MATH 525b ASSIGNMENT 5 SOLUTIONS
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Chapter 7

(17) Let X be LCH and let μ be Radon with $\mu(X) = \infty$. Since X is open in itself, there exist compact sets $A_1, A_2, \dots$ with $\mu(A_n) \to \infty$ and $\cup_n A_n = X$. Let $K_n = A_1 \cup \cdots \cup A_n$ so $K_1 \subset K_2 \subset \dots$ By removing some K_n's from the list if necessary, we may assume $\sum_n \frac{1}{\mu(K_n)} < \infty$. By 4.32, for every n there exists $f_n \in C_c(X)$ with $f_n = 1$ on K_n and $0 \leq f_n \leq 1$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{\mu(K_n)} f_n.$$

Then $x \notin K_n$ implies $f(x) \leq \sum_{m>n} \frac{1}{\mu(K_m)} \to 0$ as $n \to \infty$, which shows $f \in C_0(x)$. But

$$\int f \, d\mu = \sum_n \frac{1}{\mu(K_n)} \int f_n \, d\mu \geq \sum_n \frac{1}{\mu(K_n)} \int_{K_n} 1 \, d\mu = \infty.$$

Now if I is a positive linear functional on $C_0(X)$ then I is also a positive linear functional on the subspace $C_c(X)$ so by 7.2, there exists a Radon ν such that $I(g) = \int g \, d\nu$ for all $g \in C_c(X)$.

We claim that $\nu(X) < \infty$. Suppose not; then let f_n, f be as above and let $g_n = \sum_{k=1}^{n} \frac{1}{\mu(K_k)} f_k$ be the partial sum. Since I is positive and $f \geq g_n$, we have

$$I(f) \geq \sup_n I(g_n) = \sup_n \int g_n \, d\nu = \int f \, d\nu = \infty,$$

where the second equality follows from Montone Convergence. This contradicts the fact I is finite-valued, so the claim is proved.

Next we claim that I is bounded. Suppose not; then there exist functions $h_n \in C_0(X)$ with $\|h_n\|_\infty \leq 1$ and $|I(h_n)| > 2^n$. Since $I(h_n) = I(h_n^+) - I(h_n^-)$, we may assume $h_n \geq 0$, so also $I(h_n) \geq 0$. Let $G = \sum_{k=1}^{\infty} 2^{-k} h_k$ and $G_n = \sum_{k=1}^{n} 2^{-k} h_k$. Then $G_n \in C_c(X), G_n \to G$ uniformly, so $G \in C_0(X)$. Since $G \geq G_n$ we have

$$I(G) \geq \sup_n I(G_n) = \sup_n \sum_{k=1}^{n} 2^{-k} I(h_k) > \sup_n \sum_{k=1}^{n} 2^{-k} 2^k = \infty,$$

which is a contradiction since I is finite-valued. Thus I must be bounded.

(20)(a) For $\mu \in M(X)$ let $\Phi(\mu) = \sum_{x\in X} \mu(\{x\})$. Since μ is finite, for any finite A we have $\sum_{x\in A} |\mu|(\{x\}) < |\mu|(X) < \infty$, so there can be at most countably many x with $|\mu|(\{x\}) > 0$, so $\sum_{x\in X} \mu(\{x\})$ is actually a countable sum which is at most $|\mu|(X)$. Thus $\Phi(\mu)$ is well-defined. Let $B_\mu = \{x \in X : |\mu|(\{x\}) > 0\}$. Clearly Φ is linear on $M(X)$. To show it is continuous, suppose $\mu_n \to \mu$ in norm, that is, $|\mu_n - \mu|(X) \to 0$. Let $D = B_\mu \cup (\cup_n B_{\mu_n})$, which is at most countable. Then

$$\begin{aligned}
|\Phi(\mu_n) - \Phi(\mu)| &= \left|\sum_{x\in D}(\mu_n(\{x\}) - \mu(\{x\})\right| \\
&\le \sum_{x\in D} |\mu_n(\{x\}) - \mu(\{x\})| \\
&\le \sum_{x\in D} |\mu_n - \mu|\,(\{x\}) \\
&= |\mu_n - \mu|(D) \\
&\le |\mu_n - \mu|(X) \\
&\to 0.
\end{aligned}$$

Thus $\Phi \in M(X)^*$.

