1) Which of the following is equivalent to $\sim (P \land Q) \land Q$?

(i) $Q$
(ii) $\sim P$
(iii) $\sim P \land Q$
(iv) $P \lor Q$
(v) none of these

Ans. We can simplify the statement by

$$\sim (P \land Q) \land Q \equiv (\sim P \lor \sim Q) \land Q \quad \text{De Morgan's law}$$
$$\equiv (\sim P \land Q) \lor (\sim Q \land Q) \quad \text{distributive law}$$
$$\equiv (\sim P \land Q) \lor F \quad \text{idempotent law}$$
$$\equiv \sim P \land Q \quad \text{identity law}$$

So the correct answer is (ii).

2) Show that $P \rightarrow (P \lor Q)$ is a tautology, using truth tables and using equivalent statements.

Ans. We can resume the truth table in the following way.

- if $P$ is false, then $P \rightarrow (P \lor Q)$ is true (since the antecedent of the conditional is false).
- if $P$ is true, then the consequent $(P \lor Q)$ is also true, and the whole statement $P \rightarrow (P \lor Q)$ is true.

We conclude that $P \rightarrow (P \lor Q)$ is a tautology.

Using equivalences, we have

$$P \rightarrow (P \lor Q) \equiv \sim P \lor (P \lor Q)$$
$$\equiv (\sim P \lor P) \lor Q \quad \text{associative law}$$
$$\equiv T \lor Q \quad \text{identity law}$$

3) Simplify the following statements, justifying each passage:

(i) $(B \land \sim A) \rightarrow A$

Ans. Using the relation $P \rightarrow Q \equiv \sim P \lor Q$, we get

$$(B \land \sim A) \rightarrow A \equiv \sim (B \land \sim A) \lor A \quad \text{De Morgan's law}$$
$$\equiv (\sim B \lor A) \lor A \quad \text{associative law}$$
$$\equiv \sim B \lor A \quad \text{idempotent law}$$

(ii) $\{[P \land (P \lor Q)] \lor \sim P\} \land Q$

Ans. The equivalence relations are

$$\{[P \land (P \lor Q)] \lor \sim P\} \land Q \equiv \{[P \land (P \lor Q)] \lor \sim P\} \land Q \quad \text{distributive law}$$
$$\equiv \{P \lor \sim P\} \land Q \quad \text{idempotent law}$$
$$\equiv T \land Q \quad \text{absorption law}$$
$$\equiv Q \quad \text{identity law}$$

4) Which of the following statements is NOT equivalent to $\sim P \rightarrow Q$?
\[
\begin{array}{ll}
\text{(i)} & \sim Q \rightarrow P \equiv \sim P \rightarrow Q \quad \text{contrapositive} \\
\text{(ii)} & \sim P \rightarrow Q \\
\text{(iii)} & \sim Q \rightarrow \sim P \equiv P \rightarrow Q \quad \text{contrapositive} \\
\text{(iv)} & \sim Q \lor \sim P \equiv P \rightarrow \sim Q
\end{array}
\]

So the correct answer is (iii) and (iv).

5) Is \( P \leftrightarrow \sim P \) a tautology, a contradiction or neither? Explain.

**Ans.** \( P \) and \( \sim P \) have always different truth values, so the statement \( P \leftrightarrow \sim P \) is always false, then it is a contradiction.

6) Negate the statement: *If it rains, then I will not go out for dinner.*

**Ans.** Set \( R \) for *it rains.* and \( D \) for *I will go out for dinner.* Then *If it rains, then I will not go out for dinner.* corresponds to \( R \rightarrow \sim D \equiv \sim R \lor \sim D \).

\[
\sim (R \rightarrow \sim D) \equiv \sim (\sim R \lor \sim D) \equiv R \land D \quad \text{De Morgan’s law}
\]

So the negation of the previous statement is: *it rains and I will go out for dinner.*

7) Draw a circuit for \( (P \lor Q) \rightarrow R \).

**Ans.** \( (P \lor Q) \rightarrow R \equiv \sim (P \lor Q) \lor R \equiv (\sim P \land \sim Q) \lor R \) and the corresponding circuit is:

\[
\begin{array}{c}
\sim P \\
\sim Q \\
R
\end{array}
\]

8) Simplify the statement \( [(P \land Q) \land P] \lor \sim Q \). Justify each passage.

**Ans.** The statement is equivalent to
\[
\begin{align*}
[(P \land Q) \land P] \lor \sim Q & \equiv [(Q \land P) \land P] \lor \sim Q \quad \text{commutative law} \\
& \equiv [Q \land (P \land P)] \lor \sim Q \quad \text{associative law} \\
& \equiv [Q \land P] \lor \sim Q \quad \text{idempotent law} \\
& \equiv (Q \lor \sim Q) \land (P \lor \sim Q) \quad \text{distributive law} \\
& \equiv T \land (P \lor \sim Q) \quad \text{identity law} \\
& \equiv (P \lor \sim Q)
\end{align*}
\]

9) For the following circuit, write the corresponding logical statement, simplify the statement (state the ruled used) and draw the simplified circuit.

**Ans.** The circuit corresponds to the statement \( [(P \lor Q) \land P] \lor (Q \lor R) \) that we can simplify as

\[
\begin{array}{c}
P \\
\sim P \\
Q \\
\sim P \\
R
\end{array}
\]
\[[(P \lor Q) \land \sim P] \lor (Q \lor R) \equiv [(P \land \sim P) \lor (Q \land \sim P)] \lor (Q \lor R) \]
\[\equiv [F \lor (Q \land \sim P)] \lor (Q \lor R) \]
\[\equiv (Q \land \sim P) \lor (Q \lor R) \equiv [(Q \land \sim P) \lor Q] \lor R \equiv Q \lor R \]

The simplified circuit is then

10) Draw the circuit for \(P \land [R \lor (Q \land \sim R)] \land R\)

\textbf{Ans.} The corresponding circuit is given by

11) Find the match between equivalent statement:

\begin{align*}
A) & \quad (P \land Q) \lor P & 1) & \quad Q \to P \\
B) & \quad \sim (P \lor \sim P) & 2) & \quad P \lor (Q \land R) \\
C) & \quad \sim P \to \sim Q & 3) & \quad P \\
D) & \quad (P \lor Q) \land (P \lor R) & 4) & \quad P \land Q \\
E) & \quad \sim (\sim (P \land Q)) & 5) & \quad F
\end{align*}

\textbf{Ans.} It’s easy to check that

\begin{align*}
(P \land Q) \lor P & \equiv P \quad \text{absorption law} & \text{(A)} = (3) \\
\sim (P \lor \sim P) & \equiv F \quad \text{negation of tautology} & \text{(B)} = (5) \\
\sim P \to \sim Q & \equiv Q \to P \quad \text{contrapositive} & \text{(C)} = (1) \\
(P \lor Q) \land (P \lor R) & \equiv P \lor (Q \land R) \quad \text{distributive law} & \text{(D)} = (2) \\
\sim (\sim (P \land Q)) & \equiv P \land Q \quad \text{double negation} & \text{(E)} = (4)
\end{align*}

12) Answer to the questions:

1. How many rows does the truth table of the statement \([P \land (Q \land R)] \to (\sim Q \lor R)\) have?
2. If the consequent of an \textit{if} statement is true, is the \textit{if} statement true or false?
3. Fill up with \(\land\), \(\lor\) or \(\to\) so that \(P \cdots Q\) is true when \(Q\) is false and \(P\) is true.
4. Give the simplest expression equivalent to \([(P \land P \land P) \lor P] \lor P\).

\textbf{Ans.} 1. \(2^3 = 8\), since there are three statements involved, P,Q,R.
2. According to its truth table, if \(Q\) is \textit{true}, then \(P \to Q\) is \textit{true}.
3. If \(P\) is true and \(Q\) is false, then \(P \to Q\), \(P \land Q\) are both false. Instead, \(P \lor Q\) is still true.
4. Applying idempotent law four times, we get \([(P \land P \land P) \lor P] \lor P \equiv P\).

13) Simplify the statement \([P \to (\sim P)] \lor [Q \land P \land (\sim Q)]\).

\textbf{Ans.} The simplification is obtained via

\begin{align*}
[P \to (\sim P)] \lor [Q \land P \land (\sim Q)] & \equiv [\sim P \lor (\sim P)] \lor [(Q \land \sim Q) \land P] \quad \text{commutative law} \\
& \equiv \sim P \lor [F \land P] \quad \text{idempotent law} \\
& \equiv \sim P \lor F \quad \text{identity law for} \land \\
& \equiv \sim P \quad \text{identity law for} \lor
\end{align*}
1) Use truth tables to determine if the following arguments are valid or not.

