MATH 445 SAMPLE MIDTERM EXAM 1
Fall 2009
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(1)(a) Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y,z) = (x+y, y-x, 0)$, and let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ where $z \geq 0$. Calculate $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dA$.

(b) (Not related to (a).) Suppose a scalar field $f(x,y)$ represents the elevation of a landscape at the point (x,y). If $\mathbf{q} = (x,y)$ and $\mathbf{r} = (u,v)$ are two points, and

$$|\operatorname{grad} f(\mathbf{q})| < |\operatorname{grad} f(\mathbf{r})|,$$

this means that (circle one):

(i) you can go uphill over a longer distance from $\mathbf{r}$ than from $\mathbf{q}$;

(ii) the steepest slope uphill at $\mathbf{r}$ (i.e. steepest in any direction) is steeper than the steepest slope uphill at $\mathbf{q}$;

(iii) $\mathbf{r}$ is at a higher elevation than $\mathbf{q}$;

(iv) the slope averaged over all directions (from 0 to 2π) is greater at $\mathbf{r}$ than at $\mathbf{q}$.

(2) Find the Fourier series of the function

$$f(x) = \begin{cases} |\sin 2\pi x|, & -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ 0, & -1 \leq x < -\frac{1}{2} \text{ or } \frac{1}{2} < x \leq 1, \end{cases}$$

which has period 2. Express your answer by showing the sine and/or cosine terms out to $n = 4$. HINT: Is f even, odd, or neither? Be careful of $n = 2$, where formulas valid for other n might not apply.

(3)(a) Find the Fourier transform of

$$h(x) = \begin{cases} |x|, & |x| \leq a, \\ 0, & |x| > a. \end{cases}$$

(b) Using your answer from (a), express $h(x)$ as a "dw" integral of a specific formula.

(c) An engineer wants to design a filter that will change an input function (signal) $f(x)$ so that the Fourier transform of f gets multiplied by $\frac{1}{\sqrt{2\pi}(4+iw)}$. (In other words, if the input is a signal $f(x)$ with Fourier transform $\hat{f}(w)$, the output should have Fourier transform $\frac{\hat{f}(w)}{\sqrt{2\pi}(4+iw)}$.) The filter will accomplish this by convolving the input $f(x)$ with a fixed function $g(x)$, that is, the output will be the modified signal $(f * g)(x)$. What function $g(x)$ should the engineer use? HINT: No computation required.

(4) Let $q > 0$ and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = \left(\frac{qx}{(x^2+y^2+z^2)^{3/2}}, \frac{qy}{(x^2+y^2+z^2)^{3/2}}, \frac{qz}{(x^2+y^2+z^2)^{3/2}} \right),$$

let S be a closed surface enclosing the origin, and let S_a be the sphere of radius a around the origin (that is, $x^2 + y^2 + z^2 = a^2$), with a small enough so that S_a fits inside S. Let T denote the region enclosed between S_a and S. The boundary of T is in two pieces, S_a and S, which may or may not be relevant.

(a) Find the flux integral $\int\int \mathbf{F} \cdot \mathbf{n}\, dA$, over the boundary of T (outward from T). HINT: You don't need to parametrize–use an appropriate theorem. The boundary of T is in two pieces, S_a and S, which may or may not be relevant.

(b) Show by direct calculation that the (outward) flux integral $\int\int_{S_a} \mathbf{F} \cdot \mathbf{n}\ dA = 4\pi q$. HINT: Again, you don't need to parametrize, if you are observant. What is $\mathbf{n}$? What is $\mathbf{F} \cdot \mathbf{n}$?

(c) Show that also $\int\int_S \mathbf{F} \cdot \mathbf{n}\ dA = 4\pi q$. Note this is true for *all* surfaces S enclosing the origin. HINT: Use (a) and/or (b)—then very little computation is required.

As a note, if q represents an electric charge at the origin, then the integral in (c) represents the outward flux of the corresponding electric field.

Some useful formulas (these will be included with the midterm):

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)], \qquad \sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)],$$

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$