MATH 445 SAMPLE FINAL EXAM SOLUTIONS
Fall 2009
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(1)(a) Calculate the Laplacian:

$$\frac{\partial^2 f}{\partial x^2} = 2z + 2, \quad \frac{\partial^2 f}{\partial y^2} = 2z - 2, \quad \frac{\partial^2 f}{\partial z^2} = -4z,$$

and adding these gives $\Delta f = 0$.

(b) By (a),

$$\iint_S \nabla f \cdot n \; dA = \iiint_D \Delta f \; dV = 0.$$

(c) S_1 is flat, so on S_1 the normal is the downward vertical: $n = (0, 0, -1)$. Hence

$$\nabla f \cdot n = -\frac{\partial f}{\partial z} = -x^2 - y^2 + 2z^2 = -x^2 - y^2,$$

since $z = 0$ on S_1. Therefore

$$\iint_{S_1} \nabla f \cdot n \; dA = \iint_{S_1} -(x^2 + y^2) \; dA = \int_0^2 \int_0^{2\pi} -r^3 \; dr \; d\theta = -8\pi.$$

Then using (b),

$$\iint_{S_2} \nabla f \cdot n \; dA = \iint_S \nabla f \cdot n \; dA - \iint_{S_1} \nabla f \cdot n \; dA = 8\pi.$$

(2)(a) Calculating Fourier coefficients gives

$$a_0 = \frac{1}{4}, \quad a_1 = \frac{1}{\pi}, \quad a_2 = 0, \quad b_1 = \frac{1}{\pi}, \quad b_2 = \frac{1}{\pi},$$

so

$$P_2(x) = \frac{1}{4} + \frac{1}{\pi}\left(\cos x + \sin x + \sin 2x\right).$$

(b) $\int_{-\pi}^{\pi} f(x)^2 \; dx = \pi/2$ so

$$E^* = \int_{-\pi}^{\pi} f(x)^2 \; dx - \pi\left(2a_0^2 + a_1^2 + a_2^2 + b_1^2 + b_2^2\right) = \frac{3\pi}{8} - \frac{3}{\pi}.$$

(3)(a) From the table we get

$$iw\mathcal{F}(f) = \mathcal{F}(f') = \frac{1}{\sqrt{6}}e^{-w^2/12},$$

so

$$\mathcal{F}(f) = -\frac{i}{\sqrt{6}w}e^{-w^2/12}.$$

(b)

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}}\int_0^2 x\cos wx\, dx = \sqrt{\frac{2}{\pi}}\left(\frac{2\sin 2w}{w} + \frac{\cos 2w - 1}{w^2}\right).$$

(4) We have $u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin(nx/2)$, where $\lambda_n = cn\pi/L = 5n/2$ and

$$B_n = \frac{1}{\pi}\int_0^{2\pi}(1-\cos x)\sin\frac{nx}{2}\, dx = \begin{cases} 0, & n \text{ even}, \\ \frac{1}{\pi}\left(\frac{4}{n} - \frac{2}{n+2} - \frac{2}{n-2}\right), & n \text{ odd}. \end{cases}$$

That is,

$$u(x,t) = \sum_{n \text{ odd}} \frac{1}{\pi}\left(\frac{4}{n} - \frac{2}{n+2} - \frac{2}{n-2}\right)\cos\frac{5nt}{2}\sin\frac{nx}{2}.$$

(5)(a) $b_0 = 7, c_0 = 9$ so the characteristic equation is $r^2 + 6r + 9 = 0$, and $r = -3$ is the only root, so there is a solution $y_1(x) = x^{-3}\sum_{m=0}^{\infty} a_m x^m$. The recurrence is

$$[(m+r)(m+r-1) + 7(m+r) + 9]a_m + 2(m+r-1)a_{m-1} = 0,$$

or for $r = -3$,

$$a_m = -\frac{2(m-4)}{m^2}a_{m-1}.$$

This gives $a_2 = a_1 = 6a_0, a_3 = \frac{4}{3}a_0$ and then $a_4 = a_5 = \cdots = 0$, so

$$y_1(x) = a_0 x^{-3}\left(1 + 6x + 6x^2 + \frac{4}{3}x^3\right).$$

(b) $y_2(x) = y_1(x)\ln x + x^{-3}(A_0 + A_1 x + A_2 x^2 + \dots)$

(6)(a) One can check the Cauchy-Riemann equations are satisfied, so yes it is analytic.

(b) $2e^{i5\pi/6} = -\sqrt{3} + i$ and $2e^{i3\pi/2} = -2i$.

(c) $z_0 = \sqrt{3} + i = 2e^{i\pi/6}$ so $f'(z_0) = 3z_0^2 = 12e^{i\pi/3}$. $\Delta z = .001i = .001e^{i\pi/2}$ so

$$\Delta f \approx f'(z_0)\Delta z = .012e^{i5\pi/6},$$

so the direction is $5\pi/6$.

(7)(a) Parametrize as $z(t) = (1+3i)t,\quad 0 \le t \le 1$, and then $\dot{z}(t) = 1+3i$ and Im $z(t) = 3t$. Therefore

$$\int_C \text{Im } z \; dz \int_0^1 3t(1+3i)\; dt = \frac{3+9i}{2}.$$

(b) (Solution for the problem as modified in my email: find $\oint_\Gamma \frac{4}{z-i}\, dz$.) Let $\tilde{\Gamma}$ be the circle $\{z : |z-i| = 2\}$ centered at the singularity at i. Both circles enclose this singularity so the integral is the same for both. Parametrize $\tilde{\Gamma}$ as $z(t) = i + 2e^{it},\; 0 \le t \le 2\pi$ so $\dot{z}(t) = 2ie^{it}$, and then

$$\oint_\Gamma \frac{4}{z-i}\; dz = \oint_{\tilde{\Gamma}} \frac{4}{z-i}\; dz = \int_0^{2\pi} \frac{4}{2e^{it}} 2ie^{it}\; dt = 8\pi.$$

(c) Not the same—the new circle does not enclose the singularity at $z = i$, so the integral is 0.