(a) **Simplification:** \[ \frac{P \land Q}{P} \]

**Ans.** If \( P \land Q \) is true, then automatically \( P \) and \( Q \) are both true. \( P \) in particular is true. In other words, the statement

\[
(P \land Q) \rightarrow P \equiv T
\]

is a tautology. Indeed,

\[
(P \land Q) \rightarrow P \equiv \sim (P \land Q) \lor P \\
\equiv (\sim P \lor \sim Q) \lor P \\
\equiv \sim Q \lor (\sim P \lor P) \quad \text{commutative and associative law} \\
\equiv \sim Q \lor T \\
\equiv T \quad \text{identity law}
\]

(b) **Amplification:** \[ \frac{P}{P \lor Q} \]

**Ans.** If \( P \) is true, then automatically \( P \lor Q \) is true. In other words, the statement

\[
P \rightarrow (P \lor Q) \equiv T
\]

is a tautology. Indeed,

\[
P \rightarrow (P \lor Q) \equiv \sim P \lor (P \lor Q) \\
\equiv (\sim P \lor P) \lor Q \quad \text{associative law} \\
\equiv T \lor Q \\
\equiv T \quad \text{identity law}
\]

(c) **Conjunction:**

\[ \frac{1. \quad P}{2. \quad Q} \frac{P \land Q}{P \land Q} \]

**Ans.** If \( P \) and \( Q \) are both true, then automatically \( P \land Q \) is true. In other words, the statement

\[
(P \land Q) \rightarrow (P \land Q)
\]

is obviously a tautology, since

\[
(P \land Q) \rightarrow (P \land Q) \equiv \sim (P \land Q) \lor (P \land Q) \\
\equiv T
\]

(d) What can you say about the following argument?

\[ \frac{1. \quad P}{2. \quad \sim P} \frac{Q}{Q} \]

**Ans.** The argument is valid because the statement

\[
(P \land \sim P) \rightarrow Q \equiv T
\]

is a **tautology**. It is interesting to notice that, however, there is no possibility for the premises to be true at the same time. So in fact there is no condition to check. This reflects the idea that, if we assume a contradiction to be true, then we can conclude whatever we want.

2) Determine whether each argument is **valid** or **invalid**. If it is valid, give a proof. If it is invalid, give an assignment of truth values to the variables that makes the premises true and the conclusion false.
1. \( P \land Q \)
2. \( P \)
   \[ \sim Q \]

**Ans.** The argument is obviously invalid. Indeed, in order to have all the premises true, we need \( P=T \) and \( P \land Q=T \), so consequently \( Q=T \). But then at this point the conclusion \( \sim Q \) is false.

1. \( P \rightarrow Q \)
2. \( R \rightarrow \sim Q \)
   \[ P \rightarrow \sim R \]

**Ans.** The premise 2 is equivalent, by contra-positivity, to \( Q \rightarrow \sim R \). So, applying transitivity to 1,2, we get the conclusion \( P \rightarrow \sim R \).

1. \( P \land Q \)
2. \( Q \rightarrow \sim R \)
3. \( P \)
4. \( R \lor S \)
   \[ S \]

**Ans.** The solution is given by:

1. \( P \land Q \)
2. \( Q \rightarrow \sim R \)
3. \( P \)
4. \( R \lor S \)
5. \( Q \quad 1, \text{simplification} \)
6. \( \sim R \quad 2,5, \text{modus ponens} \)
7. \( S \quad 4,6, \text{disjunction syllogism} \)

It’s interesting to notice that the premise 3, \( P \), is unnecessary. Moreover, \( P \) and \( Q \) are consequences of premise 1, \( P \land Q \), by simplification.

1. \( P \lor Q \)
2. \( \sim P \)
3. \( R \rightarrow \sim Q \)
   \[ \sim R \]

**Ans.** The solution is given by:

1. \( P \lor Q \)
2. \( \sim P \)
3. \( R \rightarrow \sim Q \)
4. \( Q \quad 1,2, \text{disjunctive syllogism} \)
5. \( \sim R \quad 3,4, \text{modus tollens} \)

3) Which is the conclusion of the following arguments? Show all your work and justify your answer.

(a) 1. Babies are illogical.
2. Nobody is despised who can manage a crocodile.
3. Illogical persons are despised.

**Conclusion:** Babies cannot manage crocodiles.

**Ans.** The first step is the translation from English to symbols:
- The premise 1 corresponds to *If you are a baby, then you are illogical.*, Setting (B) *You are a baby.* and (I) *You are illogical.*, we get
  \[ B \rightarrow I. \]
- The premise 2 corresponds to *If you can manage a crocodile, then you are not despised.* Setting (C) *You can manage a crocodile.* and (D) *You are despised*, we get
  \[ C \rightarrow \sim D. \]
- The premise 3 corresponds to If you are illogical, then you are despised. We get
  \[ I \rightarrow D. \]

We can conclude that the premises are

1. \( B \rightarrow I \)
2. \( C \rightarrow \sim D \)
3. \( I \rightarrow D \)

By contrapositivity, \( C \rightarrow \sim D \equiv D \rightarrow \sim C \). So, applying transitivity to 1,3, we get

4. \( B \rightarrow D \)

and applying transitivity to 2,4 we get

\[ B \rightarrow \sim C \]

that is, if you are a baby, then you cannot manage crocodiles. or Babies cannot manage crocodiles.

(b)

- The premise 1 corresponds to If you are a promise-breaker, then you are untrustworthy., Setting (P) You are a promise-breakers and (U) You are untrustworthy., we get
  \[ P \rightarrow U. \]
- The premise 2 corresponds to If you are a wine-drinker, then you are very communicative. Setting (W) You are a wine-drinker. and (C) You are very communicative., we get
  \[ W \rightarrow C. \]
- The premise 3 corresponds to If you don’t break your promises, then you are honest. Setting (H) You are honest., we get
  \[ \sim P \rightarrow H. \]
- The premise 4 corresponds to If you are a pawnbroker, then you are not a teetotaler. Setting (B) You are a pawnbroker, we can identifying teetotaler and not wine-drinker and we get
  \[ B \rightarrow W \]
- The premise 5 corresponds to If you are a very communicative person, then you are trustworthy. that is
  \[ C \rightarrow \sim U \]

We can conclude that the premises are

1. \( P \rightarrow U \)
2. \( W \rightarrow C \)
3. \( \sim P \rightarrow H \)
4. \( B \rightarrow W \)
5. \( C \rightarrow \sim U \)

By contrapositivity, \( P \rightarrow U \equiv \sim U \rightarrow \sim P \). So, applying transitivity to 4,2, we get

6. \( B \rightarrow C \)

and applying transitivity to 6,5 we get

7. \( B \rightarrow \sim U \)

and applying transitivity to 7,1 we get

8. \( B \rightarrow \sim P \)

and applying transitivity to 8,3 we get

9. \( B \rightarrow H \)
that is, if you are a pawnbroker, then you are honest. or No pawnbroker is dishonest. or All pawnbrokers are honest.

4) Decide if the following statements are true or false.

(a) $3 \not\in A = \{a, b, c, d, 4, 5, 19\}$

Ans. True.

(b) $\emptyset = \{0\}$

Ans. False.

(c) $\{a, b, 4\} \subseteq A$

Ans. True.

(d) $\{x \mid x \text{ is an odd integer number}, 6 \leq x < 13\} = \{7, 9, 11, 13\}$

Ans. False.
Some useful notations:

(i) Difference of sets. Given $A$ and $B$, define the set

$$A \setminus B = \{ x \in U \mid x \in A \wedge x \not\in B \} = A \cap B'$$

The corresponding region in the Venn diagram is:

![Venn diagram](attachment:venn_diagram.png)

(ii) The power set of a set $A$ is the set of all subsets of $A$. This includes the subsets formed from all the members of $A$ and the empty set. If a finite set $A$ has cardinality $k$, then the power set of $A$ has cardinality $2^k$. The power set can be written as $\mathcal{P}(A)$.

- If $A = \emptyset$, that is $k = 0$, then there is only one subset, the empty set, that is $A$ itself:

$$\mathcal{P}(A) = \{ \emptyset \} \quad |\mathcal{P}(A)| = 1$$

- If $|A| = 1$ and $A = \{a\}$, then we have the empty set and the subset $\{a\} = A$. So

$$\mathcal{P}(A) = \{ \emptyset, \{a\} \} \quad |\mathcal{P}(A)| = 2$$

- If $|A| = 2$, $A = \{a, b\}$, we have:

$$\mathcal{P}(A) = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \} \quad |\mathcal{P}(A)| = 4$$

- If $|A| = 3$, $A = \{a, b, c\}$, we have:

$$\mathcal{P}(A) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \} \quad |\mathcal{P}(A)| = 8$$

(iii) We can represent the power set with an ordered diagram. For example, if $A = \{a, b, c\}$, then $\mathcal{P}(A)$ is:

```
   3
  / \  \
 2   2
 / \ / \ \
1   1  1
/ \ / \ / \ \
0   0   0
```

where a subset $P$ on a level is connected with a subset $Q$ in the upper level if and only if $P \subseteq Q$. So, for example, since $\{a\} \subseteq \{a, b\}$, then $\{a\}$ on level 1 is connected with $\{a, b\}$ on level 2. Then number of the level is the cardinality of the subsets on that level.
Exercises:

1) Consider the universal set
\[ U = \{1, 2, 3, 4, 5, 6, 7, 8\} \]
and the subsets
\[ E = \{1, 3, 5, 6\} \quad F = \{2, 4, 5, 7\} \]

(a) Describe the following subsets:
   a) \( E' \cup F = \{2, 4, 7, 8\} \cup \{2, 4, 5, 7\} = \{2, 4, 5, 7, 8\} \)
   b) \( E' \cap E = \emptyset \)
   c) \( E \cup F' = \{1, 3, 5, 6, 8\} \)
   d) \( E \cap F' = \{1, 3, 5, 6\} \cap \{1, 3, 6, 8\} = \{1, 3\} \)
   e) \( F' \cup F = U \)

(b) Draw the Venn diagram for \( U, E, F \) and write the number of elements for each region:

2) Let \( U \) be the universal set and \( A, B, C \) subsets. Assume \(|U| = 100, |A| = 39, |B| = 11, |C| = 43 \)
and \(|A \cup B \cup C| = 74, |A \cup B| = 41, |A \cup C| = 73, |B \cup C| = 49\). Remember that the union rule is given by:
\[ |A \cup B| = |A| + |B| - |A \cap B| \]
and, in the case of three sets,
\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

(a) Draw the Venn diagram for \( U, A, B, C \) and write the number of elements for each region.