Suppose now that $\mu_0 \in M(X)$ is nonzero but $\mu_0(\{x\}) = 0$ for all $x \in X$, i.e. μ_0 has no point masses. Then the measure $g\, d\mu_0$ also has no point masses, for all bounded measurable g, meaning $\Phi(g\, d\mu_0) = 0$. For each $f \in C_0(X)$ there is a corresponding $\hat{f} \in M(X)^* = C_0(X)^{**}$ given by $\hat{f}(\mu) = \int f\, d\mu$, which means that $\hat{f}(g\, d\mu_0) = \int fg\, d\mu_0$. Hence for each point mass δ_x with $x \in X$, we have $\Phi(\delta_x) = \delta_x(\{x\}) = 1$ while $\hat{f}(\delta_x) = \int f\, d\delta_x = f(x)$. If $\hat{f} = \Phi$ this would mean $f(x) = 1$ for all x, and therefore for all bounded measurable g, $0 = \Phi(g\, d\mu_0) = \hat{f}(g\, d\mu_0) = \int fg\, d\mu_0 = \int g\, d\mu_0$. But (by taking g of form χ_E) this implies that μ_0 is the 0 measure, a contradiction. This shows that Φ cannot be of form $\hat{f}$, so $C_0(X)$ is not reflexive.

(22) Let $\{f_n\}$ be a sequence in $C_0(X)$, where X is LCH.

Suppose $f_n \to f$ weakly. By Chapter 5 #47b (from Assignment 3), $\sup_n ||f_n||_u < \infty$. For each $x \in X$, the point mass δ_x at x is a finite Radon measure, so $\varphi_x(f) = \int f\, d\delta_x = f(x)$ defines a bounded linear functional on $C_0(X)$, by the Riesz Representation Theorem. Hence $\varphi_x(f_n) \to \varphi_x(f)$ for all x, that is, $f_n(x) \to f(x)$ for all x.

Conversely suppose $f_n \to f$ pointwise and $M = \sup_n ||f_n||_u < \infty$. By the Riesz Representation Theorem, every bounded linear functional on $C_0(X)$ has the form $\varphi_\mu(g) = \int g\, d\mu$

for some (finite) complex Radon μ. Let $\epsilon > 0$ and $A_{n,\epsilon} = \{x : |f_n(x) - f(x)| < \epsilon\}$. Then

$$\begin{aligned}
|\varphi_\mu(f_n) - \varphi_\mu(f)| &\leq \int |f_n - f| \; d|\mu| \\
&= \int_{A_{n,\epsilon}} |f_n - f| \; d|\mu| + \int_{A(n,\epsilon)^c} |f_n - f| \; d|\mu| \\
&\leq \epsilon|\mu|(X) + 2M|\mu|(A^c_{n,\epsilon})
\end{aligned}$$

Since $f_n \to f$ pointwise, we have $|\mu|(A^c_{n,\epsilon}) \to 0$, so

$$\limsup_n |\varphi_\mu(f_n) - \varphi_\mu(f)| \leq \epsilon|\mu|(X).$$

Since ϵ is arbitrary, this shows $\varphi_\mu(f_n) \to \varphi_\mu(f)$. Thus $f_n \to f$ weakly.

(A)(a) Suppose $x \in \mathcal{R}(P)$. Then $x = Py$ for some y, so $Px = P^2y = Py = x$. Conversely, by definition of $\mathcal{R}(P)$, $Px = x$ implies $x \in \mathcal{R}(P)$. Hence $\mathcal{R}(P) = (P - I)^{-1}(\{0\})$ which is closed since $\{0\}$ is closed in $\mathbb{R}$.

