MATH 445 SAMPLE FINAL EXAM
Fall 2009
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(1)(a) Show that the function

$$f(x,y,z) = (x^2+y^2)z + x^2 - y^2 - \frac{2}{3}z^3$$

satisfies Laplace's Equation, $\nabla^2 f = 0$.

(b) Let us take the (x, y)-plane to be "horizontal" and the z axis to be "vertical". Let D be the upper half of the ball of radius 2 centered at the origin in $\mathbb{R}^3$, that is, the part of the ball where $z \geq 0$, and let S be the boundary of D. Show that

$$\int_S \nabla f \cdot n \, dA = 0,$$

where n is the unit normal. HINT: Almost no calculation is needed–you don't have to actually integrate anything.

(c) Let $S_1 = S \cap \{(x,y,z) : z = 0\}$ (the flat part of S), and let S_2 be the rest of S (the curved part.) Find the flux integral

$$\int_{S_2} \nabla f \cdot n \, dA.$$

HINT: What is n, at points of S_1? How can you make use of (b)?

(2)(a) Find the trigonometric polynomial of degree 2 that is closest to the function f on $[-\pi, \pi]$ given by

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi/2, \\ 0, & -\pi \leq x < 0 \text{ and } \pi/2 < x \leq \pi. \end{cases}$$

Here "closest" means the square error is minimized.

(b)(6 points) In (a), what is the square error between f and your trigonometric polynomial? (Remember, you need not simplify.)

(3)(a) Suppose $f'(x) = e^{-3x^2}$. What is the Fourier transform $\hat{f}(w)$? HINT: Don't try to calculate $f(x)$.

(b) Find the Fourier cosine transform of the function

$$g(x) = \begin{cases} x, & 0, \leq x \leq 2 \\ 0 & x > 2. \end{cases}$$

(4) Find the displacement $u(x,t)$ of a string of length $L = 2\pi$ (tied down to displacement 0 at both ends) when the initial displacement is $f(x) = 1 - \cos x$ and the initial velocity is 0. Assume the constant in the wave equation is $c = 5$. Express your answer as an infinite series, and give a formula for the nth coefficient for n even and for n odd. Simpify these formulas as much as you can.

(5) Consider the equation

$$x^2y'' + (7x + 2x^2)y' + 9y = 0.$$

(a) Find one solution of this equation (other than $y \equiv 0$!) For maximum credit, express it in closed form (meaning without any infinite series.)

(b) Show the form that a second solution would take. (You don't have to find the second solution.)

(c) Suppose we have another differential equation with series solution $f(x) = \sum_{n=1}^{\infty} \frac{m(m+1)}{3^m} x^{2m}$. What is the radius of convergence of this series? In what interval containing 0 could the series be a valid solution? HINT: This is not related to parts (a) and (b).

(6)(a) Determine whether the function $g(z) = \overline{z}/|z|^2$ is analytic, for $z \neq 0$.

(b) One of the cube roots of $8i$ is $\sqrt{3} + i$. Find the other cube roots. Express them in the form $a + ib$.

(c) Let $f(z) = z^3$. If you move from $f(\sqrt{3}+i) = 8i$ to $f(\sqrt{3}+1.001i) = (\sqrt{3}+1.001i)^3$, in approximately what direction are you going? Use an appropriate approximation to get your answer, not an exact calculation. Your answer should be an angle relative to the positive horizontal axis, for example direction $\pi/2$ means you are going upward and $-\pi/4$ means you are going downward diagonally to the right.

(7)(a) Calculate $\int_C \text{Im } z \, dz$ for C a straight line from 0 to $1 + 3i$.

(b) Find $\oint_\Gamma \frac{z^2+4}{z-i} \, dz$ for Γ the circle$\{z : |z| = 2\}$, traversed counterclockwise. HINT: Almost no calculation is needed.

(c) Would the value of the integral in (b) be the same, if instead you integrate around $\{z : |z - 2| = 2\}$? If so, how do you know? If not, what would the new integral be?