**Ans.** The first information we can get is about the **gray region** (see below):
\[ |(A \cup B \cup C)'| = |U| - |A \cup B \cup C| = 100 - 74 = 26 \]

Then, using the union rule, we get the following informations:
\[ |A \cap B| = |A| + |B| - |A \cup B| = 39 + 11 - 41 = 9 \]
\[ |B \cap C| = |B| + |C| - |B \cup C| = 11 + 43 - 49 = 5 \]
\[ |A \cap C| = |A| + |C| - |A \cup C| = 39 + 43 - 73 = 9 \]

In particular, we can determine the **brown region**:
\[ |A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| = 74 - 39 - 43 + 9 + 5 = 4 \]

Now, using the fact that \(|A \cap B \cap C| = 4\), we can say that:
- **Violet region:** \(|(A \cap B) \setminus C| = |A \cap B| - |A \cap B \cap C| = 9 - 4 = 5\)
- **Orange region:** \(|(B \cap C) \setminus A| = |B \cap C| - |A \cap B \cap C| = 5 - 4 = 1\)
- **Green region:** \(|(A \cap C) \setminus B| = |A \cap C| - |A \cap B \cap C| = 9 - 4 = 5\)

Up to now, we have the following situation:
Then, it is easy to find:

**Blue region:** \(|A \setminus (B \cup C)| = 39 - 14 = 25

**Red region:** \(|B \setminus (A \cup C)| = 11 - 10 = 1

**Yellow region:** \(|C \setminus (A \cup B)| = 43 - 10

Finally, we have:

(b) Shade the following regions and write their cardinality:

(a) \(|A'| = |U| - |A| = 100 - 39 = 61

(b) \(|A \cap B \cap C| = 4

(c) \(|(A \cap B) \cup C| = |(A \cup C) \cap (B \cup C)| = 43 + 5 = 48
3) Decide if the following statements are true or false, where $U$ is the universal set and $A, B$ are sets:

(a) If $A \subseteq B$, then $|A| < |B|$.  
(b) $A \cup A' = U$  
(c) If $a \in A$, then $a \in (A \cap B)$  
(d) The empty set has no subset.  
(e) A set of 4 elements has 8 subsets.  
(f) If $|A| = 5$, then $|P(A)| = 32$.  
(g) $|A \cup B| + |A \cap B| = |A| + |B|$  

4) (a) Write the union rule for sets $A, B$

Ans.  

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(b) Assume $A$ and $B$ are disjoint, with $|A| = 18$ and $|B| = 4$. Find $|A \cup B|$.

Ans. Since $A, B$ are disjoint, $|A \cap B| = 0$, then $|A \cup B| = |A| + |B| = 18 + 4 = 22$.

(c) Let $|A| = 8, |B'| = 10, |A \cap B| = 5, |A \cup B| = 12$. Find $|U|$.

Ans. Using union rule,  

$$|B| = |A \cup B| + |A \cap B| - |A| = 12 + 5 - 8 = 9$$  
then, since $B, B'$ are disjoint and $U = B \cup B'$,  

$$|U| = |B| + |B'| = 9 + 10 = 19$$
5) An Italian restaurant surveyed 120 customers to find out what topping they ordered for their pizzas among olives (O), artichokes (A), and mushrooms (M). It was found that:

(a) Use the information to fill in the number of elements for each region in the following Venn diagram.

<table>
<thead>
<tr>
<th>Region</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A ∩ O ∩ M</td>
<td>14 (brown region)</td>
</tr>
<tr>
<td>(A ∪ O ∪ M)'</td>
<td>13 (gray region)</td>
</tr>
<tr>
<td>A ∩ O</td>
<td>26</td>
</tr>
<tr>
<td>A ∩ M</td>
<td>18</td>
</tr>
<tr>
<td>O ∩ M</td>
<td>30</td>
</tr>
<tr>
<td>A</td>
<td>67</td>
</tr>
<tr>
<td>A ∪ M</td>
<td>90</td>
</tr>
</tbody>
</table>

Using (a), we get the data for colors violet, green, orange:

- \(|(A ∩ O) \setminus M| = |A ∩ O| - |A ∩ O ∩ M| = 26 - 14 = 12\)
- \(|(A ∩ M) \setminus O| = |A ∩ M| - |A ∩ O ∩ M| = 18 - 14 = 4\)
- \(|(M ∩ O) \setminus A| = |M ∩ O| - |A ∩ O ∩ M| = 30 - 14 = 16\)

Up to now we have the following information:

Now, using (a), we get:

\(|A ∪ O ∪ M| = |U| - |(A ∪ O ∪ M)'| = 120 - 13 = 107\)

and by (f) we get (blue region):

\(|A \setminus (M ∪ O)| = |A| - (12 + 14 + 4) = 67 - 30 = 37\)

and by (g), we have (yellow region):

\(|M \setminus (A ∪ O)| = |A ∪ M| - (67 + 16) = 90 - 83 = 7\)

and finally (red region):

\(|O \setminus (A ∪ M)| = |A ∪ O ∪ M| - |A ∪ M| = 107 - 90 = 17\)

So, the Venn diagram is:
(b) How many customers did not order olives?

**Ans.** The request corresponds to find the cardinality of $O'$, that is

and the cardinality is given by:

$$|O'| = |U| - |O| = 120 - 59 = 61$$

6) A local town has its own electric company which also provides both digital cable TV (C) and high-speed internet access (I). The electric company has 200 customers. 70 customers subscribe to digital cable TV, 50 customers subscribe to high-speed internet access, and 110 customers subscribe to neither. How many customers subscribe to both digital cable TV and high-speed internet access?

**Ans.** In order to answer the question we have to compute $|I \cap C|$, knowing the value for $|U| = 200, |C| = 70, |I| = 50, |(I \cup C)'| = 110$. Now,

$$|(I \cup C)'| = |U| - |I \cup C| \implies |I \cup C| = 200 - 110 = 90$$

therefore, using *union rule*, we get:

$$|I \cap C| = |I| + |C| - |I \cup C| = 70 + 50 - 90 = 30$$

7) Some colleges were surveyed to determine sports teams. 34 had Football teams ($F$), 39 had Basketball teams ($B$), 51 had Volleyball ($V$) or Basketball teams, 9 had Football and Volleyball teams, 11 had Volleyball and Basketball teams, 13 had Football and Basketball teams, 6 had all three and 11 colleges had none of these 3 teams.

(a) Fill in the given Venn Diagram.

**Ans.** List all the information we have:

- $|F| = 34$
- $|B| = 39$
- $|V \cup B| = 51$
- $|F \cap V| = 9$
- $|V \cap B| = 11$
- $|F \cap B| = 13$
- *(brown)* $|F \cap B \cap V| = 6$
- *(gray)* $|(F \cup B \cup V)'| = 11$
These are the information we can recover from the above:

- **(green)** $(F \cap V) \setminus B = |F \cap V| - |F \cap V \cap B| = 9 - 6 = 3$
- **(orange)** $(B \cap V) \setminus F = |B \cap V| - |F \cap V \cap B| = 11 - 6 = 5$
- **(violet)** $(F \cap B) \setminus V = |F \cap B| - |F \cap V \cap B| = 13 - 6 = 7$

that fit in the Venn diagram:

Then

- **(yellow)** $|V| = |V \cup B| - |B| + |V \cap B| = 51 - 39 + 11 = 23$
- **(red)** $|B \setminus (F \cup V)| = |B \cup V| - (23 + 7) = 51 - 30 = 21$
- **(blue)** $|F \setminus (B \cup V)| = |F| - (3 + 6 + 7) = 34 - 16 = 18$

and the complete Venn diagram is:

(b) How many colleges were surveyed?
**Ans.** $|U| = 11 + 9 + 3 + 6 + 5 + 18 + 7 + 21 = 80$

(c) How many had only one of these teams?
**Ans.** The question concerns the area

that is formed by $18 + 9 + 21 = 38$ elements.

(d) How many had Football or Basketball teams?
**Ans.** $|F \cup B| = |F| + |B| - |F \cap B| = 34 + 39 - 13 = 60$

(e) How many had Basketball teams but not Football teams?
8) A group of 150 people go to Haymarket on Saturday morning between 10am-11am.

