{A}^1$-homotopy groups. Let $U$ be a smooth scheme, and consider the inclusion map $U \rightarrow \mathbb{G}_m \times U$ induced by the identity $e : \text{Spec } k \rightarrow \mathbb{G}_m$. For any sheaf of groups $G$, define a presheaf by

$$U \mapsto G_{-1}(U) = \text{Ker}(G(\mathbb{G}_m \times U) \xrightarrow{e} G(U)).$$

One may check that $G_{-1}$ is a sheaf of groups. We define the sheaf of groups $G_{-n}$ inductively by setting $G_{-n} = (G_{-n+1})_{-1}$. The functors $G \mapsto G_{-n}$ are closely connected with (iterated) $\mathbb{G}_m$-loop spaces via the following result.

Theorem 3.8 (Morel [Mor06 Thm. 3.11]). For any pointed $\mathbb{A}^1$-connected space $(\mathcal{X}, x)$, one has canonical isomorphisms

$$\pi^i_1(\Omega^j \mathcal{X}, x) \sim \pi^i_1(\mathcal{X}, x)^{-j}.$$

Thus, one has canonical isomorphisms

$$[\Sigma^i_s \wedge \mathbb{G}_m^j, (\mathcal{X}, x)]_{\mathbb{A}^1} \sim \pi^i_1(\mathcal{X}, x)^{-j}(k).$$
Corollary 3.9. Suppose \( r \) and \( n \) are integers. If \( r = 2m \), one has
\[
[Q_r, Q_n]_{A^1} \cong \pi^A_m(Q_n)^{-m}(k),
\]
and if \( r = 2m + 1 \), one has
\[
[Q_r, Q_n]_{A^1} \cong \pi^A_m(Q_n)^{-(m+1)}(k).
\]
In particular, if \( r = n \), we see that
\[
[Q_n, Q_n]_{A^1} \cong K_{MW}^0(k).
\]

Proof. Let us treat the case where \( r = 2m \) first; the case \( r = 2m + 1 \) is similar. Since \( Q_{2m} \cong S^m_s \wedge \mathbb{G}_m^m \) by Theorem 3.4, adjunction shows that one has
\[
[Q_r, Q_n]_{A^1} \cong [\Sigma^m_s, \Omega^m_t Q_n]_{A^1}.
\]
The latter group is by definition \( \pi^A_m(\Omega^m_t Q_n)(k) \), and Theorem 3.8 shows that we have a canonical isomorphism
\[
\pi^A_m(\Omega^m_t Q_n) \cong \pi^A_m(Q_n)^{-m}.
\]
The second statement follows from Morel’s observation (see the proof of Corollary 3.43 of [Mor06]) that \( (K_{MW}^0)_{-1} \cong K_{MW}^{A^1} \) together with induction.

Remark 3.10. Over a field \( k \) of characteristic unequal to 2, \( K_{MW}^0(k) \) is isomorphic to the Grothendieck-Witt ring of quadratic forms over \( k \). We will investigate the relationship between \( A^1 \)-homotopy groups and quadratic forms more in the sequel. Note that pre- and post-composition by \( [Q_r, Q_r]_{A^1} \) or \( [Q_n, Q_n]_{A^1} \) define a \( K_{MW}^0(k) \)-bi-module structure on \( [Q_r, Q_n]_{A^1} \). On the other hand, any morphism of quadrics \( Q_n \to Q_r \) gives rise to a class in \( [Q_r, Q_n]_{A^1} \). The following fundamental question naturally arises.

Question 3.11. What is the subgroup of \( [Q_r, Q_n]_{A^1} \) generated by morphisms of quadrics?

Motives of some affine quadrics

We can also use Theorem 3.4 to compute the motives of the quadrics \( Q_n \). We follow the conventions for motives from [MVW06]. In particular, we write \( DM_Z(k) \) for Voevodsky’s integral derived category of motives over a field \( k \) (we use this notation for the category considered in [MVW06] Definition 14.1). The functor that sends a smooth scheme \( X \) to \( C_*(Ztr(X)) \) extends to a functor \( H(k) \to DM_Z(k) \) that preserves \( A^1 \)-weak equivalences between smooth schemes and sends the functor \( \Sigma^1_k \) to the shift functor: \( X \mapsto X[1] \).

