MATH 475 FINAL EXAM SOLUTIONS
Spring 2010
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(1)(a) Find the power series expansion of $f(z) = \frac{8-z}{6-z}$ about $z_0 = 1$. HINT: First re-express f as $f(z) = (\text{constant}) + (\text{another term})$.

(b) Find the largest disc centered at $z_0 = 1$ in which your power series expansion is valid (that is, the series converges.)

SOLUTION: (a)

$$\begin{aligned} f(z) &= 1 + \frac{2}{6-z} = 1 + \frac{2}{5-(z-1)} = 1 + \frac{2}{5}\,\frac{1}{1-\frac{z-1}{5}} \\ &= 1 + \frac{2}{5}\sum_{n=0}^{\infty}\left(\frac{z-1}{5}\right)^n = \frac{7}{5} + \sum_{n=1}^{\infty}\frac{2}{5^{n+1}}(z-1)^n. \end{aligned}$$

(b) There is a singularity at $z = 6$, which is distance 5 from $z = 1$, so the series converges for $|z-1| < 5$.

(2) Find all 4th roots of $-8 + 8\sqrt{3}i$ (which is $16e^{2\pi i/3}$ in polar form.) Express your answers in the form $a + bi$.

SOLUTION: Using the polar form, the roots are

$$\begin{aligned} w_0 &= 2e^{i\pi/6} = \sqrt{3} + i, \\ w_1 &= w_0 e^{2\pi i/4} = 2e^{2\pi i/3} = -1 + \sqrt{3}i, \\ w_2 &= w_0 e^{4\pi i/4} = 2e^{7\pi i/6} = -\sqrt{3} - i, \\ w_3 &= w_0 e^{6\pi i/4} = 2e^{5\pi i/3} = 1 - \sqrt{3}i. \end{aligned}$$

(3) Let D be a domain and $f : D \to \mathbb{C}$ a nonconstant function. Show that $f(z)$ and $\overline{f(z)}$ cannot both be analytic functions.

SOLUTION 1: If $f(z)$ and $\overline{f(z)}$ were both analytic, then $g(z) = f(z)\overline{f(z)} = |f(z)|^2$ would also be analytic. But g is real-valued so its range is not open, as is required for an analytic function on a domain D, so g is not analytic. Thus $f(z)$ and $\overline{f(z)}$ aren't both analytic.

SOLUTION 2: Let $f = u + iv$, so $\overline{f} = u - iv$. Then

$$f \text{ satisfies Cauchy-Riemann} \iff u_x = v_y, u_y = -v_x,$$

$$\overline{f} \text{ satisfies Cauchy-Riemann} \iff u_x = -v_y, u_y = v_x.$$

If both of these hold, then $u_x = -u_x$ and $u_y = -u_y$. This means $u_x = u_y = 0$ everywhere in D, so $f'(z) = u_x + iu_y = 0$ in D, so f is constant in D. Therefore for nonconstant f, Cauchy-Riemann must fail for either f or $\overline{f}$, meaning one function is not analytic.

(4) Let $f(z) = \frac{1}{1-\frac{1}{z}}$ and let γ_R be the circle $|z| = R$.

(a) Show that $|f(z)| \leq 2$ whenever $|z| > 2$.

(b) Use (a) to show that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{f(z)}{z^2}\, dz = 0.$$

SOLUTION: (a)

$$|z| > 2 \implies \left|\frac{1}{z}\right| < \frac{1}{2} \implies \left|1 - \frac{1}{z}\right| > 1 - \left|\frac{1}{z}\right| > \frac{1}{2} \implies |f(z)| < 2.$$

(b)

$$\left|\int_{\gamma_R} \frac{f(z)}{z^2} dz\right| \leq (\text{length } \gamma_R) \cdot \max_{z\in\gamma_R} \frac{|f(z)|}{|z|^2} \leq 2\pi R \cdot \frac{2}{R^2} = \frac{4\pi}{R} \to 0 \text{ as } R \to \infty.$$

(5) Determine the number of zeroes of $f(z) = (z - 2i)^2 + e^{iz}$ in the upper half plane.

