

MATH 445 SAMPLE MIDTERM EXAM 2
Fall 2009
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(1) Let

$$f(x) = \begin{cases} 1, & 0 < x \leq \pi \\ -1, & -\pi \leq x < 0 \\ 0, & |x| > \pi \text{ or } x = 0. \end{cases}$$

(a) Solve the heat equation in an infinite rod with $c = 2$ and initial temperature $u(x, 0) = f(x)$. (Do not use convolution—use another method.)

(b) Suppose we add a constant to the initial temperature in (a), so the initial condition becomes $u(x, 0) = f(x) + T_0$. Can we obtain the solution to this new problem by just adding T_0 to the solution in (a)? Explain.

(2) For a $2\pi \times 2\pi$ square membrane, with $c = 3$, the initial deflection is

$$f(x, y) = u(x, y, 0) = 5(1 - \cos x)(1 - \cos y), \quad 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi$$

and the initial velocity is 0.

(a) Find the deflection $u(x, y, t)$. For maximum credit, simplify any coefficients (A_n, B_m, B_{mn} , etc.) into a formula that does not involve sines or cosines. Different formulas for even and odd indices are OK if necessary.

(b) What is the lowest frequency of vibration among all the eigenfunctions that appear in your solution?

(c) If the initial deflection is instead a product of two sine waves, for example if $f(x, y) = \sin 3x \sin 5y$, then the resulting solution $u(x, y, t)$ is periodic in time with frequency λ_{35} . (You need not show this.) Find another choice of initial deflection $f(x, y)$ which makes the solution periodic in time, but which is *not* a product of two sine waves. HINT: Calculate $\lambda_{11,2}$ and $\lambda_{10,5}$.

(3) For heat flow, in some situations the heat is in a moving substance, like water flowing through a pipe, rather than in a fixed substance like a rod. If $u(x, t)$ is the temperature at the point x along the pipe at time t , and the water is moving at constant velocity v , then u can be shown to satisfy

$$u_t = c^2 u_{xx} - v u_x.$$

(a) We wish to find solutions u of the form $u(x, t) = F(x)G(t)$ to this equation. Write the differential equation F must satisfy. Write the differential equation G must satisfy, and find the general solution G . (The equations and the solution may involve unspecified constant(s), besides c and v . Note you are not asked to solve for F yet, and you don't have to say whether unspecified constants are positive or negative.)

(b) Suppose the system is in equilibrium, meaning G is not changing in time. In this special case, the equation for F becomes simpler. Write this simpler equation, and find the general solution (which may involve arbitrary constant(s).) HINT: Let $H = F'$, and rewrite the equation for F as an equation for H . First solve for H .

(c) Is the equation in (a) elliptic, hyperbolic, or parabolic? (Show why.)

Some useful formulas:

$$\sin a \sin b = \frac{1}{2}[\cos(a - b) - \cos(a + b)], \quad \sin a \cos b = \frac{1}{2}[\sin(a + b) + \sin(a - b)],$$

$$\cos a \cos b = \frac{1}{2}[\cos(a + b) + \cos(a - b)]$$