- 90 bought apples \((A)\);
- 50 bought bananas \((B)\);
- 70 bought cherries \((C)\);
- 15 bought apples and cherries;
- 12 bought bananas and cherries;
- 10 bought all three products;
- 3 bought none of them.

Determine how many people bought apples and bananas, using the Venn diagram.

**Ans.** List all the information we have:

\[
\begin{align*}
|U| &= 150 \\
|A| &= 90 \\
|B| &= 50 \\
|C| &= 70 \\
|A \cap C| &= 15 \\
|B \cap C| &= 12 \\
|A \cap B \cap C| &= 10 \text{ (brown)} \\
|(A \cup B \cup C)'| &= 3 \text{ (gray)}
\end{align*}
\]

Here there are the first information we can deduce:

\[
\begin{align*}
|A \cup B \cup C| &= |U| - |(A \cup B \cup C)'| = 150 - 3 = 147 \\
&(\text{green}) \ |(A \cap C) \setminus B| = |A \cap C| - |A \cap B \cap C| = 15 - 10 = 5 \\
&(\text{orange}) \ |(B \cap C) \setminus A| = |B \cap C| - |A \cap B \cap C| = 12 - 10 = 2 \\
&(\text{yellow}) \ |C \setminus (A \cup B)| = |C| - (10 + 5 + 2) = 70 - 17 = 53
\end{align*}
\]

We end up with the following diagram:

Now, call \(x\) the cardinality of the *violet* region, that is, \(x = |(A \cap B) \setminus C|\). We know that:

\[
\begin{align*}
&(\text{blue}) \ |A \setminus (B \cup C)| = |A| - |A \cap C| - x = 90 - 15 - x = 75 - x \\
&(\text{red}) \ |B \setminus (A \cup C)| = |B| - |B \cap C| - x = 50 - 12 - x = 38 - x
\end{align*}
\]

So everything is reduced to \(x\). But, we know that the \(|U| = 150\), that is,

\[
150 = 3 + 53 + 5 + 10 + 2 + (75 - x) + x + (38 - x) = 186 - x \implies x = 186 - 150 = 36
\]

Finally then

\[
(\text{violet}) \ |(A \cap B) \setminus C| = 36
\]
\( (\text{blue}) \ |A \setminus (B \cup C)| = 75 - 36 = 39 \)
\( (\text{red}) \ |B \setminus (A \cup C)| = 38 - 36 = 2 \)

and the complete Venn diagram is:

![Venn Diagram](image)

The number of people that bought apples and bananas is given by

\[ |A \cap B| = 36 + 10 = 46 \]

9) Represents all the subset of \( A = \{1, 2, 3, 4\} \) using the diagram described at the beginning of this worksheet.

**Ans.** We will have \( 2^4 = 16 \) subsets, ordered on five levels. The resulting diagram is:
Problem 1 Write the following rules:

(a) Union rule for probability:
\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) \]

(b) Complement rule for probability:
\[ P(E') = 1 - P(E) \]

(c) Product rule for probability (use conditional probability):
\[ P(E \cap F) = P(E)P(F|E) = P(F)P(E|F) \]

(d) Product rule for independent events:
\[ P(E \cap F) = P(E)P(F) \]

Problem 2 Complete the following definitions:

(a) Events \( E \) and \( F \) are mutually exclusive events if \( E \cap F = 0 \), that is, \( P(E|F) = 0 \) (equivalently, \( P(F|E) = 0 \)).

(b) Events \( E \) and \( F \) are independent if \( P(E|F) = P(E) \) (equivalently \( P(F|E) = P(F) \)).

Problem 3 Circle the correct answer:

(1) Let \( E \) and \( F \) be mutually exclusive events. Which of the following are true?

(i) \( E \) and \( F \) are independent events.

(ii) \( P(E \cap F) = 0 \).

(iii) \( P(E \cup F) = P(E) + P(F) \).

A. (i) B. (i) and (ii) C. (ii) and (iii) D. (ii) only E. All of them.

(2) \( P(E|F) = \)?

A. \( \frac{P(E)}{P(F)} \) B. \( \frac{P(F)}{P(E)} \) C. \( \frac{P(E \cap F)}{P(F)} \) D. \( \frac{P(E \cap F)}{P(E)} \) E. \( P(E \cap F)P(E) \)

(3) Suppose that \( E \) and \( F \) are independent and \( P(E) = 0.3 \), \( P(F) = 0.5 \). What is \( P(E|F) \)?

A. 0.3 B. 0 C. 0.15 D. 0.6 E. Insufficient information

(4) Suppose that \( E \) and \( F \) are two independent events and \( P(E) = \frac{4}{10} \), \( P(F) = \frac{2}{10} \). What is \( P(E \cup F) \)?

A. \( \frac{4}{10} \) B. \( \frac{7}{10} \) C. \( \frac{2}{10} \) D. \( \frac{9}{10} \) E. \( \frac{8}{10} \)

Indeed, using union rule and the fact that \( E \) and \( F \) are independent,
\[ P(E \cup F) = P(E) + P(F) - P(E)P(F) = \frac{4}{10} + \frac{5}{10} - \frac{20}{100} = \frac{7}{10} \]

(5) If the odds in favor of an event \( E \) are 2 to 3 then \( P(E') \) is equal to:
Indeed, since \( m = 2, n = 3 \),

\[
P(E') = \frac{n}{m+n} = \frac{3}{2+3} = \frac{3}{5}
\]

**Problem 4** For two events \( A \) and \( B \), we know that

(i) \( P(A) = \frac{7}{10} \)

(ii) \( P(B') = 0.3 \)

(iii) \( P(A' \cap B) = \frac{1}{5} \)

Using a Venn diagram, find the following probabilities:

(a) \( P(A \cup B) = \frac{9}{10} \)

(b) \( P(A' \cup B') = \frac{1}{2} \)

(c) \( P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{2}{3} = \frac{2}{3} \)

Indeed, the Venn diagram is given by:

![Venn Diagram](image)

The information we have correspond to:

(i) \( x + y = 0.7 \)

(ii) \( P(B) = 0.7 \), then \( x + z = 0.7 \)

(iii) \( z = 0.2 \)

So, we get \( x = 0.5, y = 0.2 \) and \( t = 1 - (x + y + z) = 1 - 0.9 = 0.1 \).

![Venn Diagram](image)

From this diagram we can easily pick up the values for (a) – (b) – (c).

**Problem 5** For two events \( X, Y \) such that \( P(Y) = 0.7, P(X') = \frac{4}{5}, P(Y|X) = \frac{1}{2} \).

(a) \( P(X) = 1 - P(X') = \frac{2}{5} = \frac{4}{10} \)

(b) \( P(X \cap Y) = P(X)P(Y|X) = \frac{4}{10} \cdot \frac{5}{10} = \frac{20}{100} = 0.2 \)

(c) Complete the Venn diagram with probabilities:
(d) \( P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{0.2}{0.5} = \frac{2}{5} \)

(e) Are \( X \) and \( Y \) independent? Explain.
Since \( P(X|Y) \neq P(X) \), the two events are not independent.

Problem 6 Two dice are rolled. Let \( E \) be the event that the first die shows an even number and \( F \) be the event that the second die shows 6.

(a) Are \( E, F \) independent events?

Since \( P(E) = \frac{3}{6} = 0.5 \) and \( P(E|F) = \frac{3}{6} = 0.5 \), we have
\[
P(E|F) = P(E)
\]
that is, the two events are independent. (Indeed, if the second die shows 6, the probability that the first die shows an even number still remains the same).

(b) Are \( E, F \) mutually exclusive events?

Of course not. For example, \((4, 6)\) is an event in \( E \cap F \).

(c) \( P(E|F) = \frac{1}{2} \)

(d) \( P(E \cap F) = P(F)P(E|F) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12} \)

Problem 7 Two cards are drawn without replacement from a standard deck.

(a) Find the probability of a \( S_2 \) given \( R_1 \), that is, the probability that the second card is a spade, given that the first card is red.

\[
P(S_2|R_1) = \frac{13}{51}, \text{ since spades are not red.}
\]

(b) Find the probability that both cards are diamonds.

\[
P(D_1 \cap D_2) = P(D_1)P(D_2|D_1) = \frac{13}{52} \cdot \frac{12}{51} = \frac{3}{51}
\]

Problem 8 Three cards are drawn without replacement from a standard deck.

(a) Find the probability of a tris of aces, that is:

\[
P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_2 \cap A_1) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50}
\]

(b) Find the probability of a tris.
In a standard deck, we have 13 possible cards (up to a suit). So, the event \( T = \{\text{tris}\} \) is the union of thirteen events, each of one corresponds to one of those cards, that is:

\[ T = \{\text{AAA, 111, 222, 333, 444, 555, \ldots}\} \]

It’s easy to see that these events have the same probability, so

\[ P(T) = 13 \cdot P(\text{AAA}) = 13 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{3}{25 \cdot 51} \approx 0.0023 \]

**Problem 9** Consider the following diagram:

\[ \text{Diagram with probabilities} \]

(a) \( P(B|A') = 0.1 \)

(b) \( P(A \cap B) = P(A)P(B|A) = \frac{6}{10} \cdot \frac{8}{10} = \frac{48}{100} \)

(c) \( P(B) = P(B \cap A') + P(B \cap A) = P(A')P(B|A') + P(B \cap A) = \frac{4}{10} \cdot \frac{1}{10} + \frac{48}{100} = \frac{52}{100} \)

**Problem 10** According to a survey of a group of college students, 40% of the students have heard Tchaikovsky’s Piano Concert No.1 (T), of those 50% have read a novel by Dostoyevsky (D). Of those who have not heard the concerto, 25% have read a novel by Dostoyevsky.