Lemma 3.12. Suppose \( n \) is an integer \( \geq 0 \). One has canonical isomorphisms
\[
M(Q_n) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}(m)[2m] & \text{if } n = 2m, \text{ and} \\
\mathbb{Z} \oplus \mathbb{Z}(m)[2m-1] & \text{if } n = 2m-1,
\end{cases}
\]
in \( DM_Z(k) \).
Proof. The case where $n$ is odd is essentially proved in, e.g., [MVW06] Corollary 15.3. It follows from an analysis of the Mayer-Vietoris triangle for the open covering of $\mathbb{A}^n \setminus 0$ described in the proof of Theorem 3.4. Let us thus treat the case where $n$ is even. First, if $x$ denotes a rational point of $Q_{2n}$, then one has a canonical splitting (see [MVW06] p. 17):

$$M(Q_{2n}) \cong \mathbb{Z} \oplus \tilde{M}(Q_{2n}).$$

We now use the fact that taking the reduced motive commutes with simplicial suspension. Theorem 3.4 implies that $Q_{2m} \cong \Sigma^1 Q_{2m-1}$ and thus we have

$$\tilde{M}(Q_{2n}) \cong \tilde{M}(Q_{2n-1})[1].$$

The result then follows immediately from the computation in the odd-dimensional case.

Example 3.13. If $X$ is a smooth scheme, one knows that $\text{Hom}_{DM_\mathbb{Z}(k)}(M(X), \mathbb{Z}(q)[p]) \sim \rightarrow H^{p,q}(X, \mathbb{Z})$. In particular, one knows that $CH^n(X) \cong H^{2n,n}(X, \mathbb{Z})$. It follows immediately from the above computation that $\text{Hom}_{DM_\mathbb{Z}(k)}(M(X), M(Q_{2m})) \cong CH^0(X) \oplus CH^m(X)$. In particular, any non-trivial morphism $X \rightarrow Q_{2m}$ defines a class in $CH^m(X)$.

**Vector Bundles, K-theory, and Hermitian K-theory**

In topology, the computations of (topological) complex K-theory ($KU$) and real K-theory ($KO$) of spheres are closely related to complex and real Bott periodicity. We refer the reader to [Hus94] Chapter 9 Corollary 5.2 and Notation 5.3 for the explicit computations of K-theory of spheres and to Chapter 11 Theorem 1.3 of ibid. for the relationship with Bott periodicity. The algebro-geometric analogs of these two theories are, respectively, algebraic K-theory and Hermitian K-theory (in combination with some closely related cousins).

We begin by studying the algebraic K-theory of the quadrics $Q_n$. Let us note that the algebraic K-theory of general smooth quadrics was studied in the beautiful paper of Swan ([Swa85]). We have nothing to add to this study. However, we do provide some completely elementary computations of algebraic K-theory that can be used to deduce existence results for non-trivial vector bundles on the $A^1$-contractible smooth schemes $X_{2m}$ extending the results of [AD07c]. The simplicity of the next statement is closely connected to Bott periodicity for algebraic K-theory: algebraic K-theory is a “$(2,1)$-periodic” theory.

**Lemma 3.14.** For any integer $n \geq 0$, we have isomorphisms

$$K_i(Q_n) \cong \begin{cases} K_i(k) \oplus K_{i-1}(k) & \text{if } n = 2m - 1 \\ K_i(k) \oplus K_i(k) & \text{if } n = 2m. \end{cases}$$

**Proof.** We will provide a more “homotopic” proof of this fact than Swan. We know algebraic K-theory is $A^1$-representable. Indeed by [MV99] Theorem 4.3.13 we know that as long as $k$ is a regular ring, we have

$$[\Sigma^m \mathbb{G}_m \wedge X_+, (\mathbb{Z} \times BGL_\infty, \ast)]_{A^1} \cong K_{n-m}(X).$$
Together, these observations give the result.

In the case $\beta$, we assume that $2$ is invertible in $k$. Lemmas 2.2, 2.3 and 2.4 of [AD07c] to conclude the proof in the same way.

**Proof.** Observe that the hypotheses stated above are sufficient to guarantee that one may apply sufficiently large ranks.

is $X$ gives a class in $G$ if $X,x \in \mathcal{X}$. Algebraic K-theory of a pointed smooth scheme $(X,x)$ decomposes as a sum $K_i(X) \cong K_i(\text{Spec } k)$, and we refer to the first summand as *reduced* algebraic K-theory.

