RESEARCH STATEMENT

ANDREA APPEL

1. Overview

1.1. My research lies in the area of representation theory. It touches upon several areas, such as the representation theory of Kac–Moody algebras and quantum groups, Artin–Tits braid groups, quantum integrable systems, hyperplane arrangements, knot invariants, and categorification.

It revolves around a principle which first appeared in the work of T. Kohno and V. Drinfeld and describes the quantum groups as analytic objects, natural receptacles for the monodromy representations of certain flat connections arising in representation theory [30, 8].

More specifically, it states that the universal $R$–matrix of the quantum group $U_\hbar \mathfrak{g}$, where $\mathfrak{g}$ is a semisimple Lie algebra, describes the monodromy of the Knizhnik-Zamolodchikov connection (a flat connection over the configuration of $n$–points in $\mathbb{C}$ with logarithmic singularities on the coordinates hyperplanes $\{z_i = z_j\}$) and links two distinct appearances of the braid group of $n$ strands in the representation theory of $\mathfrak{g}$. This monodromic description was generalised to arbitrary symmetrisable Kac–Moody by P. Etingof and D. Kazhdan in [15, 16, 17].

The link between quantum groups and differential equations was subsequently extended by V. Toledano Laredo in [38, 40, 41] providing the interpretation of the quantum Weyl groups in terms of the monodromy of the Casimir connection, a flat connection over the Cartan subalgebra with logarithmic singularities on the root hyperplanes. This result links two appearances of the generalised braid group (or Artin–Tits braid group) of type $\mathfrak{g}$.

The results of Drinfeld and Toledano Laredo are obtained by the mean of cohomological methods which are unift for the infinite–dimensional setting, due to the lack of cohomological rigidity of arbitrary Kac–Moody algebras.

In a recent series of papers, in collaboration with V. Toledano Laredo, we generalised the monodromic description of the quantum Weyl group to arbitrary symmetrisable Kac–Moody algebras [1, 2, 3], with an interesting blend of representation theory, category theory, and geometry, enhancing the results obtained by Toledano Laredo, just as Etingof and Kazhdan enhanced Drinfeld’s work.

Our program can be summarized in the following steps:

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• Define a suitable categorical structure, called Coxeter structure, encoding the joint action of the generalised and the standard braid groups [1].
• Encode the monodromy of the KZ and the Casimir connection (resp. the $R$–matrix and the quantum Weyl operators) into a Coxeter structure on $U\mathfrak{g}$ (resp. $U_h\mathfrak{g}$) [3].
• Construct an equivalence of Coxeter categories [1, 2].

1.2. In [1], we defined the notion of quasi–Coxeter category, which is informally a tensor category carrying commuting actions of Artin’s braid groups and a given generalised braid group on the tensor product of its objects. We showed that such a structure arises from the quantum group $U_h\mathfrak{g}$ of a symmetrisable Kac–Moody algebra $\mathfrak{g}$, specifically on the category $\mathcal{O}_h$ of integrable, highest weight representations of $U_h\mathfrak{g}$. The corresponding Artin group actions are given by the universal $R$–matrices of the Levi subalgebras of $U_h\mathfrak{g}$, and the action of the generalised braid group of $\mathfrak{g}$ by the quantum Weyl group operators of $U_h\mathfrak{g}$. The main result of [1] is that this structure can be transferred to the category $\mathcal{O}_h$ of integrable, highest weight modules for $\mathfrak{g}$. The proof of this fact relies on the construction of a relative version of the Etingof–Kazhdan quantisation functor which takes as input an inclusion of Lie bialgebras $\mathfrak{a} \subset \mathfrak{b}$, and allows in particular to construct an equivalence $\mathcal{O}_h \cong \mathcal{O}_h$ which is compatible with a given chain of Levi subalgebras of $\mathfrak{g}$.

1.3. In [2], we proved that $\mathcal{O}_h$ possesses, up to unique equivalence, a unique quasi–Coxeter structure with prescribed $R$–matrices and local monodromies. The uniqueness of quasi–Coxeter structures on $\mathcal{O}_h$ is obtained from a cohomological rigidity result, as was done in the case of a semisimple Lie algebra. The proof of this result, however, differs significantly from that given in [39, 40]. Indeed, the latter relies on the well–known computation of the Hochschild (coalgebra) cohomology of the enveloping algebra $U\mathfrak{g}$ in terms of the exterior algebra $\wedge \mathfrak{g}$. For an arbitrary Kac–Moody algebra $\mathfrak{g}$, the tensor powers of $U\mathfrak{g}$ need to be replaced by their completion $\hat{U}\mathfrak{g}$ with respect to category $\mathcal{O}$, since $U\mathfrak{g}$ and $U\mathfrak{g}^{\otimes 2}$ do not contain the Casimir operator $C$ of $\mathfrak{g}$ and the invariant tensor $\Omega = \Delta(C) - C \otimes 1 - 1 \otimes C$ respectively, and are therefore not an appropriate receptacle for the coefficients of the Casimir and KZ connections. While the computation of the Hochschild cohomology of $U\mathfrak{g}$ holds for an arbitrary Lie algebra, it is not known to do so, and may in fact fail, for the topological Hopf algebra $\hat{U}\mathfrak{g}$, which seems to have a rather unwieldy cohomology.