(a) Draw a tree diagram to summarize the results of the survey.
(b) What is the probability that a random student has not read a novel by Dostoyevsky?

\[ P(D) = P(D \cap T) + P(D \cap T') = P(T)P(D|T) + P(T')P(D|T') = \frac{4}{10} \cdot \frac{5}{10} + \frac{6}{10} \cdot \frac{2.5}{10} = \frac{20+15}{100} = \frac{35}{100} \]

(c) What is the probability that a random student has heard Tchaikovsky's Piano Concerto No.1, given that he/she has not read a novel by Dostoyevsky?

First, we can compute:

\[ P(T \cap D') = P(T)P(D'|T) = \frac{4}{10} \cdot \frac{5}{10} = \frac{2}{10} \]
\[ P(T' \cap D') = P(T')P(D'|T') = \frac{6}{10} \cdot \frac{7.5}{10} = \frac{45}{100} \]
\[ P(D') = P(T \cap D') + P(T' \cap D') = \frac{20}{100} + \frac{45}{100} = \frac{65}{100} \]

Then

\[ P(T|D') = \frac{P(T \cap D')}{P(D')} = \frac{\frac{20}{100}}{\frac{65}{100}} = \frac{20}{65} = \frac{4}{13} \approx 0.307 \]
Problem 1 Write the following rules:

(a) Union rule for probability:
\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) \]

(b) Complement rule for probability:
\[ P(E') = 1 - P(E) \]

(c) Product rule for probability (use conditional probability):
\[ P(E \cap F) = P(E)P(F|E) = P(F)P(E|F) \]

(d) Product rule for independent events:
\[ P(E \cap F) = P(E)P(F) \]

Problem 2 Complete the following definitions:

(a) Events \( E \) and \( F \) are mutually exclusive events if \( E \cap F = 0 \), that is, \( P(E|F) = 0 \) (equivalently, \( P(F|E) = 0 \)).

(b) Events \( E \) and \( F \) are independent if \( P(E|F) = P(E) \) (equivalently \( P(F|E) = P(F) \)).

Problem 3 Circle the correct answer:

1. Let \( E \) and \( F \) be mutually exclusive events, such that \( P(E) = 0.3, P(F) = 0.4 \). Which is \( P(E \cup F) \)?

   A. 0.3     B. 0.9     C. 0.7     D. 0.1     E. None of them.

2. Let \( E \) and \( F \) be mutually exclusive events, such that \( P(E) = 0.8, P(F) = 0.4 \). Which is \( P(E \cap F) \)?

   A. 0.1     B. 0     C. 0.4     D. 0.15     E. 0.9

3. Suppose that \( E \) and \( F \) are independent and \( P(E) = 0.45, P(F) = 0.15 \). What is \( P(E|F) \)?

   A. 0.3     B. 0     C. 0.15     D. 0.45     E. Insufficient information

4. Suppose that \( E \) and \( F \) are two independent events and \( P(E) = \frac{4}{9}, P(F) = \frac{5}{9} \). What is \( P(E \cup F) \)?

   A. \( \frac{29}{30} \)     B. \( \frac{1}{10} \)     C. \( \frac{27}{28} \)     D. \( \frac{91}{100} \)     E. \( \frac{84}{100} \)

5. If the odds in favor of an event \( E \) are 6 to 3 then \( P(E') \) is equal to:

   A. \( \frac{1}{3} \)     B. \( \frac{2}{3} \)     C. 1     D. \( \frac{3}{2} \)     E. \( \frac{9}{10} \)

Problem 4 For two events \( A \) and \( B \), we know that
\[ P(A) = 0.7 \]
\[ P(B') = 0.3 \]
\[ P(A' \cap B') = 0.2 \]

Using a Venn diagram, find the following probabilities:

(a) \[ P(A \cup B) = 0.8 \]

(b) \[ P(A' \cup B') = 0.4 \]

(c) \[ P(A \cap B') = 0.1 \]

(c) \[ P(A'|B) = \frac{0.1}{0.5} = \frac{1}{5} \]

(d) \[ P(B|A) = \frac{0.6}{0.7} = \frac{6}{7} \]

(e) Are \( A, B \) independent?

Indeed, the Venn diagram is given by:

The information we have correspond to:

(i) \[ P(A) = x + y = 0.7 \]

(ii) \[ P(B) = 0.7, \text{ then } x + z = 0.7 \]

(iii) \[ P(A' \cap B') = P((A \cup B)') = t = 0.2 \]

So, we get \( x + y + z = 0.8 \), then \( z = 0.1, x = 0.6, y = 0.1 \).

Problem 5 For two events \( X, Y \) such that \( P(Y) = 0.7, P(X') = 0.3, P(Y|X) = 0.6 \), find:

(a) \[ P(X) = 0.7 \]

(b) \[ P(X \cap Y) = P(X)P(Y|X) = \frac{7}{10} \cdot \frac{6}{10} = \frac{42}{100} \]

(c) Complete the Venn diagram with probabilities:
(d) \[ P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{0.42}{0.70} = \frac{42}{70} \]

(e) Are \( X \) and \( Y \) independent? Explain.

No, because \( P(X|Y) \neq P(X) \).

**Problem 6** Three cards are drawn without replacement from a standard deck.

(a) Find the probability of three 3: \( P(333) = ? \)

\[ P(333) = P(3_1|3_1 \cap 3_2)P(3_2|3_1)P(3_1) = \frac{4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50} \]

(b) Find the probability of getting at least one king.

\[ P(K) = 1 - P(\text{no king}) = 1 - \frac{4 \cdot 47 \cdot 46}{52 \cdot 51 \cdot 50} \]

(c) Find the probability of getting at least a pair.

\[ P(\text{at least a pair}) = 1 - P(\text{no pair}) = 1 - \frac{52 \cdot 48 \cdot 44}{52 \cdot 51 \cdot 50} \]

**Problem 7** Consider the following diagram:

\[ \begin{align*}
A & \rightarrow B \\
A' & \rightarrow B' \\
B & \rightarrow \frac{0.4}{0.6} \\
B' & \rightarrow \frac{0.2}{0.8} \\
A' & \rightarrow \frac{0.3}{0.7} \\
B & \rightarrow \frac{0.7}{0.3} \\
B' & \rightarrow \frac{0.6}{0.4} \\
\end{align*} \]

(a) \( P(A' \cap B') = \frac{2}{10} \cdot \frac{7}{10} \)

(b) \( P(B) = \frac{8}{10} \cdot \frac{4}{10} + \frac{2}{10} \cdot \frac{3}{10} = \frac{38}{100} \)

(c) \( P(A|B) = \frac{32}{38} \)

**Problem 8** Consider the following table:

<table>
<thead>
<tr>
<th>Age</th>
<th>Population</th>
<th>Married (M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 25</td>
<td>16%</td>
<td>5%</td>
</tr>
<tr>
<td>25-34</td>
<td>25%</td>
<td>36%</td>
</tr>
<tr>
<td>35-44</td>
<td>32%</td>
<td>52%</td>
</tr>
<tr>
<td>&gt; 45</td>
<td>26%</td>
<td>61%</td>
</tr>
</tbody>
</table>
The table encodes the following information: for example, the first line express the fact that 16% of the population is less than 25 yo and 5% of them are married.

(a) Put all the information in a diagram.

(b) \( P(M'|35 - 44) = \frac{48}{100} \)

(c) 
\[
P(> 45|M') = \frac{26.39}{16.95 + \frac{25.62}{100-100} + \frac{32.48}{100-100} + \frac{26.39}{100-100}}
\]

Problem 9 The following information pertains to three shipping terminals:

<table>
<thead>
<tr>
<th>Terminal</th>
<th>Cargo Handled</th>
<th>Error (E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land (L)</td>
<td>0.5</td>
<td>0.02</td>
</tr>
<tr>
<td>Air (A)</td>
<td>0.4</td>
<td>0.04</td>
</tr>
<tr>
<td>Sea (S)</td>
<td>0.1</td>
<td>0.14</td>
</tr>
</tbody>
</table>
(a) Put all the information in a diagram.

(b) \( P(E) = \frac{52}{1000} + \frac{44}{1000} + \frac{14}{1000} = \frac{4}{100} \)

(c) \( P(S|E) = \frac{14}{40} \)
Problem 1 True/False Questions:

1. \( P(n, n) = 1 \) \quad \text{F}
2. \( 24! = 8! \) \quad \text{F}
3. \( ^n1 = n \) \quad \text{T}
4. \( ^n0 = n! \) \quad \text{F}
5. \( ^{10}7 = \binom{10}{3} \) \quad \text{T}
6. \( ^{n}r = \frac{P(n, r)}{r!} \) \quad \text{T}

Problem 2 Multiple Choice Questions:

1. How many 5-character sequences can be constructed, if the first and the last characters are digits and the three middle characters are distinct upper case letters?