SOLUTION: In the upper half plane, for $z = Re^{i\theta}$ we have $\theta \in [0, \pi]$ so $\sin\theta \geq 0$, hence

$$|e^{iz}| = \left|e^{iR\cos\theta - R\sin\theta}\right| = e^{-R\sin\theta} \leq 1.$$

Let the semicircular contour γ_R be the union of $[-R, R]$ and the semicircle α_R. For large R, the closest point to $2i$ on γ_R is $z = 0$, at distance 2. Therefore $|z - 2i|^2 \geq 4$ for all $z \in \gamma_R$. Letting $g(z) = (z - 2i)^2$ we then have

$$|f(z) - g(z)| = |e^{iz}| \leq 1, \quad |g(z)| = |z - 2i|^2 \geq 4 > |f(z) - g(z)| \quad \text{for all } z \in \gamma_R,$$

so by Rouché's Theorem, f and g have the same number of zeroes inside γ_R. g has two zeroes (both at $z = 2i$) hence so does f. Letting $R \to \infty$ shows f has two zeroes in the upper half plane.

(6) Recall that *entire* means analytic on all of $\mathbb{C}$. Suppose $f : \mathbb{C} \to \mathbb{C}$ is one-to-one and entire. Show that f is onto, that is, $f(\mathbb{C}) = \mathbb{C}$.

SOLUTION: Suppose $f(\mathbb{C}) \neq \mathbb{C}$. The Riemann Mapping Theorem then says there is a 1-1 analytic function $g : \Delta \to f(\mathbb{C})$ (for Δ = unit disk.) Then g^{-1} is also analytic, and $g^{-1} \circ f$ maps $\mathbb{C}$ 1-1 onto Δ. But then $g^{-1} \circ f$ is bounded (since all its values are in Δ) and entire, so is constant by the Liouville Theorem. This is not possible since $g^{-1} \circ f$ is 1-1. This contradiction means we must have $f(\mathbb{C}) = \mathbb{C}$.

(7)Let $a > 1$. Let $f(z)$ be the Schwarz-Christoffel transformation which maps the upper half plane U onto the rectangle D with vertices $-2+3i, -2, 2$ and $2+3i$ and satisfies $f(-1) = -2, f(1) = 2, f(a) = 2+3i$. Because of the symmetry of the rectangle, it can be shown that f is symmetric between positive and negative values, so that also $f(-a) = -2+3i$. (You need not show this.)

(a) Find a formula for $f(z)$. Your formula may involve an integral of some function $g(w)$, and two unspecified constants (call them A, B), but otherwise it should be as explicit as possible. You do not need to do the integration, and you need not find A, B yet.

(b) Let $g(w)$ be the integrand from part (a), and let

$$S = \int_{-1}^{1} |g(w)|\ dw, \quad T = \int_{1}^{a} |g(w)|\ dw.$$

Express $f(1) - f(-1)$ and $f(a) - f(1)$ each in terms of A, S, T (though you might not need all 4), and find B.

SOLUTION: (a) The angle $\theta_i = \pi/2$ for all $i = 1, 2, 3, 4$ so the exponent $\alpha_i = -1/2$ for all i. Therefore

$$f'(z) = A(z+a)^{-1/2}(z+1)^{-1/2}(z-1)^{-1/2}(z-a)^{-1/2}$$

so

$$f(z) = A\int_{-1}^{z} (w+a)^{-1/2}(w+1)^{-1/2}(w-1)^{-1/2}(w-a)^{-1/2}\ dw + B.$$

