MATH 475 ASSIGNMENT 15 (SUGGESTED PROBLEMS)
SPRING 2010
Prof. Alexander
Not to turn in.

Section 3.5 #8–12, 15

HINTS

(8) See the discussion around Figure 3.7. $1/(z-1)$ will map each of the circles to a line; you just have to check a couple of points to determine what line it is, in each case. So the region between the circles is mapped to (what region R?) Now you need a mapping to take R to the unit disk. For that, one option is to map R first to the upper half plane, and then map the upper half plane to the unit disk. It may be useful to remember how the exponential function maps a strip—see section 1.5.

(9) The solution in the back of the book is incorrect: it should be $\frac{4}{\pi}(\text{Log } z) - i$. (Thanks to Dayton for noticing.)

FINAL EXAM NOTES:

I did not cover the part about circulation produced by an LFT, p. 201–203 in 3.3, so it's not on the final.

STUDY SUGGESTIONS:

These are some themes that run through the course, which you should be sure you understand well. I have chosen them because they are not confined to a particular section of the book, but spread over multiple sections.

(A) *Properties of analytic functions:* By definition, "analytic" means "differentiable." Analytic functions (in a region D) satisfy the Cauchy-Riemann equations, and conversely, any function satisfying the Cauchy-Riemann equations is analytic, provided the partial derivatives are continuous (section 2.1.) An analytic function is always differentiable infinitely many times, and it can be expressed as a converging power series (its Taylor series) in a disk around any given point of D (section 2.4.) Analytic functions are conformal, meaning that if you have 2 curves in the domain that meet at an angle α, and you use an analytic function to map these curves to another region, then they still meet at the same angle α. This is because if the derivative at some point, in polar form, is $re^{i\theta}$, then the curves are each approximately rotated by angle θ (section 3.3.)

(B) *Line integrals and analytic functions:* If a curve γ and its inside are contained in a region

D where f is analytic, then $\int_\gamma f(z)\ dz = 0$. The converse is also true for continuous f: if $\int_\gamma f(z)\ dz = 0$ for all SC curves γ in a domain, then f is analytic—in fact, it is enough that you get 0 when integrating around triangles. This is Morera's Theorem, section 2.4. You won't necessarily get 0 for the integral if f fails to be analytic everywhere inside γ. For example, if there are poles inside γ, then $\int_\gamma f(z)\ dz$ is determined by the residues at those poles. If f is analytic inside γ, then the values of f at all points *inside* γ are determined by the values of f *on* γ, via Cauchy's Formula (section 2.3.)

(C) *Antiderivatives:* An analytic function in a region D does not necessarily have an antiderivative that exists in all of D. (For example, there is $f(z) = 1/z$ in $D = \mathbb{C}\backslash\{0\}$ for which the hoped-for antiderivative $\log z$ cannot be defined as a continuous function in all of D.) If D is simply connected then the antiderivative does exist (section 2.3.) Whenever an antiderivative F exists, you can evaluate a line integral as F(ending point) $-$ F(starting point).

(D) *Special properties of analytic functions:* There are various properties of analytic functions which do not have analogs in the world of real-valued functions. Here are some: An entire function (i.e. analytic in all $\mathbb{C}$) cannot be bounded (section 2.4.) If a function is analytic in a punctured disk $0 < |z - z_0| < r$ and stays bounded as $z \to z_0$, then the function actually must have a limit as $z \to z_0$ (section 2.5.) Analytic functions satisfy the Argument Principle and Rouché's Theorem (section 3.1) and Schwarz's Lemma (section 3.2.) If f is analytic in a region D, then $|f|$ cannot have a local maximum in D (section 3.2), so if ∂D is part of the region where f is analytic, then the maximum of $|f|$ must occur on ∂D.