   A. \( 26^3 \cdot 10^2 \)  
   B. \( 10^2 (26 \cdot 25 \cdot 24) \)  
   C. \( P(10, 2) P(26, 3) \)  
   D. \( \frac{10!}{2!} \left(\frac{26}{3}\right) \)  
   E. None of the preceding.

2. A family of 2 grandparents, 2 parents and 3 children are seated in a row of 7 seats. How many ways are there to seat them if the parents sit together and the grandparents sit together?

   A. \( 7! \)  
   B. \( \frac{7!}{2!2!3!} \)  
   C. \( 5!2!2! \)  
   D. \( 3!2!2!3! \)  
   E. None of the precedings.

3. A basketball team has 14 players. How many ways are there to choose a captain and 4 other players?

   A. \( P(14, 5) \)  
   B. \( \frac{14!}{4!} \)  
   C. \( \binom{14}{1} \binom{14}{4} \)  
   D. \( 14 \binom{13}{4} \)  
   E. None of the precedings.

4. In a club of 10 men and 15 women, how many ways are there to choose a committee consisting of 2 men and 3 women?

   A. \( \binom{10}{2} \binom{15}{3} \)  
   B. \( P(10,2)P(15,3) \)  
   C. \( 10 \cdot 9 \cdot 15 \cdot 14 \cdot 13 \)  
   D. \( \frac{10! \cdot 15!}{2!3!} \)  
   E. None of the precedings.

5. In how many ways can a hand of 13 cards contain exactly one ace?

   A. \( P(4,1) P(48,12) \)  
   B. 13!  
   C. \( \binom{13}{1} \)  
   D. \( \frac{4!}{1!} \left(\frac{48}{12}\right) \)  
   E. None of the precedings.

6. In how many ways can 2 red, 2 blue and 2 green marble be drawn from a bag containing 2 red, 4 blue and 6 green marbles?

   A. 90  
   B. 48  
   C. 720  
   D. 8  
   E. None of the precedings.

7. How many ways are there to answer 10 multiple choice questions, each with 4 choices for the answer?
A. 40  B. $10^4$  C. $4^{10}$  D. $\binom{10}{4}$  E. None of the precedings.

8. Suppose you draw 6 cards from the card deck without replacement. What is the probability that you draw 3 hearts and 3 spades?

A. $\frac{\binom{13}{3}\binom{13}{3}}{\binom{52}{6}}$  B. $\frac{\binom{13}{3}\binom{13}{3}}{\binom{39}{6}}$  C. $\frac{3!3!}{6!}$  D. $\frac{P(13,3)P(13,3)}{P(52,6)}$  E. None of the precedings.

9. Suppose you draw 3 times with replacement from a bag that contains 10 digit tiles: 0, 1, 2, . . . , 8, 9. What is the probability that you draw three different digits?

A. 0.99  B. 0.72  C. 0.5  D. $\frac{1}{3}$  E. None of the precedings.

10. Suppose a student take a test that consists of three T/F questions. If he guess the answers, what is the probability that his answers are all correct?

A. 0.125  B. $\frac{1}{6}$  C. 0.5  D. $\frac{1}{3}$  E. None of the precedings.

11. If 5 children are picked randomly from a group of 6 boys and 6 girls, what is the probability that all the 5 children are girls? (Round your answer to 4 decimal place)

A. 0.4167  B. 0.1667  C. 0.0313  D. 0.0076  E. None of the precedings.

12. If the letters of "CEDED" are rearranged randomly, what is the probability that the same word is resulted?

A. 0.1667  B. 0.0333  C. 0.0167  D. 0.0083  E. None of the precedings.

Remark 0.1 A nice and useful formula is given by the following problem. We assume to repeat an experiment $n$ times, with a probability of success $p$. Call $S$ the number of successes. Then

$$P(S = k) = \binom{n}{k}p^k(1-p)^{n-k}$$

Problem 3 Short Answer problems.

1. There are 4 Freshman, 6 Sophomores, 3 Seniors in a club.

   a. In how many ways it is possible to choose a committee of 3 members?

   $$\binom{13}{3}$$

   b. In how many ways it is possible to choose a committee of 3 members such that there are 1 Freshman, 1 Sophomore, 1 Senior?

   $$\binom{4}{1}\binom{6}{1}\binom{3}{1}$$

   c. In how many ways is it possible to choose a committee of 3 members such that there are no Senior?

   $$\binom{10}{3}$$
d. In how many ways is it possible to choose a committee of 3 members, find the probability that there are no Freshman in the committee?

\[ \frac{\binom{8}{3}}{\binom{13}{3}} \]

2. There are 120 Men and 80 Women in a club. In how many ways can you form a committee of 30 Men and 20 Women?

\[ \binom{120}{30} \binom{80}{20} \]

3. Using all the letters in ALABAMA exactly once.

a. How many distinct words of 7 letters can be constructed?

\[ \frac{7!}{4!1!1!1!1!} = \frac{7!}{6!} = 7 \cdot 6 \cdot 5 = 210 \]

b. How many distinct 2-letter words can be constructed?

We can find the answer thinking about the number of A’s present: 0, 1, 2. Then the number of possible 2-letters words is

\[ p(3, 2) + \binom{2}{1} \binom{3}{1} + 1 = 6 + 6 + 1 = 13 \]

4. When you rearrange all the letters in the word KAYAK randomly, find the probability that the rearrangement is again KAYAK.

\[ \frac{1}{2!2!} = \frac{2!2!}{5!} = \frac{4}{120} = \frac{1}{30} \]

5. When you pick a hand of 5 cards from a deck of 52 cards.

a. Find the probability that all of them are clubs.

\[ \frac{\binom{13}{5}}{\binom{52}{5}} \]

b. Find the probability that none of them are clubs.

\[ \frac{\binom{39}{5}}{\binom{52}{5}} \]

c. Find the probability that there are two spades and three diamonds.

\[ \frac{\binom{13}{2} \binom{13}{3}}{\binom{52}{5}} \]

d. Find the probability that none of them are aces.

\[ \frac{\binom{48}{5}}{\binom{52}{5}} \]

e. Find the probability that at least one of them is an ace.

Set \( A = k \) = {there are \( k \) aces}

\[ P(A \geq 1) = 1 - P(A = 0) = 1 - \frac{\binom{48}{5}}{\binom{52}{5}} \]

f. Find the probability of a full house of K and Q.
6. When you flip a fair coin 80 times, find the probability that there are exactly 40 Heads.

\[ \binom{80}{40} \frac{1}{2^{80}} \]

7. When you roll a 6-die faces 12 times, find the probability that you roll a 6 exactly three times.

\[ \binom{12}{3} \left( \frac{1}{6} \right)^3 \left( \frac{5}{6} \right)^9 \]

8. In a certain city, the probability that a new born baby is a girl is 0.6. If 200 babies were born last month in this city, what is the probability that exactly 120 of them are girls?

\[ \binom{200}{120} (0.6)^{120} (0.4)^{80} \]
Remark 0.1 Binomial Probability Formula:

\[ P(S = k) = \binom{n}{k} p^k (1-p)^{n-k} \]

Problem 1 Multiple Choice Questions:

1. Suppose you draw 6 marbles (without replacement) from a bag containing 2 red, 3 blue, 4 green marbles. What is the probability that you draw 2 red, 2 blue and 2 green marbles?
   
   A. \( \frac{2 \cdot 2 \cdot 2}{2 \cdot 3 \cdot 4} \)  
   B. \( \frac{2!2!2!}{2!3!4!} \)  
   C. \( \frac{P(2,2)P(3,2)P(4,2)}{P(9,6)} \)  
   D. \( \binom{2}{2} \binom{3}{2} \binom{4}{2} \binom{9}{6} \)  
   E. None of the precedings.

2. Jessica is in a basketball team with 9 players (including Jessica). Five players are chosen at random. What is the probability that Jessica is picked?
   
   A. \( \frac{P(8,4)}{P(9,5)} \)  
   B. \( \frac{\binom{8}{4}}{\binom{9}{5}} \)  
   C. \( \frac{1}{9} \)  
   D. \( \frac{1}{5} \)  
   E. None of the precedings.

Indeed, we are assuming that Jessica is picked and we need to choose 4 other player out of 8.

3. Suppose you draw 3 times with replacement from a bag that contains 4 digits tiles: 1,2,3,4, what is the probability that you draw three same digits?
   
   A. \( \frac{1}{2} \)  
   B. \( \frac{1}{4} \)  
   C. \( \frac{1}{16} \)  
   D. \( \frac{1}{64} \)  
   E. None of the precedings.

Indeed, we have 4 configuration \( \{111, 222, 333, 444\} \) out of \( 4^3 = 64 \) possible configuration.

4. If two letters in ACT are rearranged randomly, what is the probability that the word AT is resulted?
   
   A. \( \frac{1}{6} \)  
   B. \( \frac{1}{2} \)  
   C. \( \frac{1}{3} \)  
   D. \( \frac{2}{3} \)  
   E. None of the precedings.