By construction, reduced algebraic K-theory is also $\mathbb{A}^1$-representable and we have

$$\tilde{K}_{n-m}(X) \cong [\Sigma S^m_\ast \mathbb{G}_m^\wedge m(X,x), (\mathbb{Z} \times BGL_\infty, *)]_{\mathbb{A}^1}$$

In the case $X = Q_{2m-1} \cong S^{m-1} \wedge \mathbb{G}_m^\wedge m$, we get

$$\tilde{K}_n(Q_{2m-1}) \cong [S^{n+m-1} \wedge \mathbb{G}_m^\wedge m, (\mathbb{Z} \times BGL_\infty, *)]_{\mathbb{A}^1} \cong \tilde{K}_{n-1}(\text{Spec } k_+)$$

In the case $X = Q_{2m}$, we see that

$$\tilde{K}_n(Q_{2m}) \cong [S^{n+m} \wedge \mathbb{G}_m^\wedge m, (\mathbb{Z} \times BGL_\infty, *)] \cong \tilde{K}_n(\text{Spec } k_+)$$

Together, these observations give the result.

**Remark 3.15.** Let us note that the reduced K-theory $\tilde{K}_0(Q_{2m}) \cong K_0(k)$ has a canonical generator if $k$ is a field. Indeed, consider the inclusion $E_m \hookrightarrow Q_{2m}$. The class of the push-forward $[\pi_*(\mathcal{O}_{E_{m}})]$ gives a class in $G_0(Q_{2m})$ that is not a multiple of the trivial class. By Poincaré duality gives a class $\beta_{2m}$ in $\tilde{K}_0(Q_{2m})$. In case $m = 1$, this class corresponds to the class of $O(-1)$. There are external product maps $\tilde{K}_0(Q_{2m}) \times \tilde{K}_0(Q_{2n}) \longrightarrow \tilde{K}_0(Q_{2(m+n)})$ induced, via Theorem 3.4, by the $\mathbb{A}^1$-weak equivalence $Q_{2n} \wedge Q_{2m} \cong Q_{2(n+m)}$. We denote the external product by $\cdot$ and one can check that $\beta_{2m} \cdot \beta_{2n} = \beta_{2(n+m)}$.

We can also significantly strengthen Theorem 1.2(iv) of [AD07c] on existence of non-trivial vector bundles on non-affine $\mathbb{A}^1$-contractible varieties. Note that in combination with Conjecture 2.15 the following result (almost) implies the dichotomy that affine $\mathbb{A}^1$-contractible smooth schemes have no non-trivial vector bundles, while non-affine $\mathbb{A}^1$-contractible smooth schemes have many non-trivial vector bundles.

**Proposition 3.16.** Suppose $f : \mathbb{A}^n \longrightarrow X$ is a Zariski locally trivial smooth morphism with affine space fibers. Assume that, for some integer $d \geq 2$, $X$ can be realized as the complement of a (non-empty) codimension $d$ closed subscheme of a smooth affine scheme $X$. The variety $X$ is $\mathbb{A}^1$-contractible, quasi-affine yet not affine, and admits non-trivial vector bundles of all sufficiently large ranks.

**Proof.** Observe that the hypotheses stated above are sufficient to guarantee that one may apply Lemmas 2.2, 2.3 and 2.4 of [AD07c] to conclude the proof in the same way.

We now study Hermitian K-theory of split affine quadrics. Throughout the rest of this section, we assume that $2$ is invertible in $k$. Recall that Hermitian $K_0$ of a ring $A$ is the Grothendieck group of the monoid of isomorphism classes of projective $A$-modules equipped with symmetric bilinear forms under orthogonal direct sum; this is also called the Grothendieck-Witt group of $A$. Tensor product of symmetric bilinear forms equips this group with the structure of a ring. Hermitian K-theory provides an extension of the Grothendieck-Witt ring in the same sense that higher K-theory extends the functor $K_0$. In order to discuss (higher) Hermitian K-theory on the same footing as algebraic K-theory, we need to introduce some additional notation.
We use the $\mathbb{A}^1$-representability of Hermitian K-theory proved by Hornbostel (see [Hor05]). Unlike algebraic K-theory, even for regular schemes, Hermitian K-theory doesn’t vanish in negative degrees. This fact, coupled with the “(8, 4)-periodicity” of Hermitian K-theory, makes the statement of the result more complicated than Lemma 3.14.