1.4. Instead of using $\hat{U}\mathfrak{g}$, we rely on a universal analogue $\mathfrak{U}$, constructed in the framework of PROP (product–permutation categories), which contains all the information necessary to extend, simplify, and strengthen the cohomological rigidity, leading to an essential uniqueness of the Coxeter structure. This raises the hope that the equivalences we construct may be convergent as series in the deformation parameter $\hbar$, and could in particular be specialised to non–rational values of $\hbar$.

1.5. Finally, in [3], we prove that the monodromy of the rational Casimir connection of a symmetrizable Kac–Moody algebra $\mathfrak{g}$ (and in particular affine) is described by the quantum Weyl group operators of $U_h\mathfrak{g}$, by showing that the monodromy of the rational KZ and Casimir connections of $\mathfrak{g}$ arise from a quasi–Coxeter structure on $\mathcal{O}_h$. 
1.6. Many interesting questions remain open about the Casimir connection, and in the last section I will illustrate few of them which naturally stem from the outlined program. In particular, I will discuss the extension of the monodromy theorem to the numerical setting and specifically at roots of unit. Such result would also be relevant in the framework of Conformal Field Theory. The Casimir connection is, in fact, strongly related to a certain class of quantum integrable systems known as irregular Gaudin models, whose spectrum is described by differential operators (Opers) on $\mathbb{P}^1$ with irregular singularities [18, 20]. These have also appeared in the work of Gaiotto and Witten on the gauge–theoretic description of Khovanov homology [24]. Finally, I will briefly describe two ongoing projects in the framework of categorified quantum group, started when I joined USC in 2014.

2. Monodromy theorems in the infinite–dimensional setting

2.1. Motivation: the KZ connection. Around 1990, Kohno [30] and Drinfeld [8] proved a rather astonishing result, now known as the Kohno-Drinfeld Theorem. The theorem states that quantum groups can be used to describe the monodromy of certain first order Fuchsian PDE’s known as the Knizhnik-Zamolodchikov (KZ) equations. Given a simple Lie algebra $\mathfrak{g}$, a representation $V$ of $\mathfrak{g}$ and a positive integer $n$, the KZ equations are the following system of PDEs

$$\frac{\partial \Phi}{\partial z_i} = \hbar \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Phi$$

where

(i) $\Phi$ is a function on the configuration space of $n$ ordered points in $\mathbb{C}$

$$X_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}$$

with values in $V^\otimes n$,

(ii) $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir tensor,

(iii) $\hbar$ is a complex deformation parameter.

This system is integrable and equivariant under the natural $\mathfrak{S}_n$–action and hence defines a one-parameter family of monodromy representations of Artin’s braid group $B_n = \pi_1(X_n/\mathfrak{S}_n)$. The Kohno–Drinfeld theorem asserts that this representation is equivalent to the $R$-matrix representation of $B_n$ on $V^\otimes n$ arising from the quantum group $U_{\hbar}\mathfrak{g}$. Here $V$ is a quantum deformation of $V$, that is a $U_{\hbar}\mathfrak{g}$-module such that $V/\hbar V \simeq V$.

The Kohno–Drinfeld theorem can be stated in the following category theoretical way: there exists an equivalence of braided tensor categories

$$\text{Rep}_{KZ} U_{\hbar}\mathfrak{g}[[\hbar]] \stackrel{\sim}{\longrightarrow} \text{Rep}_{R_{\hbar}} U_{\hbar}\mathfrak{g}$$

where the braided tensor structures are induced by the KZ–equations on $\text{Rep} U_{\hbar}\mathfrak{g}[[\hbar]]$ and by the universal $R$–matrix on $\text{Rep} U_{\hbar}\mathfrak{g}$.

This equivalence has rather striking implications. It implies that the monodromy of the KZ–equations is defined over $\mathbb{Q}$ (as the $R$–matrix is). In knot theory, it establishes a connection between the Reshetikhin–Turaev link invariants and the universal Vassiliev invariants. Its specialization at roots of unity, constructed by Kazhdan and Lusztig in [26, 27], provides an explanation for the coincidence between the
fusion rules arising in Conformal Field Theory and those arising from quantum groups at roots of unity.

2.2. The Casimir connection. In subsequent work, J. Millson and V. Toledano Laredo [33, 38], and independently C. De Concini (unpublished) and Felder–Markov–Tarasov–Varchenko [22], constructed another flat connection $\nabla_C$, now known as the Casimir connection, which is described as follows. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $\mathfrak{h}_{\text{reg}}$ be the complement of the root hyperplanes in $\mathfrak{h}$, i.e.,

$$
\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi} \ker(\alpha)
$$

where $\Phi = \{\alpha\} \subset \mathfrak{h}^*$ is the set of roots of $\mathfrak{g}$. For a finite-dimensional $\mathfrak{g}$-module $V$, $\nabla_C$ is the following connection on the trivial vector bundle $\mathfrak{h}_{\text{reg}} \times V \to \mathfrak{h}_{\text{reg}}$:

$$
\nabla_C = d - \hbar \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} C_\alpha
$$

where

(i) the summation is over a set of chosen system of positive roots $\Phi^+ \subset \Phi$;

(ii) $C_\alpha$ is the Casimir operator of the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ corresponding to the root $\alpha$.