The question is slightly unclear. With "the word AT is resulted" we mean that AT is contained in a possible rearrangement of ACT. The number of all possible rearrangement is \( 3! = 6 \) and AT is contained only in CAT and ATC. So the probability is \( \frac{1}{3} \).

5. There is a 5% probability for a transistor manufactured in a plant to be defective. Consider a sample of 10 transistor. What is the probability that at most one of them is defective?
   
   A. 0.3151  
   B. 0.5987  
   C. 0.6302  
   D. 0.9138  
   E. None of the precedings.

Denote by \( D \) the number of defective transistor.

\[ P(D \leq 1) = P(D = 0) + P(D = 1) = \binom{10}{0}(0.05)^0(0.95)^{10} + \binom{10}{1}(0.05)^1(0.95)^9 = 0.5987 + 0.3151 = 0.9138 \]

On the previous file, answer D was slightly wrong, since it was 0.9137.

Problem 2 Short Answer Problems.
(a) A fair die is rolled 4 times. What is the probability that the number is different each time?

We have $6^4 = 1296$ possible outcomes. Of them, $P(6, 4) = 360$ are made by four different numbers. So the probability is

$$\frac{360}{1296} = \frac{5}{18} \approx 0.278$$

(b) A bridge hand consists of 13 cards. What is the probability that a bridge hand contains no aces?

We have \( \binom{52}{13} \) possible outcomes and, of them, \( \binom{48}{13} \) contain no aces (we have to choose 13 cards out of 48 - where the aces are excluded). Then the probability is

$$\frac{\binom{48}{13}}{\binom{52}{13}} \approx 0.303$$

(c) Suppose you guess the answer of 5 multiple choice questions, each with 4 choices for the answer. Which is the probability that you have at least 4 right answers?

We have $4^5 = 1024$ possible answers. We want to use binomial probability. The probability that an answer is right is \( \frac{1}{4} \). Call $A$ the number of right answers:

$$P(A \geq 4) = P(A = 4) + P(A = 5) = \binom{5}{4} \left( \frac{1}{4} \right)^4 \left( \frac{3}{4} \right)^1 + \binom{5}{5} \left( \frac{1}{4} \right)^5 \left( \frac{3}{4} \right)^0$$

$$= 5 \cdot \frac{3}{1024} + \frac{1}{1024} = \frac{16}{1024} = \frac{1}{64} \approx 0.015$$

(d) An unfair coin with a probability of $1/3$ coming up heads is tossed 5 times. Find the probability that it comes up heads at least once.

Call $H$ the number of heads, after tossing the coin five times. Probability of having head in a single trial is $\frac{1}{3}$. Using binomial probability:

$$P(H \geq 1) = 1 - P(H < 1) = 1 - P(H = 0) =$$

$$= 1 - \binom{5}{0} \left( \frac{1}{3} \right)^0 \left( \frac{2}{3} \right)^5 = 1 - \frac{2^5}{3^5} = \frac{243 - 32}{243} = \frac{211}{243} \approx 0.868$$

Problem 3 Long Answer Problems.

(a) A class of 12 students consist of 5 Freshmen, 4 Sophomores and 3 Senior. A committee of 3 students is chosen at random. Find the probability of having 0, 1, 2, 3 or 4 Sophomores. Check that this is a probability distribution.

Call $S$ the random variable (start getting used with this terminology) that counts the number of Sophomores in the chosen committee. We have $\binom{12}{3} = 220$ possible committees. Then,

$$P(S = k) = \frac{\binom{4}{k} \binom{8}{3-k}}{\binom{12}{3}}$$

$$P(S = 0) = \frac{\binom{4}{0} \binom{8}{3}}{220} = \frac{1}{220} = \frac{56}{220}$$

$$P(S = 1) = \frac{\binom{4}{1} \binom{8}{2}}{220} = \frac{4}{220} = \frac{112}{220}$$

$$P(S = 2) = \frac{\binom{4}{2} \binom{8}{1}}{220} = \frac{6}{220} = \frac{48}{220}$$

$$P(S = 3) = \frac{\binom{4}{3} \binom{8}{0}}{220} = \frac{4}{220}$$

$$P(S = 4) = 0$$
Of course, \( P(S = 4) = 0 \) since the committee is formed by 3 members. This is a probability distribution, since it is made by mutually exclusive events such that the sum of probabilities is
\[
\frac{56}{220} + \frac{112}{220} + \frac{48}{220} + \frac{4}{220} = 1
\]

(b) In a bag there are 9 candies and 3 poison pills. Four kids can draw from the bag. Find the probability that 0, 1, 2, 3, or 4 of them will die. Check that this is a probability distribution.

Call \( P \) the random variable that counts the number of poison pills. The problem reduces to subsets of cardinality 4 out of a set with 12 elements. The total number of possible outcomes is \( \binom{12}{4} = 495 \). As before,
\[
P(P = k) = \frac{\binom{3}{k} \binom{9}{4-k}}{\binom{12}{4}}
\]
\[
P(P = 0) = \frac{\binom{3}{0} \binom{9}{4}}{495} = \frac{1 \cdot 126}{495} = \frac{126}{495}
\]
\[
P(P = 1) = \frac{\binom{3}{1} \binom{9}{3}}{495} = \frac{3 \cdot 84}{495} = \frac{252}{495}
\]
\[
P(P = 2) = \frac{\binom{3}{2} \binom{9}{2}}{495} = \frac{3 \cdot 36}{495} = \frac{108}{495}
\]
\[
P(P = 3) = \frac{\binom{3}{3} \binom{9}{1}}{495} = \frac{1 \cdot 9}{495} = \frac{9}{495}
\]
\[
P(P = 4) = 0
\]

As before, \( P(P = 4) = 0 \) since we have only 3 poisoned pills. This is a probability distribution, since it is made by mutually exclusive events such that the sum of probabilities is
\[
\frac{126}{495} + \frac{252}{495} + \frac{108}{495} + \frac{9}{495} = 1
\]

(c) An unfair coin with probability of heads \( \frac{2}{3} \) is tossed three times.

(1) Describe the sample space and for each simple event find the probability.

<table>
<thead>
<tr>
<th>( E )</th>
<th>( P(E) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>hhh</td>
<td>( \frac{8}{27} )</td>
</tr>
<tr>
<td>hht</td>
<td>( \frac{4}{27} )</td>
</tr>
<tr>
<td>hth</td>
<td>( \frac{4}{27} )</td>
</tr>
<tr>
<td>htt</td>
<td>( \frac{2}{27} )</td>
</tr>
<tr>
<td>thh</td>
<td>( \frac{2}{27} )</td>
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<td>( \frac{2}{27} )</td>
</tr>
<tr>
<td>tth</td>
<td>( \frac{2}{27} )</td>
</tr>
<tr>
<td>ttt</td>
<td>( \frac{1}{27} )</td>
</tr>
</tbody>
</table>

(2) Find the probability that the number of occurrence of heads is 0, 1, 2 or 3. Check that this is a probability distribution.

Call \( H \) the random variable that counts the number of heads. Use binomial probability.
\[
P(H = k) = \binom{3}{k} \left( \frac{2}{3} \right)^k \left( \frac{1}{3} \right)^{3-k}
\]
\[
P(H = 0) = \binom{3}{0} \left( \frac{2}{3} \right)^0 \left( \frac{1}{3} \right)^3 = \frac{1}{27}
\]
\[
P(H = 1) = \binom{3}{1} \left( \frac{2}{3} \right)^1 \left( \frac{1}{3} \right)^2 = \frac{2}{27}
\]
\[
P(H = 2) = \binom{3}{2} \left( \frac{2}{3} \right)^2 \left( \frac{1}{3} \right)^1 = \frac{4}{27}
\]
\[
P(H = 3) = \binom{3}{3} \left( \frac{2}{3} \right)^3 \left( \frac{1}{3} \right)^0 = \frac{8}{27}
\]
The sum of all these probabilities is:

\[
\frac{1 + 6 + 12 + 8}{27} = 1
\]

(3) Find the expected value of the random variable \( x = \{ \text{number of heads} \} \). Recall that the expected value of a probability distribution is given by the following formula: if \( X \) is a random variable with value \( x_1, x_2, \ldots, x_n \) and probability \( P(x_i) = p_i \), then the expected value \( E(X) \) is given by

\[
E(X) = x_1 \cdot p_1 + x_2 \cdot p_2 + \cdots + x_n \cdot p_n
\]

According to the values we just found at (2), we have:

\[
E(H) = 0 \cdot \frac{1}{27} + 1 \cdot \frac{6}{27} + 2 \cdot \frac{12}{27} + 3 \cdot \frac{8}{27} = \frac{0 + 6 + 24 + 24}{27} = \frac{54}{27} = 2
\]
Problem 1 Compute the expected value of this probability distribution:

<table>
<thead>
<tr>
<th>x</th>
<th>3</th>
<th>8</th>
<th>14</th>
<th>20</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(x)</td>
<td>0.2</td>
<td>0.05</td>
<td>0.3</td>
<td>0.25</td>
<td></td>
</tr>
</tbody>
</table>

Clearly, a value is missing from the table. In order to find \( P(x = 35) \) I should remember that the table describes a probability distribution. Then the sum should be equal to 1 and

\[
P(x = 35) = 1 - (0.2 + 0.05 + 0.3 + 0.25) = 1 - 0.8 = 0.2
\]

So the expected value is

\[
E(x) = 3 \cdot 0.2 + 8 \cdot 0.05 + 14 \cdot 0.3 + 20 \cdot 0.25 + 35 \cdot 0.2 = 0.6 + 0.4 + 4.2 + 5 + 7 = 17.2
\]

Problem 2 Consider tossing 2 times an unfair coin with a probability of 0.3 turning up heads.