(b) If P is self-adjoint, then from the definitions, P is normal. Suppose P is normal. By a lemma from lecture and (B)(b) below, $\mathcal{R}(P)^\perp = \mathcal{N}(P^*) = \mathcal{N}(P)$. Since by (a) $\mathcal{R}(P)$ is closed, we then have $\mathcal{R}(P) = (\mathcal{R}(P)^\perp)^\perp = \mathcal{N}(P)^\perp$. Finally suppose $\mathcal{R}(P) = \mathcal{N}(P)^\perp$. Then for all x, y we have $\langle Px, y\rangle = \langle Px, (y - Py) + Py\rangle$ and $y - Py \in \mathcal{N}(P) = \mathcal{R}(P)^\perp$, which means $\langle Px, y - Py\rangle = 0$. Therefore $\langle Px, y\rangle = \langle Px, Py\rangle$. Similarly $\langle x, Py\rangle = \langle Px, Py\rangle$, so $\langle Px, y\rangle = \langle x, Py\rangle$ which means $P = P^*$. Thus all three are equivalent.

(B)(a) Suppose $T \in L(X, X)$ and $\langle Tx, x\rangle = 0$ for all x. Then for all x, y,

$$\begin{aligned}
0 &= \langle T(x + y), x + y\rangle \qquad (1) \\
&= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle T(iy), iy\rangle \\
&= \langle Tx, y\rangle + \langle Ty, x\rangle,
\end{aligned}$$

and

$$\begin{aligned}
0 &= \langle T(x + iy), x + iy\rangle \qquad (2) \\
&= \langle Tx, x\rangle - i\langle Tx, y\rangle + i\langle Ty, x\rangle + \langle Ty, y\rangle \\
&= -i\langle Tx, y\rangle + i\langle Ty, x\rangle.
\end{aligned}$$

Multiplying (1) by i and adding it to (2), we get $\langle Tx, y\rangle = 0$. Since x, y are arbitrary, this means $T = 0$.

(b) By (a),

$$\begin{aligned} T \text{ normal} &\iff TT^* - T^*T = 0 \\ &\iff \langle (TT^* - T^*T)x, x\rangle = 0 \text{ for all } x \\ &\iff \langle TT^*x, x\rangle = \langle T^*Tx, x\rangle \text{ for all } x \\ &\iff \langle T^*x, T^*x\rangle = \langle Tx, Tx\rangle \text{ for all } x \\ &\iff ||T^*x|| = ||Tx|| \text{ for all } x. \end{aligned}$$

Thus for T normal we have $||T^*x|| = 0 \iff ||Tx|| = 0$, which implies $\mathcal{N}(T^*) = \mathcal{N}(T)$.

(C) Suppose $T \in L(X,X)$ and $\langle Tx, x\rangle \geq 0$ for all x. Then for all x, $\langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} = \langle Tx, x\rangle$, so $\langle (T - T^*)x, x\rangle = 0$. By (B)(a) this means $T - T^* = 0$, that is, T is self-adjoint.

Let $\lambda > 0$. Then for all x,

$$\lambda ||x||^2 = \langle \lambda x, x\rangle \leq \langle \lambda x + Tx, x\rangle \leq ||(\lambda I + T)x||\ ||x||, \tag{3}$$

so $||(\lambda I + T)x|| \geq \lambda ||x||$. This shows that $\mathcal{N}(\lambda I + T) = \{0\}$. Since T is self-adjoint there is no residual spectrum, so $\lambda I + T$ is a bijection and by (3) its inverse is bounded. Thus $\lambda I + T$ is invertible, so $-\lambda \notin \sigma(T)$. This shows that $\sigma(T) \subset [0, \infty)$.

(D) Let X be a separable Hilbert space and suppose $T \in L(X,X)$ is normal. For each $\lambda \in \sigma_p(T)$ let x_λ be an eigenvector of norm 1. By a proposition from lecture, $\{x_\lambda : \lambda \in \sigma_p(T)\}$ is an orthonormal set. By 5.29, $\{x_\lambda\}$ is at most countable, so $\sigma_p(T)$ is at most countable.

(E) Let S_L, S_R denote the left and right shifts on ℓ^2.

Claim 1. $S_R^* = S_L$. *Proof*: For all x, y, $\langle S_R x, y\rangle = \sum_{i \geq 0} x_i \overline{y_{i+1}} = \langle x, S_L y\rangle$.

