MATH 445 SAMPLE MIDTERM EXAM 2 SOLUTIONS
Fall 2009
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(1)(a) The solution has form

$$u(x,t) = \int_0^\infty \left(A(p)\cos px + B(p)\sin px\right) e^{-c^2p^2t}\ dp.$$

Calculate: since f is odd,

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos px\ dx = 0.$$

Also

$$\begin{aligned} B(p) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin px\ dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin px\ dx \\ &= -\frac{2}{\pi p}\cos px\Big|_0^{\pi} \\ &= \frac{2(1-\cos p\pi)}{p\pi}, \end{aligned}$$

so

$$u(x,t) = \int_0^\infty \frac{2(1-\cos p\pi)}{p\pi}\sin px\ e^{-4p^2t}\ dp.$$

(b) Yes, you can just add T_0. Here's why: if $u(x,t)$ is the solution you found in (a), then $u(x,t)+T_0$ is also a solution to the heat equation, since adding T_0 doesn't change the derivatives u_t or u_{xx}. The new solution $u(x,t)+T_0$ satisfies the desired initial condition $f(x)+T_0$.

(2)(a) The solution has form

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{mx}{2}\sin\frac{ny}{2}.$$

$B^*_{mn} = 0$ since the initial velocity is 0.

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{4\pi^2}+\frac{n^2}{4\pi^2}} = \frac{3}{2}\sqrt{m^2+n^2}$$

and

$$\begin{aligned}
B_{mn} &= \frac{4}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)\sin\frac{mx}{2}\sin\frac{ny}{2}\ dx\ dy\\
&= \frac{5}{\pi^2}\left(\int_0^{2\pi}(1-\cos x)\sin\frac{mx}{2}\ dx\right)\left(\int_0^{2\pi}(1-\cos y)\sin\frac{ny}{2}\ dy\right).
\end{aligned}$$

To calculate these integrals, use the formulas from the last page of the exam:

$$\begin{aligned}
\int_0^{2\pi}(1-\cos x)\sin\frac{mx}{2}\ dx &= \int_0^{2\pi}\sin\frac{mx}{2}\ dx - \int_0^{2\pi}\cos x\sin\frac{mx}{2}\ dx\\
&= -\frac{2\cos\frac{mx}{2}}{m}\Bigg|_0^{2\pi} - \frac{1}{2}\int_0^{\pi}\left(\sin(\frac{m}{2}+1)x+\sin(\frac{m}{2}-1)x\right)\ dx\\
&= 2\frac{1-\cos m\pi}{m} + \frac{1}{2}\left(\frac{2\cos\frac{(m+2)x}{2}}{m+2}+\frac{2\cos\frac{(m-2)x}{2}}{m-2}\right)\Bigg|_0^{2\pi}\\
&= 2\frac{1-(-1)^m}{m} - \left(\frac{1-\cos(m+2)\pi}{m+2}+\frac{1-\cos(m-2)\pi}{m-2}\right)\\
&= 2\frac{1-(-1)^m}{m} - \left(\frac{1-(-1)^{m+2}}{m+2}+\frac{1-(-1)^{m-2}}{m-2}\right)\\
&= \begin{cases} 0, & m \text{ even},\\ \frac{4}{m}-\frac{4m}{m^2-4}, & m \text{ odd}.\end{cases}
\end{aligned}$$

Therefore

$$B_{mn} = \begin{cases} \frac{5}{\pi^2}\left(\frac{4}{m}-\frac{4m}{m^2-4}\right)\left(\frac{4}{n}-\frac{4n}{n^2-4}\right), & m, m \text{ both odd},\\ 0, & \text{otherwise},\end{cases}$$

and

$$u(x,y,t) = \sum_{m\text{ odd}}\sum_{n\text{ odd}}\frac{5}{\pi^2}\left(\frac{4}{m}-\frac{4m}{m^2-4}\right)\left(\frac{4}{n}-\frac{4n}{n^2-4}\right)\cos(\frac{3}{2}\sqrt{m^2+n^2}t)\ \sin\frac{mx}{2}\sin\frac{ny}{2}.$$

(b) $\lambda_{11} = \frac{3}{2}\sqrt{2}$, so the lowest frequency is $\lambda_{11}/2\pi = 3\sqrt{2}/4\pi$.

(c) $\lambda_{11,2} = \frac{3}{2}\sqrt{11^2+2^2} = \frac{3}{2}\sqrt{125}$, $\lambda_{10,5} = \frac{3}{2}\sqrt{10^2+5^2} = \frac{3}{2}\sqrt{125}$ so a linear combination of the corresponding eigenfunctions will work, for example $f(x,y) = \sin 11x\sin 2y + \sin 10x\sin 5y$.

(3)(a) Plug $u(x,t) = F(x)G(t)$ into the PDE:

$$F(x)G'(t) = c^2F''(x)G(t) - vF'(x)G(t),$$

so

$$\frac{G'(t)}{G(t)} = c^2\frac{F''(x)}{F(x)} - v\frac{F'(x)}{F(x)} = \alpha$$

for some constant α. This gives the equation for G,

$$G'(t) - \alpha G(t) = 0, \quad \text{solution } G(t) = Ke^{\alpha t},$$

and the equation for F,

$$c^2F''(x) - vF'(x) - \alpha F(x) = 0.$$

(b) If G is not changing in time then $G'(t) = 0$, so the first equation in (a) becomes

$$G(t)(c^2F''(x) - vF'(x)) = 0, \quad \text{or just } c^2F''(x) - vF'(x) = 0.$$

(You can also get this from (a) by noting that α must be 0 in this case.) Letting $H = F'$ the equation becomes $c^2H' - vH = 0$, or

$$H'(x) - \frac{v}{c^2}H(x) = 0, \quad \text{solution } H(x) = Ae^{\frac{v}{c^2}x}.$$

Thus $F'(x) = Ae^{-\frac{v}{c^2}x}$, and taking the antiderivative we get

$$F(x) = A_1e^{\frac{v}{c^2}x} + A_2.$$

(c) It is parabolic, the same as the usual heat equation. Adding the term $-vu_x$ does not change the type of the equation.