This connection is equivariant under the action of the Weyl group $W$ of $\mathfrak{g}$, and gives rise to a one-parameter family of monodromy representations on $V$ of the generalised braid group

$$
B_\mathfrak{g} = \pi_1(\mathfrak{h}_{\text{reg}}/W)
$$

If $\mathcal{V}$ is a quantum deformation of $V$, the quantum Weyl group operators $S_{\hbar,i} \in U_\hbar \mathfrak{g}$ defined by G. Lusztig, A.N. Kirillov–N. Reshetikhin and Y. Soibelman, [32, 29, 36] define a representation of $B_\mathfrak{g}$ on $\mathcal{V}$. In this setting a Kohno–Drinfeld type theorem was obtained by V. Toledano Laredo, stating the equivalence of the above two representations of $B_\mathfrak{g}$ [37, 38, 39, 40, 41].

2.3. Extension of rational Casimir connection to Kac–Moody algebras.

It is natural to ask if it is possible to generalize the previous construction to the affine setting and possibly to arbitrary Kac–Moody algebras.

If $\mathfrak{g}$ is a symmetrisable Kac-Moody algebra (in particular, affine), and $U_\hbar \mathfrak{g}$ is its quantized universal enveloping algebra, the corresponding quantum Weyl group operators give an action of the braid group of type $\mathfrak{g}$ on any integrable $U_\hbar \mathfrak{g}$-module.

It is natural to expect the existence of a monodromic description of the quantum Weyl group action of the generalised braid group $B_\mathfrak{g}$ on integrable $\mathfrak{g}$-modules in category $\mathcal{O}$. The main difficulty in carrying out this problem is the fact that the Kac–Moody algebra $\mathfrak{g}$ is not cohomologically rigid, in general. Moreover, since $U\mathfrak{g}[[\hbar]]$ and $U_\hbar \mathfrak{g}$ are not isomorphic as algebras (only their completions with respect to category $\mathcal{O}$ are), we need to rephrase the problem in a category theoretical way.
2.4. **Categorical reformulation.** This proceeds in close analogy with the Kohno–Drinfeld theorem: in the Kac–Moody case, the theorem was proved by Etingof–Kazhdan [17] and presented as an equivalence of braided tensor categories

\[ \text{EK} : \text{Rep}_{KZ} U \mathfrak{g}[[h]] \xrightarrow{\sim} \text{Rep}_{R, U \mathfrak{g}} \]

where \( \text{Rep}_{KZ} U \mathfrak{g}[[h]] \) denotes the category \( \mathcal{O} \) for \( U \mathfrak{g}[[h]] \), with braided tensor structure induced by the KZ–equations, and \( \text{Rep}_{R, U \mathfrak{g}} \) denotes the category \( \mathcal{O} \) for \( U_h \mathfrak{g} \) with braided tensor structure induced by the universal \( R \)–matrix.

The approach towards addressing the problem can be divided in the following steps.

- Define a suitable categorical structure on \( \text{Rep} U \mathfrak{g}[[h]] \) and \( \text{Rep} U_h \mathfrak{g} \) which encodes the joint actions of \( B_n \) and \( B_\mathfrak{g} \) (Coxeter categories)
- Construct an equivalence of Coxeter categories

\[ \text{Rep}_{\nabla \mathcal{KZ}, \nabla C U \mathfrak{g}} [[h]] \xrightarrow{\sim} \text{Rep}_{R, S, U \mathfrak{g}} \]

where \( \text{Rep}_{\nabla \mathcal{KZ}, \nabla C U \mathfrak{g}} [[h]] \) denotes the category \( \mathcal{O} \) for \( U \mathfrak{g}[[h]] \) with Coxeter structure induced by the KZ–equations and the Casimir connection \( \nabla C \), while \( \text{Rep}_{R, S, U \mathfrak{g}} \) denotes the category \( \mathcal{O} \) for \( U_h \mathfrak{g} \) with Coxeter structure induced by the universal \( R \)–matrix and the quantum Weyl group operator.

3. **COXETER CATEGORIES**

3.1. We give a precise description of Coxeter categories as a generalised braid group analogue of a braided tensor category. Namely, a Coxeter category is to the generalised braid group what braided tensor categories are to braid group on \( n \)–strands. Just as braided tensor categories are related to the Deligne–Mumford compactification of the configuration space of \( n \) points in \( \mathbb{C} \), Coxeter categories are related to the compactification of hyperplane arrangements constructed by De Concini–Procesi in [6, 7].

3.2. Let us fix a labeled diagram \( D \). A **Coxeter category of type D** is a tuple

\[ \mathcal{C} = (C_B, F_{BB'}, \Upsilon_{FG}, S_i) \]

where

- \( C_B = (C_B, \otimes_B, \Phi_B, \beta_B) \) are braided tensor categories indexed by subdiagrams \( B \subset D \);  
- \( F_{BB'} = (F_{BB'}, J_{BB'}) : C_B \rightarrow C_{B'} \) are tensor restriction functors, corresponding to the inclusion \( B \subset B' \), with a tensor structure depending upon the choice of a maximal chain of subdiagrams \( \mathcal{F} : B = B_0 \subset B_1 \subset \cdots \subset B_n = B' \);  
- A collection of invertible endomorphisms \( S_i \in \text{End} (F_i) \) indexed by the vertices \( i \in D \) (local monodromies) and natural transformations (De Concini–Procesi associators)

\[ \Upsilon_{FG} \in \text{Nat} (F_{BB'}, F_{BB'})^\times \]

for any pair of maximal chains \( \mathcal{F}, \mathcal{G} \) from \( B \) to \( B' \), satisfying

