MATH 525b ASSIGNMENT 7 SOLUTIONS
SPRING 2009
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(A) Let $\tilde{F}(x) = \lim_{y \searrow x} F(y) - \lim_{y \searrow 0} F(y)$, so $\tilde{F}$ if right-continuous with $\tilde{F}(0) = 0$, and let μ be the associated Borel measure on $\mathbb{R}$. For $\varphi \in \mathcal{D}(\mathbb{R})$, since $F = \tilde{F}$ a.e. we have

$$\begin{aligned}
-(\partial F)(\varphi) &= \int_{\mathbb{R}} F\varphi' \, dm \\
&= \int_{\mathbb{R}} \tilde{F}\varphi' \, dm + \left(\lim_{y \searrow 0} F(y)\right) \int_{\mathbb{R}} \varphi' \, dm \\
&= \int_{\mathbb{R}} \tilde{F}\varphi' \, dm \\
&= \int_{[0,\infty)} \left[\int_{(0,x]} \mu(dy)\right] \varphi'(x) \, dx - \int_{(-\infty,0)} \left[\int_{(x,0]} \mu(dy)\right] \varphi'(x) \, dx \\
&= \int_{(0,\infty)} \int_{(y,\infty)} \varphi'(x) \, dx \, \mu(dy) - \int_{(-\infty,0]} \int_{(-\infty,y)} \varphi'(x) \, dx \, \mu(dy) \\
&= -\int_{(0,\infty)} \varphi(y) \, \mu(dy) - \int_{(-\infty,0]} \varphi(y) \, \mu(dy) \\
&= -\int_{\mathbb{R}} \varphi(y) \, \mu(dy).
\end{aligned}$$

(B) If $T \equiv 0$, take $\varphi_0 = 0$. Otherwise, choose φ_0 with $T\varphi_0 = 1$. Then $T(\varphi - (T\varphi)\varphi_0) = T\varphi - (T\varphi)(T\varphi_0) = 0$ for all φ, so given φ if we let $\psi = \varphi - (T\varphi)\varphi_0$ then we have $\psi \in \mathcal{N}(T)$ and $\varphi = (T\varphi)\varphi_0 + \psi$.

(C) Suppose first that $f \in W^{1,\infty}(\mathbb{R}^n)$. This means that the distributional derivative $\partial_i f$ is a bounded (a.e.) function g_i, which implies that $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$. By Corollary 6 from lecture, for each $y \in \mathbb{R}^n$, for a.e. $x \in \mathbb{R}^n$, letting $M = \max_{i \le n} ||g_i||_\infty$ we have using the Schwartz Inequality,

$$\begin{aligned}
|f(x+y) - f(x)| &= \left| \int_0^1 \sum_{i=1}^n y_i g_i(x+ty) \, dt \right| \\
&\le \sum_{i=1}^n M|y_i| \int_0^1 \, dt \qquad (1) \\
&\le M\sqrt{n}|y|.
\end{aligned}$$

By Fubini, the set of (x, y) where this statement fails is a null set in $\mathbb{R}^2$, so for a.e. x, this statement is true for a.e. y. This means that if we fix x_0 (outside a null set) and set

$$\tilde{f}(x_0 + y) = f(x_0) + \int_0^1 \sum_{i=1}^n y_i g_i(x_0 + ty)\ dt,$$

we have $\tilde{f}$ continuous and $\tilde{f} = f$ a.e. Therefore (1) holds for $\tilde{f}$ in place of f, for all x, y, meaning $\tilde{f}$ is Lipschitz.

Conversely suppose $f = h$ a.e. for some h satisfying $|h(x) - h(y)| \leq c|x - y|$ for all x, y. Then h is absolutely continuous in each variable separately, so the (usual, not distributional) partial derivatives $\partial_i h$ exist a.e. Where these partials exist they are necessarily bounded by c. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Wherever the (usual) partial $\partial_i h$ exists we have $\partial_i(\varphi h) - \varphi\partial_i h + h\partial_i\varphi$. h and φ are absolutely continuous (in each variable), and hence so is $h\varphi$. Therefore

$$0 = \int_{-\infty}^{\infty} \partial_i(\varphi h)\ dx_i = \int_{-\infty}^{\infty} \varphi(\partial_i h)\ dx_i + \int_{-\infty}^{\infty} (\partial_i\varphi)h\ dx_i,$$

and hence

$$(\partial_i T_h)(\varphi) = -\int_{\mathbb{R}^n} (\partial_i\varphi)h\ dx = \int_{\mathbb{R}^n} \varphi(\partial_i h)\ dx.$$

This means that the distributional derivative $\partial_i T_h$ is given by the function $\partial_i h$, which is in $L^\infty(\mathbb{R}^n)$. Equivalently, we have h (and f) in $W^{1,\infty}(\mathbb{R}^n)$.