For our notation, we refer to [Hor05] §4.1. We let $K\tilde{O}_n(X)$ denote the $n$-th Hermitian K-theory group of a smooth scheme $X$ and $\tilde{K}O_n$ the Hermitian K-theory of anti-symmetric forms. The latter is perhaps better described as the algebro-geometric analog of quaternionic K-theory and is related to the symplectic group in the same way that $KO$ is related to the orthogonal group. We let $U_n$ denote the U-theory groups of Karoubi and $\tilde{U}_n$ the anti-symmetric U-theory groups.

If $X$ is a scheme, and $F$ is a locally free $O_X$-module, $F \oplus F^\vee$ (where $F^\vee = \text{Hom}_{O_X}(F, O_X)$) can be equipped with a hyperbolic form. This construction induces a functor from the category of vector bundles on $X$ to the category of vector bundles equipped with a symmetric bilinear form. In turn, this functor induces a morphism of the associated $Q$-constructions, and the groups $\tilde{U}_n$ are the homotopy groups of the homotopy fiber of this map. The groups $\tilde{U}_n$ admit a similar description.

Hornbostel proved that the above sequences of groups (i.e., $KO_i(X), U_i(X), \tilde{K}O_i(X), \tilde{U}_i(X)$) are the stable homotopy groups of a 4-periodic $\mathbb{P}^1$-spectrum. He also proved an appropriate unstable representability result for the resulting theories. For simplicity of formulation, recall that if $X$ is a pointed space, the Hermitian K-theory of $X$ splits as a sum $KO_i(X) \cong KO_i(\text{Spec } k) \oplus \tilde{K}O_i(X)$ where the second group is the reduced Hermitian K-theory of $X$.

**Lemma 3.17.** For any integer $n \geq 0$, we have isomorphisms

\[
K\tilde{O}_i(Q_n) \cong \begin{cases} 
KO_i(k) & \text{if } n \equiv 0 \mod 8, \\
U_{i-1}(k) & \text{if } n \equiv 1 \mod 8, \\
U_i(k) & \text{if } n \equiv 2 \mod 8, \\
-K\tilde{O}_{i-1}(k) & \text{if } n \equiv 3 \mod 8, \\
-K\tilde{O}_i(k) & \text{if } n \equiv 4 \mod 8, \\
-U_{i-1}(k) & \text{if } n \equiv 5 \mod 8, \\
-U_i(k) & \text{if } n \equiv 6 \mod 8, \\
KO_{i-1}(k) & \text{if } n \equiv 7 \mod 8.
\end{cases}
\]

**Proof.** We just want to interpret the groups

\[
K\tilde{O}_i(\Sigma^m \mathbb{G}_m \wedge m), \text{ and } K\tilde{O}_i(\Sigma^m \mathbb{G}_m \wedge m)
\]

Now, by the definition of the Hermitian K-theory spectrum, we know

\[
K\tilde{O}_i(\Sigma^m \mathbb{G}_m \wedge m) \cong K\tilde{O}_{i+m}(\text{Spec } k_+)
\]

(see [Hor05] p. 678). To deduce the result for odd-dimensional quadrics from this statement, we use [Hor05] Corollary 5.3.
Remark 3.18. Together with the localization sequence in Hermitian K-theory proved by Hornbostel and Schlichting ([HS04]) we could use these computations to deduce existence of non-trivial quadratic vector bundles on $\mathbb{A}^1$-contractible smooth schemes that are quasi-affine but not affine. One can formulate statements analogous to the main theorem of [AD07c] regarding infinitude (e.g., arbitrary dimensional moduli in large dimension) of quadratic bundles on $\mathbb{A}^1$-contractible smooth scheme that are quasi-affine yet not affine. However, unlike the situation for vector bundles, one does not expect the functor that assigns to a smooth affine scheme the set of isomorphism classes of quadratic bundles of a fixed rank to be $\mathbb{A}^1$-invariant. Indeed, even on affine space it is known that not all symmetric bilinear spaces are trivial, i.e., extended from the base field (cf. [Par78]).

Remark 3.19. Suppose $E$ is a $\mathbb{P}^1$-spectrum in the sense of [Mor04] §5.1. One can define (reduced) $E$-cohomology as in ibid. Definition 5.1.6. The (reduced) $E$-cohomology of the smooth quadrics $Q_n$ can be computed essentially by shifting degrees.

References


[Mor06] F. Morel. $\mathbb{A}^1$-algebraic topology over a field. 2006. Preprint - available at http://www.mathematik.uni-muenchen.de/~morel/A1homotopy.pdf. 4, 6, 18, 19, 20