**Orientation:** \( \Upsilon_{FG} = \Upsilon_{G,F}^{-1} \)
Transitivity: \( \Upsilon_{FH} = \Upsilon_{F} \cdot \Upsilon_{G} \)

**Vertical Decomposition:** For any two maximal chains \( F, G \) from \( B' \) to \( B \) and \( F', G' \) from \( B'' \) to \( B' \)

\[
\Upsilon_{FG} \circ \Upsilon_{F'} G' = \Upsilon_{F'} \sqcup \Upsilon_{FG'} \sqcup G
\]

A morphism of Coxeter structure is then naturally defined as the datum of

- for any \( B \subseteq D \) a functor \( H_B : \mathcal{C}_B \to \mathcal{C}'_B \);
- for any \( B \subseteq B' \) and maximal chain \( F \), a natural transformation

\[
\begin{array}{c}
\mathcal{C}_B \xrightarrow{H_B} \mathcal{C}_B' \\
\mathcal{C}_B' \xrightarrow{F_{BB'}} \mathcal{C}_B \\
\end{array}
\]

satisfying the obvious compatibility condition with respect to the Coxeter structure.

Interestingly perhaps, these notions can be concisely rephrased in terms of a 2–functor from a combinatorially defined 2–category \( qC(D) \) to the 2–categories \( \text{Cat}^\otimes \) of tensor categories. The objects of \( qC(D) \) are the subdiagrams of the Dynkin diagram \( D \) of \( B \) and, for two subdiagrams \( D' \subseteq D'' \), \( \text{Hom}_{qC(D)}(D'', D') \) is the fundamental 1–groupoid of the De Concini–Procesi associahedron for the quotient diagram \( D''/D' \).

4. **Differential Coxeter structures**

The universal \( R \)–matrix and the quantum Wey group operators defines a natural Coxeter structure on \( \text{Rep} U \mathfrak{g} \).

The construction of a differential Coxeter structure on \( \text{Rep} U \mathfrak{g}[[\hbar]] \) encoding the monodromies of the KZ and Casimir connection is more involved and it will be published in [3]. In this section, we present an anticipation of the results obtained.

4.1. **The Casimir connection.** A rational Casimir connection was defined for any symmetrisable Kac–Moody algebra in [22]. It has the form

\[
\nabla_C = d - \hbar \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \kappa_\alpha : \\
\]

where \( \alpha \) ranges over all positive (real and imaginary) roots, \( \kappa_\alpha \) is the truncated Casimir operator corresponding to \( \alpha \), and

\[
\kappa_\alpha := \sum_{i=1}^{\text{mult}(\alpha)} 2e^{-\alpha} e^i(\alpha)
\]

is its Wick ordered form. This ordering makes the sum over \( \alpha \) locally finite on a category \( \mathcal{O} \)–module, but breaks the equivariance of \( \nabla_C \) with respect to the Weyl group of \( \mathfrak{g} \). If \( \mathfrak{g} \) is affine, this equivariance can however be restored by adding to \( \nabla_C \) a closed one–form \( A \) with values in the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) [3]. The 1–form \( A \) morally represents the infinite sum of the elements \( h_\alpha \in \mathfrak{h} \) and can be expressed
in terms of dilogarithmic functions.

The monodromy of the connection $\nabla_C + A$ then gives rise to a one-parameter family of representations of the generalised affine braid group on $V$.

The De Concini–Procesi associators are weight zero, therefore the 1–form does not affect them.

**Theorem.** The combinatorics of the De Concini–Procesi associators \([6, 7]\) holds for root systems of arbitrary type. In particular, the De Concini–Procesi associators of the Casimir connection $\nabla_C$ satisfy the properties of orientation, transitivity, and vertical decomposition, and they induce an action of the generalised braid group $B_g$ on the object of $O^{int}$.

4.2. **The Fusion operator.** In the semisimple case, the Coxeter structure on $Ug[[h]]$ underlying the monodromy of the KZ and Casimir connections is obtained as follows [41].

For any $B \subseteq D_g$, the quasitriangular quasibialgebra structure on $Ug_B[[h]]$ is given by the associator of the KZ equations for $g_B$ and the corresponding $R$–matrix $\exp(i\pi h\Omega_B)$. The gauge transformations $\Upsilon_{g,F}$ and the elements $S_i$ are given by the De Concini–Procesi associators of the Casimir connection, and its local monodromies around the fundamental root hyperplanes \{$\alpha_i = 0$\}.

Finally, the twists $J_F$ are obtained as suitable limits of a joint solution $F$ of the system (cf. [22])

$$d_\mu F = \frac{h}{2} \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \left[ \Delta(\kappa_\alpha)F - F(\kappa_\alpha^{(1)} + \kappa_\alpha^{(2)}) \right] + (z_1 d\mu^{(1)} + z_2 d\mu^{(2)})F$$

$$d_z F = h \frac{dz_1 - z_2}{z_1 - z_2} \Omega_{12}F + (\mu^{(1)}dz_1 + \mu^{(2)}dz_2)F$$

where $(z, \mu) \in \mathbb{C}^2 \times h$ and $F$ has values in $Ug[[h]]^\otimes 2$.