(a) Let \( x \) denote the number of heads in the 2 tosses. Fill in the following probability distribution table for \( x \).

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(x)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using binomial probability it is easy to see that

\[
P(x = k) = \binom{2}{k} (0.3)^k (0.7)^{2-k}
\]

then the table is

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(x)</td>
<td>0.49</td>
<td>0.42</td>
<td>0.09</td>
</tr>
</tbody>
</table>

(b) Find the expected number of heads.

Since we are using binomial probability \( E(x) = 2 \cdot 0.3 = 0.6 \). That indeed is equal to \( E(x) = 1 \cdot 0.42 + 2 \cdot 0.09 = 0.6 \).

Problem 3 An unfair coin with probability of heads 0.2 is tossed 35 times. Find the expected number of heads.

Call \( x \) the number of heads in 35 tossing. As before, the probability distribution is given by the formula:

\[
P(x = k) = \binom{35}{k} (0.2)^k (0.8)^{35-k}
\]

then the expectation value is:

\[
E(x) = 35 \cdot 0.2 = 7
\]

( using the expectation value for probability distribution).

Problem 4 Suppose you choose 4 kids randomly from a group of 2 girls and 8 boys.

(a) Find the expected number of girls picked.
Call $G$ the random variable that counts the number of girls. $G$ can be \{0, 1, 2\}.

\[
P(G = 0) = \frac{\binom{5}{0}}{\binom{10}{4}} = \frac{1}{3}
\]

\[
P(G = 1) = \frac{\binom{5}{1}}{\binom{10}{4}} = \frac{8}{15}
\]

\[
P(G = 2) = \frac{\binom{5}{2}}{\binom{10}{4}} = \frac{2}{15}
\]

and the expectation value is

\[
E(G) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{2}{15} = \frac{8 + 4}{15} = \frac{12}{15} = 0.8
\]

(b) Find the expected number of boys picked.

Call $B$ the random variable that counts the number of boys. Then $B = 4 - G$ and $B$ can assume value \{4, 3, 2\}. Obviously

\[
P(B = 4) = P(G = 0) = \frac{1}{3}
\]

\[
P(B = 3) = P(G = 1) = \frac{8}{15}
\]

\[
P(B = 2) = P(G = 2) = \frac{2}{15}
\]

and the expectation value is

\[
E(G) = 4 \cdot \frac{1}{3} + 3 \cdot \frac{8}{15} + 2 \cdot \frac{2}{15} = \frac{20 + 24 + 4}{15} = \frac{48}{15} = 3.2
\]

Of course, it’s not an accident that $E(G) + E(B) = 4$.

**Problem 5** Consider the following probability distribution of $x$, the number on a biased die.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

(a) Find $P(x = 6)$.

In order to find $P(x = 6)$ I should remember that the table describes a probability distribution. Then the sum should be equal to 1 and

\[
P(x = 6) = 1 - (0.1 + 0.02 + 0.1 + 0.2 + 0.1) = 1 - 0.7 = 0.3
\]

(b) Find the expected value of $x$.

The expected value is

\[
E(x) = 1 \cdot 0.1 + 2 \cdot 0.2 + 3 \cdot 0.1 + 4 \cdot 0.2 + 5 \cdot 0.1 + 6 \cdot 0.3 = 0.1 + 0.4 + 0.3 + 0.8 + 0.5 + 1.8 = 3.9
\]

**Problem 6** Let $x$ be the number of a tiles drawn from a bag containing \{1, 2, 3, 4, 5\}.

(a) Fill in the following probability distribution for $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

since every event is equally likely so the probability is always the same, equal to $\frac{1}{5}$

(b) Find the expectation value of $x$. 


The expectation value is:

\[ E(x) = 1 \cdot 0.2 + 2 \cdot 0.2 + 3 \cdot 0.2 + 4 \cdot 0.2 + 5 \cdot 0.2 = (1 + 2 + 3 + 4 + 5) \cdot 0.2 = 15 \cdot 0.2 = 3 \]

Problem 7 (just for fun...) A raffle offers a first prize of 400$, 3 second prizes of 150$ and 10 third prizes of 60$. A total of 700$ tickets are sold. Determine the expected winnings for a person who buys 1 ticket.

(a) Let \( y \) denote the price of 1 ticket. Determine the expected winnings for a person who buys 1 ticket.

Call \( x \) the money earned by a person who buys 1 ticket. The probability distribution of \( x \) is

\[
\begin{array}{c|c|c|c|c}
 x & 399 & 149 & 59 & -y \\
P(x) & \frac{700}{700} & \frac{3}{700} & \frac{10}{700} & \frac{686}{700}
\end{array}
\]

The last column is given by the fact that that person wins no prize, so there is a loss of \( y \$ \). The expectation value is then

\[ E(x) = 399 \cdot \frac{1}{700} + 149 \cdot \frac{3}{700} + 59 \cdot \frac{10}{700} - y \cdot \frac{686}{700} = \frac{1436 - y \cdot 686}{700} \]

(b) If \( y = 3 \$ \), is it worth it?

In this case the expectation value is:

\[ E(x) = \frac{1436 - 3 \cdot 686}{700} = \frac{-622}{700} \approx -0.88 \]

Of course, the game is not fair, since on the average I will lose 88 cents, but this does not mean that it is not worth...

(c) How much would you be willing to pay for a ticket?

The best deal will be when \( E(x) = 0 \) This means \( 1436 - y \cdot 686 = 0 \) that is

\[ y = \frac{1436}{686} \approx 2.10 \]

Problem 8 In a game, you draw 2 tiles (without replacement) from a bag that contains 5 digit tiles: \{1,2,3,4,5\}. If both of your tiles are even, you win 2$. If one of your tiles is even, you win 0.5$. Otherwise, you lose 1$. Find the expected gain or loss in one game.

The random variable \( x \) can assume values \{2,0.5,-1\}. Call \( e \) the number of even tiles.

\[
\begin{align*}
P(x = 2) &= P(e = 2) = \binom{2}{2} \binom{3}{0} = \frac{1}{10} \\
P(x = .5) &= P(e = 1) = \binom{2}{1} \binom{3}{1} = \frac{6}{10} \\
P(x = -1) &= P(e = 0) = \binom{2}{0} \binom{3}{2} = \frac{3}{10}
\end{align*}
\]

Then the expectation value of \( x \) is:

\[ E(x) = 2 \cdot 0.1 + 0.5 \cdot 0.6 - 1 \cdot 0.3 = 0.2 + 0.3 - 0.3 = 0.2 \]

Problem 9 I have two job interviews the same day. I have the 80% of chance of getting the first job, which pays 20,000$ per year. Instead, I have 50% of chance of getting the second job, which pays 30,000$ per year.

(a) Which interview should I go to if I make my decision based on the expected gain of each interview?
This is the easiest case. Call $X_1$ the possible gain if I go to the first job interview and $X_2$ the possible gain if I go to the second job interview. Then,

$$E(X_1) = 20000 \cdot 0.8 = 16000$$
$$E(X_2) = 30000 \cdot 0.5 = 15000$$

and it is better if I go to the first job interview.

(b) Suppose that I have a job that pays 15,000$ per year. Does my decision change?

In this case,

$$E(X_1) = 20000 \cdot 0.8 + 15000 \cdot 0.2 = 16000 + 3000 = 19000$$
$$E(X_2) = 30000 \cdot 0.5 + 15000 \cdot 0.5 = 15000 + 7500 = 22500$$

So in this case it is better if I go to the second job interview.

(c) Assume further that I have 40% of chance of getting a bonus of 10,000$ if I get the first job, and 30% of chance of getting a bonus of 15,000$ if I get the second job. Which will be my decision then?

In order to see what it is better in this case, let’s build up a diagram to summarize the situation. Call $J_1, J_2$ the event that I got job 1 or job 2, respectively, and $B_1, B_2$ the event that I got the corresponding annual bonus.

If I go to the first interview, then

Of course, if I don’t get the job it’s impossible to get the bonus, so the probability that I gain 25,000$ per year is zero! The expectation value then is

$$E(X_1) = 30000 \cdot 0.4 \cdot 0.8 + 20000 \cdot 0.6 \cdot 0.8 + 25000 \cdot 0 + 15000 \cdot 1 \cdot 0.2$$
$$= 9600 + 9600 + 0 + 3000 = 22200$$
If I go to the first interview, then

The expectation value then is

\[
E(X_2) = 45000 \cdot 0.3 \cdot 0.5 + 30000 \cdot 0.7 \cdot 0.5 + 30000 \cdot 0 + 15000 \cdot 1 \cdot 0.5
\]

\[
= 6750 + 10500 + 0 + 7500 = 24250
\]

So, even in this case it is better to take the risk.