Lecture 4: Continuity and Differentiability

• Continuity

• The derivative

Continuity

• Let \( N(x^0) \) denote neighborhood of \( x^0 \), meaning an open interval covering \( x^0 \):
  \[
  N(x^0) = (x^0 - \varepsilon, x^0 + \varepsilon)
  \]

• We have

*Def 1* A function \( f(x), f : X \rightarrow \mathbb{R} \), is continuous at a point \( x^0 \), in the domain of \( f \), iff for any \( N(f(x^0)) \) there is a neighborhood \( N(x^0) \) such that if (for every) \( x \in N(x^0) \), we have \( f(x) \in N(f(x^0)) \), where \( x \in X \).
Continuity: A Result

**Proposition 1.** If the \( \lim_{x \to x^0} f(x) = L \) exists and is finite and the ordered pair \((x^0, L) \in \text{Gr}(f)\), then 
\( f(x) \) is continuous at \( x^0 \).

**Proof:** If \( \lim_{x \to x^0} f(x) = L \) then for every \( N(L) \exists N(x^0) \) such that if \( x \in N(x^0) \) then \( f(x) \in N(L) \).

However, we have that \((x^0, L) \in \text{Gr}(f)\) which implies that \( L = f(x^0) \). Hence, all the conditions of Def 1 are satisfied. \( \| \)

**Proposition 2.** If \( f(x) \) is continuous at \( x^0 \), then \( \exists \lim_{x \to x^0} f(x) = L \) and \((x^0, L) \in \text{Gr}(f)\).

**Proof:** See Def 1.

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Continuity

- **Def 2** A function, \( f: X \to \mathbb{R} \), is *continuous* if it is continuous at every point \( x \) in the domain of \( f \).

- **Def 3** A function, \( f: X \to \mathbb{R} \), is *continuous on the interval* \((x', x'')\), \( x' < x'' \), if it is continuous at all points in this interval.
The Derivative

• Consider a function $f : X \rightarrow \mathbb{R}$, where $X$ is an open interval of $\mathbb{R}$

• Consider a change in the independent variable $x$ and the corresponding change in the image of $x$. Let the change in $x$ be given by $\Delta x$ and let the initial value of $x$ be $x^o$

The Derivative

• Form the difference quotient

$$\frac{\Delta y}{\Delta x} \equiv \frac{f(x^o + \Delta x) - f(x^o)}{\Delta x}$$

• If the following limit exists, it is said to be the \textit{derivative of $f$ at $x^o$}

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x^o + \Delta x) - f(x^o)}{\Delta x}$$
The Derivative

• Notation: \( f'(x^o), \ \text{df}(x^o)/dx \) or \( \text{dy/dx}\rvert_{x^o} \)

• \( f \) is said to be differentiable if it is has a derivative at each point in its domain.

Illustration
The relationship between differentiability and continuity

Proposition 1. The function \( y = f(x) \) is differentiable at a point \( x^0 \) in its domain only if \( f(x) \) is continuous at \( x^0 \).

Proof

To show necessity of continuity we need to show that differentiability implies continuity. If \( f(x) \) is differentiable at \( x^0 \),

\[
f'(x^0) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x^0)}{\Delta x}.
\]

Since, \( x \to x^0 \) implies \( x \neq x^0 \), then

\[
f(x) - f(x^0) = \frac{f(x) - f(x^0)}{(x - x^0)}(x - x^0).
\]

Take limits as \( x \to x^0 \) of both sides

\[
\lim_{x \to x^0} f(x) - f(x^0) = f'(x^0)(0) = 0
\]

Hence,

\[
\lim_{x \to x^0} f(x) = f(x^0)
\]

Since \( \exists \lim_{x \to x^0} f(x) = f(x^0) \) and since \( x^0 \) is contained in the domain of \( f \), \( f(x) \) is continuous at \( x^0 \).
The converse is not true

- Although differentiability implies continuity, the converse is not true. To show this, consider the function

\[
f(x) = \begin{cases} 
  ax & \text{if } x < \bar{x} \\
  a\bar{x} & \text{if } x \geq \bar{x}
\end{cases}
\]