MATH 525b ASSIGNMENT 1 SOLUTIONS
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Chapter 5

(30)(a) We need to find a sequence $\{f_n\}$ in $C^1[0,1]$ converging (uniformly) to an $f \notin C^1[0,1]$. There are many possible examples. The one we give here is most naturally described in $C[-1,1]$ instead of $C[0,1]$, but we can then transfer it over to $C[0,1]$ by using $\tilde{f}_n(x) = f_n(2x-1)$, for example.

Define φ_n on [-1,1] by

$$\varphi_n(x) = \begin{cases} -1, & -1 \le x \le -\frac{1}{n}, \\ nx, & -\frac{1}{n} < x < \frac{1}{n}, \\ 1, & \frac{1}{n} \le x \le 1, \end{cases}$$

and f_n by

$$f_n(x) = \int_0^x \varphi_n(x)\ dx = \begin{cases} -x + \frac{1}{2n}, & -1 \le x \le -\frac{1}{n}, \\ \frac{nx^2}{2}, & -\frac{1}{n} < x < \frac{1}{n}, \\ x - \frac{1}{2n}, & \frac{1}{n} \le x \le 1. \end{cases}$$

Then $f_n' = \varphi_n$ is continuous so $f_n \in C^1[-1,1]$, but $\big|f_n(x) - |x|\big| \le 1/2n \to 0$, so for $f(x) = |x|$ we have $||f_n - f|| \to 0$ and $f \notin C^1[-1,1]$. Thus $\{f_n\}$ is Cauchy but not convergent in $C[-1,1]$, which shows that $C^1[-1,1]$ is not complete.

(b) To show d/dx is not bounded, let $f_n(x) = \frac{1}{n}\sin^2 x, x \in [0,1]$. Then $f_n'(x) = n \cos n^2 x$ so $f_n'(0) = n$. Hence $\{f_n\}$ is bounded but $||\frac{d}{dx} f_n|| \ge n \to \infty$. This shows that d/dx is not bounded.

To show $\frac{d}{dx}$ is closed, we suppose $f_n \to f$ and $f_n' \to g$ uniformly, and show that $f' = g$. In fact, for every x we have

$$f(x) - f(0) = \lim_n (f_n(x) - f_n(0)) = \lim_n \int_0^x f_n'(t)\ dt = \int_0^x g(t)\ dt,$$

which shows that f is absolutely continuous with $f' = g$, as desired.

(32) Let $\mathcal{X}_i$ denote the space $\mathcal{X}$ endowed with norm $|| \cdot ||_i, i = 1,2$. Let T be the identity map on $\mathcal{X}$, viewed as a map from $\mathcal{X}_2$ to $\mathcal{X}_1$. Then $||Tx||_1 = ||x||_1 \le ||x||_2$ so T is bounded. By Corollary 5.11, T^{-1} is bounded, that is, there exists c such that $||x||_2 = ||T^{-1}x||_2 \le c||x||_1$ for all x. Thus $||x||_1 \le ||x||_2 \le c||x||_1$ for all x, so the norms are equivalent.

(38)Let $\mathcal{A} = \{T_n : n \geq 1\} \subset L(\mathcal{X}, \mathcal{Y})$. Since $\{T_n x\}$ conveges for all x, it is bounded for all x, that is, $\sup_{T\in\mathcal{A}} ||Tx|| < \infty$ for all x. By 5.13a, $\sup_n ||T_n|| = c < \infty$ for some c. Hence for all x, $||Tx|| = \lim_n ||T_n x|| \leq c||x||$, so $T \in L(\mathcal{X}, \mathcal{Y})$.

(39) Suppose $B : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is separately continuous and bilinear. Let $B_x(y) = B^y(x) = B(x, y)$ and let $U_\mathcal{X}$ be the closed unit ball in $\mathcal{X}$. Let $\mathcal{A} = \{B_x : x \in U_\mathcal{X}\}$. For fixed $y \in \mathcal{Y}$ we have

$$\sup_{T\in\mathcal{A}} ||Ty|| = \sup_{x\in U_\mathcal{X}} ||B_x(y)|| = \sup_{x\in U_\mathcal{X}} ||B^y(x)|| \leq \sup_{x\in U_\mathcal{X}} ||B^y||\ ||x|| = ||B^y|| < \infty,$$