Since $F$ is of weight zero, the above equations can be written with the Wick ordered $\kappa_\alpha$: instead of $C_\alpha$, and therefore make sense for an arbitrary Kac–Moody $g$. The construction of the joint solution carries over to arbitrary symmetrisable Kac–Moody algebras [3]. We then have

**Theorem.** The KZ–connection and the Casimir connection induce a Coxeter structure on the integrable modules in category $O$ of an arbitrary Kac–Moody algebra.

4.3. The link between the quantum Weyl group and the Casimir connection is obtained as follows.

- We first build an equivalence of Coxeter categories using the Etingof–Kazhdan functor [1]. This endow the representation theory of $Ug[[h]]$ with an additional Coxeter structure arising from the quantum group, which is universal in the sense of [16, 13], but not necessarily related with the Casimir connection.

- We then prove a rigidity theorem proving that, up to gauge transformation, the category $O$ of $Ug[[h]]$ has a unique universal Coxeter structure.
5. The Coxeter structure arising from the quantum group

For a semisimple Lie algebra \( g \), the transfer of structure from \( U_\hbar g \) to \( U_\hbar g[[h]] \) ultimately rests on the vanishing of the first and second Hochschild cohomology groups of \( U_\hbar g[[h]] \), and in particular on the fact that \( U_\hbar g \) and \( U_\hbar g[[h]] \) are isomorphic as algebras, a fact which does not hold for infinite–dimensional Kac–Moody algebras. Rather than the cohomological methods of [40], we use instead the Etingof–Kazhdan functor [15, 16, 17].

The construction of a Coxeter equivalence amounts to providing a commutative diagram of tensor functors

\[
\begin{array}{ccc}
\text{Rep}_{\text{KZ}} U_\hbar g[[h]] & \xrightarrow{\text{EK}} & \text{Rep}(U_\hbar g) \\
\Gamma & & \\
\text{Rep}_{\text{KZ}} U_\hbar g_D[[h]] & \xrightarrow{\text{EK}_D} & \text{Rep}(U_\hbar g_D)
\end{array}
\]

and therefore requires a tensor restriction functor \( \Gamma \) on \( U_\hbar g[[h]] \) and an appropriate natural transformation compatible with the tensor structures.

5.1. The tensor restriction functor \( \Gamma \). The construction of the functor \( \Gamma \) is very much inspired by [15]. To outline our construction, which works more generally for an inclusion \((g_D, g_D, - , g_D, + ) \subset (g, g, g, g, g)\) of Manin triples over a field \( k \) of characteristic zero, recall first that the main steps of the Etingof–Kazhdan construction are as follows.

(i) One considers the Drinfeld category \( \text{Rep}_{\text{KZ}} U_\hbar g[[h]] \) of (deformation) equicontinuous \( g \)-modules, with associativity constraints given by the KZ associator \( \Phi_{\text{KZ}} \). This category can be thought of as a topological analogue of category \( \mathcal{O} \) when \( g \) is the Manin triple associated to a Kac–Moody algebra. It can equivalently be described as the category of Drinfeld–Yetter modules over a Lie bialgebra.

(ii) One constructs a tensor functor \( F \) from \( \text{Rep}_{\text{KZ}} U_\hbar g[[h]] \) to the category \( \text{Vect}_{[h]} \) of topologically free \( k[[h]] \)-modules. The algebra of endomorphisms \( H = \text{End}(F) \) is then a topological bialgebra, i.e., it is endowed with a coproduct \( \Delta \) mapping \( H \) to a completion of \( H \otimes H \).

(iii) Inside \( H \), one constructs a subalgebra \( U_h g_- \) such that \( \Delta(U_h g_-) \subset U_h g_- \otimes U_h g_- \), and which is a quantisation of \( U g_- \). The quantum group \( U_h g \) is then defined as the quantum double of \( U_h g_- \).

(iv) By construction, \( U_h g_- \) acts and coacts on any \( F(V), V \in \text{Rep}_{\text{KZ}} U_\hbar g[[h]] \), so that the functor \( F \) lifts to \( \text{EK} : \text{Rep}_{\text{KZ}} U_\hbar g[[h]] \to \text{Rep}(U_\hbar g) \) where, by definition, the latter is the category of Drinfeld–Yetter modules over \( U_h g_- \).

(v) Finally, one proves that \( \text{EK} \) is an equivalence of categories.

\(^1\)The construction however works for any fixed Lie associator \( \Phi \) over \( k \).
The principle adopted by Etingof and Kazhdan is the following. In a k-linear monoidal category $\mathcal{C}$, a coalgebra structure on an object $C \in \text{Obj}(\mathcal{C})$ induces a tensor structure on the Yoneda functor

$$h_C = \text{Hom}_\mathcal{C}(C, -) : \mathcal{C} \to \text{Vect}_k$$

If $\mathcal{C}$ is braided and $C_1, C_2$ are coalgebra objects in $\mathcal{C}$, then so is $C_1 \otimes C_2$, and there is therefore a canonical tensor structure on $h_{C_1 \otimes C_2}$.