(D) Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ in a neighborhood of 0. For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ let

$$T\varphi = \int_{\mathbb{R}^n\backslash\{0\}} (\varphi(x) - \varphi(0)\psi(x))|x|^{-n}\ dx.$$

If $\varphi \in \mathcal{D}(\mathbb{R}^n\backslash\{0\})$ then $\varphi(0) = 0$ so $T\varphi = T_f\varphi$. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then $\varphi'(0)$ exists so $|\varphi(x) - \varphi(0)| = O(|x|)$ as $x \to 0$, so the integral in the definition of $T\varphi$ does not diverge at 0; thus $T\varphi$ is well-defined. We must show that T is continuous on $\mathcal{D}(\mathbb{R}^n)$. Suppose $\varphi_k \to \varphi$ in $\mathcal{D}(\mathbb{R}^n)$. By definition this means there is a single compact K supporting all but finitely many of the functions φ_k; by enlarging K if necessary, we may assume ψ is also supported in K. Since $\varphi_k(x) - \varphi_k(0)\psi(x) \to \varphi(x) - \varphi(0)\psi(x)$ uniformly on K, these functions are 0 outside K, and $|x|^{-n}$ is bounded on $\{x : |x| \geq r\}$, we have for $r > 0$ that

$$\begin{aligned}
\int_{\{x:|x|\geq r\}} \big(\varphi_k(x) - \varphi_k(0)\psi(x)\big)|x|^{-n}\ dx &= \int_{\{x:|x|\geq r\}\cap K} \big(\varphi_k(x) - \varphi_k(0)\psi(x)\big)|x|^{-n}\ dx \\
&\to \int_{\{x:|x|\geq r\}\cap K} \big(\varphi(x) - \varphi(0)\psi(x)\big)|x|^{-n}\ dx \\
&= \int_{\{x:|x|\geq r\}} \big(\varphi(x) - \varphi(0)\psi(x)\big)|x|^{-n}\ dx.
\end{aligned}$$

Presuming r is small enough, we have $\psi(x) = 1$ for all $|x| < r$, so using the Mean Value Theorem we get

$$\begin{aligned}
&\left| \int_{\{x:|x|<r\}} \big(\varphi_k(x) - \varphi_k(0)\psi(x)\big)|x|^{-n}\, dx - \int_{\{x:|x|<r\}} \big(\varphi(x) - \varphi(0)\psi(x)\big)|x|^{-n}\, dx \right| \\
&\qquad \leq \int_{\{x:|x|<r\}} \big|(\varphi_k - \varphi)(x) - (\varphi_k - \varphi)(0)\big||x|^{-n}\, dx \\
&\qquad \leq ||(\varphi_k - \varphi)'||_\infty \int_{\{x:|x|<r\}} |x|^{-(n-1)}\, dx \\
&\qquad \to 0 \quad \text{as } r \to 0.
\end{aligned}$$

Combining these shows that $T\varphi_k \to T\varphi$, as desired.

(E) Suppose Ω is connected, $T \in \mathcal{D}'(\Omega)$, and for some m, $D^\alpha T = 0$ for all α with $|\alpha| = m+1$.

For $m = 0$, the assumption says that $\partial_i T = 0$ for all $i \leq n$. Since $0 \in C(\Omega)$, by Theorem 7 from lecture this says that $T = T_f$ for some $f \in C^1$, with $\partial f/\partial x_i \equiv 0$ for all i. Since Ω is connected, this shows f is constant. In other words, $T = T_f$ for a polynomial f of degree at most 0.

Suppose the result is true for $m-1$, for some $m \geq 1$, and suppose $D^\alpha T = 0$ for all α with $|\alpha| = m+1$. Then for all i, $\partial_i T$ satisfies the induction hypothesis for $m-1$, so $\partial_i T = T_{g_i}$ for some polynomial g_i of degree at most $m-1$. By Theorem 7, $T = T_f$ for some function $f \in C^1$ with $\partial f/\partial x_i \equiv g_i$ for all i, which means f is also a polynomial. Thus the result is true for m.