Claim 2. $\sigma_p(S_L) = \{\lambda : |\lambda| < 1\}$. *Proof*: Suppose $S_L x = \lambda x$ for some λ. Then $x_{i+1} = \lambda x_i$ for all i, so x is a multiple of $v_\lambda = (1, \lambda, \lambda^2, ...)$. Thus $\lambda \in \sigma_p(S_L) \iff v_\lambda \in \ell^2 \iff |\lambda| < 1$.

Claim 3. $\sigma_p(S_R) = \phi$. *Proof*: Suppose $(\lambda I - S_R)x = 0$ for some $x = (x_0, x_1, ...)$. Then $\lambda x_0 = 0$ and $\lambda x_i = x_{i-1}$ for all $i \geq 1$. If $\lambda \neq 0$ this show that $x_0 = 0$, and then inductively, $x_i = 0$ for all i, so $x = 0$. If $\lambda = 0$ then we get $x_{i-1} = 0$ for all $i \geq 1$, so $x = 0$. Thus $\lambda \notin \sigma_p(S_R)$, for all λ.

Claim 4. $\sigma_r(S_L) = \phi$. *Proof*: For $\lambda \in \mathbb{C}$, by Claim 1, $\mathcal{R}(\lambda I - S_L)^\perp = \mathcal{N}((\lambda I - S_L)^*) = \mathcal{N}(\overline{\lambda} I - S_R)$, which is $\{0\}$ by Claim 3. Thus $\overline{\mathcal{R}(\lambda I - S_L)} = X$, which means $\lambda \notin \sigma_r(S_L)$.

Claim 5. $\sigma_r(S_R) = \{\lambda : |\lambda| < 1\}$. *Proof*: By Claim 3, $\lambda \in \sigma_r(S_R) \iff \overline{\mathcal{R}(\lambda I - S_R)} \neq X \iff \mathcal{R}(\lambda I - S_R)^\perp \neq \{0\} \iff \mathcal{N}(\lambda I - S_L) \neq \{0\} \iff \lambda \in \sigma_p(S_L) \iff |\lambda| < 1$. (The last equivalence here is Claim 2.)

Claim 6. $\sigma_c(S_L) = \{\lambda : |\lambda| = 1\}$. *Proof*: By Claims 2 and 4, $\lambda \in \sigma_c(S_L)$ implies $|\lambda| \geq 1$, and for $|\lambda| \geq 1$ we have

$$\begin{aligned} \lambda \in \sigma_c(S_L) &\iff (\lambda I - S_L)^{-1} \text{ is unbounded} \\ &\iff \text{for every } \epsilon > 0 \text{ there exists } x : ||(\lambda I - S_L)x|| \leq \epsilon ||x||. \end{aligned}$$

Given λ with $|\lambda| = 1$, and $\epsilon \in (0, 1)$, we apply this to $x = v_{(1-\epsilon)\lambda}$ (defined in Claim 2) by noting that $||(\lambda I - S_L)v_{(1-\epsilon)\lambda}|| = ||\epsilon\lambda v_{(1-\epsilon)\lambda}|| = \epsilon||v_{(1-\epsilon)\lambda}||$; this shows that $\lambda \in \sigma_c(S_L)$. If instead $|\lambda| > 1$, then $||S_L/\lambda|| = 1/|\lambda| < 1$ so by a lemma from lecture $I - S_L/\lambda$ is invertible, so $\lambda I - S_L$ is invertible, that is, $\lambda \notin \sigma_c(S_L)$.

Claim 7. $\sigma_c(S_R) = \{\lambda : |\lambda| = 1\}$. *Proof*: By a Proposition from lecture and Claims 2, 4, 6, $\lambda \in \sigma(S_R) \iff \lambda I - S_R$ is not invertible $\iff (\lambda I - S_R)^*$ is not invertible $\iff \overline{\lambda} I - S_L$ is not invertible $\iff \overline{\lambda} \in \sigma(S_L) \iff |\lambda| \leq 1$. Since (by Claims 3 and 5) $\sigma_p(S_R) \cup \sigma_r(S_R) = \{\lambda : |\lambda| < 1\}$, we must have $\sigma_c(S_R) = \{\lambda : |\lambda| = 1\}$.