If $g$ is finite-dimensional, the polarization $Ug \simeq M_- \otimes M_+$, where $M_{\pm}$ are the Verma modules $\text{Ind}_{g_{\pm}}^g k$ realizes $Ug$ as the tensor product of two coalgebra objects in $\text{Rep}_{\text{KZ}} U[[\hbar]]$. This yields a tensor structure on the forgetful functor

$$h_{Ug} : \text{Rep}_{\text{KZ}} U[[\hbar]] \to \text{Vect}_k[[\hbar]]$$

Our starting point is to apply the same principle to the (abelian) restriction functor $i_B^* : \text{Rep}_{\text{KZ}} U[[\hbar]] \to \text{Rep}_{\text{KZ}} U_B[[\hbar]]$. We therefore factorize $Ug$ as a tensor product of two coalgebra objects in the braided monoidal category of $(g, g_D)$-bimodules. Just as the modules $M_-, M_+$ are related to the decomposition $g = g_- \oplus g_+, L_- \text{ and } N_+ \text{ are related to the asymmetric decomposition}$

$$g = m_- \oplus p_+$$

where $m_- = g_- \cap g_D^\perp$ and $p_+ = g_D \oplus m_+$. This factorization induces a tensor structure on the functor $\Gamma = h_{L_- \otimes N_+}$, canonically isomorphic to $i_B^*$ through the right $g_D$-action on $N_+$. As in [15, Part II], this tensor structure can also be defined in the infinite-dimensional case. We obtain the following [1]

**Theorem.** For any inclusion of Manin triple $i_B : g_D \to g$, there is a tensor functor

$$\Gamma : \text{Rep}_{\text{KZ}} U[[\hbar]] \to \text{Rep}_{\text{KZ}} U_D[[\hbar]]$$

canonically isomorphic to the restriction functor $i_B^*$.

The functor $E_K$ is, in fact, universal, i.e. it can be realized in the PROP category of a Lie bialgebras [16]. We obtain an analogous results for $\Gamma$ [1].

**Theorem.** The functor $\Gamma$ is universal, i.e. it can be realized in a multicolored PROP category naturally attached to a split pair of Lie bialgebras, which is a pair of Lie bialgebras $(g, g_B)$ with a Lie bialgebra idempotent representing the projection on $g_B$.

**5.2. The natural transformation.** To construct a natural transformation making the following diagram commute

$$\begin{array}{ccc}
\text{Rep}_{\text{KZ}} U[[\hbar]] & \xrightarrow{E_K} & \text{Rep}(U_h g) \\
\downarrow \Gamma & & \downarrow \\
\text{Rep}_{\text{KZ}} U_D[[\hbar]] & \xrightarrow{E_K_B} & \text{Rep}(U_h g_B)
\end{array}$$

we remark, as suggested to us by P. Etingof, that a quantum analogue $\Gamma_h$ of $\Gamma$ can be similarly defined using a quantum version $L_-^\hbar, N_-^\hbar$ of the modules $L_-, N_+$. The functor $\Gamma_h = \text{Hom}_{U_h}(L_-^\hbar \otimes N_+^\hbar, -)$ is naturally isomorphic to the restriction to $U_h g_B$ as tensor functor. Moreover, an identification

$$E_K_B \circ \Gamma \simeq \Gamma_h \circ E_K$$
is readily obtained, provided one establishes isomorphisms of \((U_\hbar g, U_\hbar g_B)\)–bimodules
\[ EK_B \circ EK(L_-) \simeq L^h_- \quad \text{and} \quad EK_B \circ EK(N_+) \simeq N^h_+ \]

The previous construction leads to the equivalence of Coxeter categories \([1]\).

**Theorem.** Let \(\mathfrak{g}\) be a symmetrisable Kac–Moody algebra and \(U_\hbar \mathfrak{g}\) the corresponding Drinfeld–Jimbo quantum group. Then the Coxeter structure of \(U_\hbar \mathfrak{g}\) is transferred by the Etingof–Kazhdan functor on the braided tensor category \(\text{Rep}_{KZ} U_\hbar [\hbar]\). In particular, the induced Coxeter structure is universal.

6. **Uniqueness of Coxeter structures on \(U_\mathfrak{g}\)**

The Coxeter category naturally associated to \(U_\hbar \mathfrak{g}\) admits an equivalent universal counterpart on \(U_\mathfrak{g}[[\hbar]]\), as illustrated in the previous section. The next step is to prove the uniqueness of such structure on \(\text{Rep}_{KZ} U_\mathfrak{g}[[\hbar]]\) \([2]\).

6.1. **Universal algebras.** The Coxeter structure on \(\text{Rep}_{KZ} U_\mathfrak{g}[[\hbar]]\) relies on three main ingredients: the braided tensor structure, given by the monodromy of the KZ connection; the tensor structure on the restriction functors, which are determined by the universal twists \(J_\Gamma\) constructed in \([1]\); and the gauge transformations \(\Upsilon_{FG}\) relating the twists and determined by the natural transformations constructed in the previous section.

The proof of the uniqueness theorem amounts to matching the universal twists and the gauge transformation. In \([2]\), inspired by the work of Enriquez in \([13]\), we construct a *universal algebra* \(\mathcal{U}\), realized as the space of morphisms of a PROP category modeling the decomposition in root spaces
\[ b = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbb{R}^+} b_{\alpha} \quad \mathfrak{g} \oplus \mathfrak{h} = b \oplus b_- \]

The algebra \(\mathcal{U}\) may also be thought of as the smallest subalgebra in \(U_\mathfrak{g}\) generated by the normally ordered Casimir elements \(\kappa_\alpha\). The \(n\)th (topological) tensor product \(\mathcal{U}^n\) naturally contains the holonomy algebra representing the joint system of the dynamical KZ equations and the \(n\)th dynamical Casimir equations. We then get the following \([2]\).