(b) Let $g(w)$ be the integrand in the formula above for $f(z)$. Then $f(1) - f(-1) = A\int_{-1}^{1} g(w)\ dw$. For all $w \in [-1, 1]$ we have $\arg(z+a) = \arg(z+1) = 0$, and $\arg(z-1) = \arg(z-a) = \pi$, so $\arg g(w) = 0 + 0 - \frac{\pi}{2} - \frac{\pi}{2} = -\pi$, which means $g(w) = e^{-i\pi}|g(w)| = -|g(w)|$. Therefore

$$f(1) - f(-1) = -A\int_{-1}^{1} |g(w)|\ dw = -AS.$$

Similarly $f(a) - f(1) = A\int_{1}^{A} g(w)\ dw$ and for $w \in [1, a]$ we have $\arg g(w) = -\frac{\pi}{2}$ and hence $g(w) = e^{-i\pi/2}|g(w)| = -i|g(w)|$. Therefore

$$f(a) - f(1) = -iA\int_{1}^{a} |g(w)|\ dw = -AiT.$$

Finally, plugging in $z = -1$ makes the integral 0 in the definition of $f(z)$, so $B = f(-1) = -2$.

(8) Let $U = \{z : \text{Im } z > 0\}$ be the upper half plane, let $\Delta = \{z : |z| < 1\}$ be the unit disc, and let $D = \{z \in U : |z| > 1\}$ be the portion of U outside $\Delta \cup \partial\Delta$. Let $f(z) = \frac{1}{2}(z + \frac{1}{z})$.

(a) We showed in lecture that f is a one-to-one map of $\{z : |z| > 1\}$ onto $\mathbb{C}\backslash[-1, 1]$. Show that $\text{Im } z > 0$ if and only if $\text{Im } f(z) > 0$. (This means that f is a one-to-one map of D onto U.)

(b) Use f to streamline D, that is, give the equation of the streamlines in the variables x, y, where $z = x + iy$. Express the equation in the form $x^2 = g(y)$ for some $g(y)$. This gives the flow over a semicircular obstacle.

SOLUTION: (a) Write $z = x + iy$, so

$$f(z) = \frac{1}{2}\left(x + iy + \frac{1}{x+iy}\right) = \frac{1}{2}\left(x + iy + \frac{x-iy}{x^2+y^2}\right),$$

so

$$\text{Im } f(z) = \frac{y}{2}\left(1 - \frac{1}{x^2+y^2}\right).$$

Now $|z| > 1$ means $x^2 + y^2 > 1$, which means $1 - \frac{1}{x^2+y^2} > 0$. Therefore for $|z| > 1$ we have $\text{Im } f(z) > 0 \iff y > 0 \iff \text{Im } z > 0$.

(b) Each streamline is given by $\text{Im } f(z) = c$, and we have

$$\begin{aligned}\text{Im } f(z) = c &\iff y\left(1 - \frac{1}{x^2+y^2}\right) = 2c \\ &\iff y(x^2+y^2-1) = 2c(x^2+y^2) \\ &\iff (y-2c)x^2 = -y^3 + 2cy^2 + y \\ &\iff x^2 = \frac{-y^3+2cy^2+y}{y-2c}.\end{aligned}$$

(9) Find the linear fractional transformation that carries the circle $|z| = 2$ onto the line $\{u + iv : v = -u\}$.

SOLUTION: The problem should really say, "Find a linear fractional transformation...", because it's not unique. We can find one which maps $2, 2i, -2$ to $-1+i, 0, 1-i$, respectively. Let R map $2, 2i, -2$ to $0, 1, \infty$, respectively:

$$R(z) = \frac{z-2}{z+2}\,\frac{2i+2}{2i-2} = -i\frac{z-2}{z+2}.$$

Let S map $-1+i, 0, 1-i$ to $0, 1, \infty$, respectively:

$$S(w) = \frac{w+1-i}{w-1+i}\,\frac{0-1+i}{0+1-i} = -\frac{w+1-i}{w-1+i}.$$

To get the desired LFT $T = S^{-1} \circ R$, solve $R(z) = S(w)$ for w:

$$\begin{aligned}-i\frac{z-2}{z+2} &= -\frac{w+1-i}{w-1+i} \\ i(w-1+i)(z-2) &= (w+1-i)(z+2) \\ w(iz-2i+z+2) &= (z-2)(1+i) + (1-i)(z+2) = 2z - 4i,\end{aligned}$$

so

$$T(z) = w = \frac{2z-4i}{(i+1)z+2-2i}.$$