so $\mathcal{A}$ is pointwise bounded. By 5.13a, $\mathcal{A}$ is uniformly bounded, that is there exists C such that $||B_x|| \leq C$ for all $x \in U_\mathcal{X}$. Since B is bilinear this implies that $||B_x|| \leq C||x||$ for all $x \in \mathcal{X}$. Therefore for all $x \in \mathcal{X}, y \in \mathcal{Y}$,

$$||B(x, y)|| = ||B_x(y)|| \leq ||B_x||\ ||y|| \leq C||x||\ ||y||.$$

Now suppose $(x_n, y_n) \to (x, y)$ in $\mathcal{X} \times \mathcal{Y}$, that is, $x_n \to x$ in $\mathcal{X}$ and $y_n \to y$ in $\mathcal{Y}$. Then $||x_n|| \to ||x||$, so $\{||x_n||\}$ is bounded, say $||x_n|| \leq K$ for all n. Therefore

$$\begin{aligned} ||B(x_n, y_n) - B(x, y)|| &\leq ||B(x_n, y_n) - B(x_n, y)|| + ||B(x_n, y) - B(x, y)|| \\ &\leq C||x_n||\ ||y_n - y|| + C||x_n - x||\ ||y|| \\ &\leq CK||y_n - y|| + C||y||\ ||x_n - x|| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus B is jointly continuous.

(I) Suppose $E_n, n \geq 1$ are disjoint closed sets. Let $F_n = E_n \backslash E_n^o$ and $F = \cup_n F_n$. Since F_n is closed and has no interior, F_n is n.d. in $[0, 1]$, and F_n^c is an open set which is dense in $[0, 1]$. Since every set open in $[0, 1]$ is a countable union of open intervals (plus perhaps intervals $[0, a)$ or $(b, 1]$), every set open in F is a union of sets of form $I \cap F$ with I an interval open in $[0, 1]$.

Suppose F_n is not n.d. in F for some n. Then $F_n \supset I \cap F \neq \phi$ for some interval I as above. This means $I \backslash F_n \subset F^c$. Suppose $x \in E_k^o \cap I$ for some k. Since $E_k^o \cap I$ is a union of open intervals, there is a maximal open interval of form (a, b) or $(a, 1]$ in $E_k^o \cap I$ containing x; the maximality means the endpoints a, b cannot be in $E_k^o \cap I$. But $a, b \in E_k$ since E_k is closed, so each of a, b is either an endpoint of I or a point of F_k. Since $F_n \supset I \cap F$, this means either $k = n$ or the maximal open interval is all of I. The second of these possibilities is ruled out by the fact that $I \cap F \neq \phi$, so we have $k = n$. Thus for $k \neq n$ we have $E_k^o \cap I = \phi$ and $F_k \cap I = \phi$, so $E_k \cap I = \phi$. Since $I \cap F_n \neq \phi$, we do not have $I \subset E_n$, so we do not have $I \subset \cup_k E_k$, meaning $\cup_k E_k$ is not all of $[0, 1]$.

The other possibility is that F_n is n.d. in F for all n. In this case, if $\cup_k E_k = [0,1]$ then $[0,1]\backslash F = \cup_k E_k^o$, which is open, so F is closed, hence compact, hence complete. By the Baire Category Theorem we have a contradiction. Thus again, $\cup_k E_k$ is not all of $[0,1]$.

(II) Let $\{f_n\}$ be a countable dense subset of X^*. By the definition of $||f_n||$, there exists $x_n \in X$ with $||x_n|| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}||f_n||$. Let $S = \text{span}(x_1, x_2, \dots)$. Since the rationals are dense in $\mathbb{R}$, the countable set

$$D = \left\{ \sum_{k=1}^{n} q_k x_k : n \geq 1, q_k \in \mathbb{Q} \text{ for all } k \right\}$$

is dense in S, so $\overline{D} = \overline{S}$. Suppose $\overline{S} \neq X$. By Theorem 5.8a, there exists $0 \neq f \in X^*$ with $f = 0$ on $\overline{S}$. In particular, $f(x_n) = 0$ for all n. Since $\{f_n\}$ is dense, there exists n with $||f_n - f|| < \frac{1}{2}||f_n||$, so $|f_n(x_n)| = |f_n(x_n) - f(x_n)| \leq ||f_n - f|| < \frac{1}{2}||f_n||$, contradicting the definition of $||x_n||$. Thus $\overline{S} = X$, so $\overline{D} = X$, i.e. D is a countable dense subset of X.