**Theorem.** The universal twists \(J_\Gamma\) and the differential twists \(J_\mathcal{F}\) produced by the fusion operator \(\mathcal{F}\) corresponds to elements in \(\mathcal{U}^2\).

6.2. **The rigidity theorem.** We have identified the differential Coxeter structure and the quantum Coxeter structure with elements in \(\mathcal{U}^n\). The tower of algebras \(\{\mathcal{U}^n\}\) is naturally endowed with a cosimplicial structure, and its cohomology is isomorphic to a universal version of the external algebra. It is important to observe that, in \(\mathcal{U}\) there is no primitive element. This two features allow us to use a cohomological argument.

**Theorem.** Any two twists in \(\mathcal{U}^2\) are equivalent by the action of a unique gauge transformation.

It follows that the gauge transformation matching two twists, automatically matches also the De Concini–Procesi associators \(\Upsilon_{\mathcal{FG}}\). Finally, we have
Theorem. Let \( \mathfrak{g} \) be an arbitrary Kac–Moody algebra. There exists an equivalence of Coxeter categories

\[
\text{Rep}_{\nabla, \nabla_c} \mathcal{K}_Z U[\hbar]) \sim \text{Rep}_{R_S, S_S} U[\hbar] \mathfrak{g}.
\]

In particular, the quantum Weyl group operators describe the monodromy of the Casimir connection.

The use of the algebra \( \mathfrak{H} \) leads to far stronger uniqueness results than had been obtained in [39, 40] for a semisimple Lie algebra \( \mathfrak{g} \). Namely, the isomorphism of two Coxeter structures is unique up to a unique gauge, and this raises the hope that the equivalences we construct may be convergent as series in the deformation parameter \( \hbar \), and could in particular be specialised to non–rational values of \( \hbar \).

7. Future Directions and Proposed Research

7.1. Quantum groups at root of unity. The methods to construct the equivalence of Coxeter categories replace entirely the cohomological constructions of [40] and suggest a strategy in the solution of the following problem.

We wish to extend the monodromy theorem for the rational Casimir connection of a semisimple Lie algebra \( \mathfrak{g} \) to numerical values of \( \hbar \), in particular to rational values of the form

\[
\hbar = \frac{1}{l + \hbar^\vee}
\]

where \( l \in \mathbb{N} \) and \( \hbar^\vee \) is the dual Coxeter number of \( \mathfrak{g} \).

These values of \( \hbar \) are of particular interest since they are related to the quantum group \( U_q \mathfrak{g} \) at the root of unity

\[
q = \exp \left( \frac{\pi i}{m(l + \hbar^\vee)} \right)
\]

where \( m \in \{1, 2, 3\} \) is the ratio of the squared lengths of the longest and shortest roots of \( \mathfrak{g} \).

A possible strategy is directly inspired by the extension of the Kohno-Drinfeld theorem to numerical values of \( \hbar \) by Kazhdan and Lusztig [27]. Their proof goes along the following lines:

(i) Regard the monodromy of the Knizhnik–Zamolodchikov equations as defining the commutativity and associativity constraints of a braided tensor category, specifically the category \( \mathcal{O}_l \) of highest weight representations of the affine Lie algebra \( \hat{\mathfrak{g}} \) at level \( l \) which are integrable as \( \mathfrak{g} \)-modules, where \( l \in \mathbb{C} \setminus \{-\hbar^\vee + Q_{\geq 0}\} \).

(ii) Regard the \( R \)-matrix representations of the braid groups \( B_n \) as defining the commutativity constraints on the (strict) tensor category \( \mathcal{E}_l \) of finite-dimensional representations of the quantum group \( U_q \mathfrak{g} \), where \( q \) is as above.

(iii) Give a constructive (i.e., non cohomological) proof of the equivalence of \( \mathcal{O}_l \) and \( \mathcal{E}_l \) for \( l \in \mathbb{Q} \), when both categories are semisimple.

(iv) Show that the equivalence between \( \mathcal{O}_l \) and \( \mathcal{E}_l \) behaves well as \( l \) tends to \( l_0 \in -\hbar^\vee + Q_{< 0} \).
M. Finkelberg [23] showed that the equivalence of $\mathcal{O}_l$ and $\mathcal{E}_l$ extends further to the case when $l \in \mathbb{N}$ provided

(i) $\mathcal{O}_l$ is replaced by the full subcategory $\mathcal{O}_l^{\text{int}}$ of integrable $\hat{g}$–modules.

(ii) $\mathcal{E}_l$ is replaced by the subcategory of tilting modules quotiented out by those of $q$–dimension equal to 0.

