MATH 525b ASSIGNMENT 6
SPRING 2009
Prof. Alexander
Due Wednesday April 15.

Done correctly, none of these are very long!

(I) Show that a normal T is unitary if and only if $\sigma(T)$ is contained in the unit circle.

(II) Suppose T is normal, and f_n, f are bounded measurable functions on $\sigma(T)$ with $f_n \to f$ uniformly. Show that $||f_n(T) - f(T)|| \to 0$.

(II) Show that for all $n \geq 2$, every normal operator T has an nth root, that is, there is an $S \in L(X, X)$ with $S^n = T$. If T is invertible, show that S is invertible.

(IV) For T normal, show that there exists a unitary U with $T^* = UT$. HINT: Take $U = f(T)$ for some choice of f. To see what $f(\lambda)$ will work, consider what happens with $T = \lambda I$.

(V)(a) In lecture we showed that T self-adjoint implies $\sigma_r(T) = \phi$. Show that this is true or normal T as well.

(b) For $(Y, \mathcal{M}, \mu)$ a measure space, for $L^2 = L^2(Y, \mathcal{M}, \mu)$ and for ψ with $||psi||_\infty < \infty$, define the multiplication operator $M_\psi : L^2 \to L^2$. Show that $\lambda \in \sigma(M_\psi)$ if and only if $\mu(\{x \in Y : |\psi(x) - \lambda| < \epsilon\}) > 0$ for all $\epsilon > 0$. (This set of λ's is called the *essential range* of ψ.)

(c) Give a corresponding description of $\sigma_p(M_\psi)$ (specific to M_ψ, not just the definition for general T.)