7.2. **Factorization of the KL functor.** The above strategy could be adapted to our case in the following way. It seems natural to approach the Kazhdan–Lusztig functor in the same way we did for the Etingof-Kazhdan functor in Section 5. In analogy with the program outlined in [1], the diagrammatic Kazhdan–Lusztig categories of the affine Kac–Moody algebra should be naturally entangled within a Coxeter structure. This additional structure should also be interpreted and described in terms of the Weiss–Zumino–Witten model and conformal blocks. Similarly to what we obtained with the Etingof–Kazhdan functor $\mathcal{E}_K$, it should be possible to look at the collection of Kazhdan–Lusztig functors $\mathcal{KL}$ as an equivalence of Coxeter categories. The construction of such structure on the family of Kazhdan–Lusztig functors may shed more light on the functor itself and clarify the possible interpretation of the latter as an *intertwiner* between the classical and quantum version of the geometric Langlands correspondence, as suggested by D. Gaitsgory in [25].

7.3. **Fusion operator and Tannakian reconstruction.** A different solution to the previous problem could possibly arise from the following construction, relying on the fusion operator described in Section 4. The joint solution of the dynamical KZ and Casimir connections yields a tensor structure on the forgetful functor from $\text{Rep}_{KZ} Ug[[\hbar]]$ to $\text{Vect}$, which differs from the one constructed by Kazhdan–Lusztig [27] in the semisimple case and by Etingof–Kazhdan [15] in the affine case. More precisely, the fusion operator may be regarded as a tensor structure on an appropriate fiber functor, from category $\mathcal{O}$ to the category of local system over $h_{\text{reg}}$. This would possibly lead to a topological realization of the quantum group (and its Coxeter structure) in the spirit of [21]. This would lead to a constructive proof of the equivalence of Coxeter categories, which would likely apply to the numerical case and relate $\mathcal{O}_l$ and $\mathcal{E}_l$ as Coxeter categories.

7.4. **Gaudin Hamiltonians.** For a simple Lie algebra, the Casimir connection is related with the irregular Gaudin model, a quantum integrable system studied in [20] and [35]. The corresponding algebra of quantum Hamiltonians is obtained as a quotient of the center of the completed enveloping algebra of the affine algebra at the critical level, and its spectrum is consequently described in terms of a special class of opers in the spirit of the Feigin–Frenkel theorem [18]. In [19], Feigin–Frenkel studied the Gaudin system associated to an affine Kac–Moody and its relation with a hypothetic center of the completed enveloping algebra of the double affine algebra. It will be interesting to study the role played by the affine Casimir connection in this setting.

7.5. **De Concini–Procesi associators and Grothendieck–Teichmuller group.** It is a natural to ask which properties may be satisfied by the De Concini–Procesi associators. The Drinfeld’s associator related to the KZ–connection, which corresponds to the De Concini–Procesi associator of type A, may be thought of as
the generating series of the multiple zeta function. The cyclotomic associator con-
structed by B. Enriquez in [14], which corresponds to the De Concini–Procesi as-
socia tor of type B, are similarly related with the cyclotomic multiple zeta function.
It will be interesting to first study the case G₂, and then eventually generalize it
to arbitrary (infinite–dimensional) type. Similarly, it would be of great interest a
generalised theory of the Grothendieck–Teichmuller groups controlling these asso-
ci at ors, which would unify the results in [10, 14].

7.6. Ongoing Projects in Categorified Quantum Groups.

7.6.1. Categorification of Yangians. I recently joined a project with A. Lauda et
al., whose aim is to categorify the Yangian of a simple Lie algebra Yℏg, a Ĉ[h]–
deformation of the current algebra g[t].

In [31, 28], M. Khovanov and A. Lauda, and independently R. Rouquier in
[34], defined an additive 2–category U, which categorifies the Beilinson–Lusztig–
MacPherson idempotent integral form U of the quantum group U = Uqg [32]. That U categorifies U means here that the split Grothendieck group K₀(U) is iso-
morphic to U.

In [5], the 2–category U (rather a slightly different version of it) is decategorified
by the mean of the 0th–Hochschild–Mitchell homology HH₀(U), better known as the trace Tr(U). Contrarily to the Grothendieck group case, this decategorification
does not yield the quantum group U. Instead Tr(U) is isomorphic to the idempo-
tented integral form Uₜ of the universal enveloping algebra of the current algebra
Ug[t].

It is then natural to ask if this method could be adapted in order to produce a
trace decategorification of the Yangian Yℏg, or rather its idempotent version Y. A first attempt has been made by construction of an appropriate deformation
2–category Uℏ. This is obtained by deformation of the nil–Hecke relations of the
2–morphisms in U. The category Tr(Uℏ) is a homogeneous deformation of Uₜ, and,
by Drinfeld’s uniqueness theorem, it is isomorphic to Y. Yet, an explicit description
of Y in terms of generators and relation seems challenging and is still undergoing
computations.

7.6.2. Loop presentation and categorified quantum groups. The difficulties in car-
rying out the computation of the generators of the Yangian in the category Tr(Uℏ)
implicitly suggested to approach the same problem with the 2–category U cor-
responding to an affine Lie algebra g. In other words, the 2–category U is defined
in terms of Kac–Moody type generators. An explicit loop presentation of U in the
affine case could be useful in the understanding of the categorified Yangian.

Preliminary computations, inspired by [4], show that the (forthcoming) categori-
fication of the Lusztig’s operators of Uqg by Abraham, Lamberto–Egan, and Lauda,
can be used to determine a loop presentation of U. This would also provide the
first categorification of a PBW basis of non–finite type.

References

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Department of Mathematics, University of Southern California, 3620 S Vermont Ave, KAP 104, Los Angeles, 90089, CA, USA
E-mail address: andrea.appel@usc.edu