MATH 445 MIDTERM EXAM 2 SOLUTIONS
Fall 2009
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(1)(a) The coefficient B_n is given (for $n \neq 4$) by

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^\pi (1-\cos 4x)\sin nx\ dx \\
&= -\frac{2}{\pi n}\cos nx\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi \cos 4x \sin nx\ dx \\
&= \frac{2}{\pi n}(1-(-1)^n) - \frac{1}{\pi}\int_0^\pi [\sin(n+4)x + \sin(n-4)x]\ dx \\
&= \frac{2}{\pi n}(1-(-1)^n) + \frac{1}{\pi}\left[\frac{\cos(n+4)x}{n+4} + \frac{\cos(n-4)x}{n-4}\right]\Big|_0^\pi \\
&= \frac{2}{\pi n}(1-(-1)^n) + \frac{1}{\pi}\left[\frac{(-1)^n-1}{n+4} + \frac{(-1)^n-1}{n-4}\right] \\
&= \frac{1-(-1)^n}{\pi}\left(\frac{2}{n} - \frac{1}{n+4} - \frac{1}{n-4}\right) \\
&= \frac{1-(-1)^n}{\pi}\left(\frac{2}{n} - \frac{2n}{n^2-16}\right) \\
&= \begin{cases} \frac{2}{\pi}\left(\frac{2}{n} - \frac{2n}{n^2-16}\right), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}
\end{aligned}$$

For $n = 4$ the second term on the 4th line inside the brackets is absent, but we still end up with $B_n = 0$, as for other even n, due to the factor $1-(-1)^n$. Therefore

$$u(x,t) = \sum_{n=1}^\infty B_n \sin nx \cos 2nt = \sum_{n \text{ odd}} \frac{2}{\pi}\left(\frac{2}{n} - \frac{2n}{n^2-16}\right)\sin nx \cos 2nt.$$

(b) $g(x) = 5\sin 3x$ is its own Fourier series, since it is a trigonometric polynomial, so we only have an $n = 3$ term. We have $\lambda_n = cn$ so $\lambda_3 = 6$ and $B_3^* = b_3/\lambda_3 = 5/6$, with all other $B_n^* = 0$. Thus the series we add to the series from (a) has only one term in it:

$$u(x,t) = \frac{5}{6}\sin 3x \sin 6t + \sum_{n \text{ odd}} \frac{2}{\pi}\left(\frac{2}{n} + \frac{2n}{n^2-16}\right)\sin nx \cos 2nt.$$

(c) This problem should have been eliminated from the exam after I altered parts (a) and (b) from a membrane problem to a string problem, because for the string, *all* solutions are periodic. So *any* $f_0(x)$ is a correct answer!

(2)(a) We calculate the Fourier sine coefficients of f:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\; dx = \frac{2}{\pi}\int_0^{\pi/2} 10\sin nx\; dx = -\frac{20}{n\pi}\cos nx\Big|_0^{\pi/2} = \frac{20}{n\pi}\left(1-\cos\frac{n\pi}{2}\right).$$

Then

$$A_n^* = \frac{b_n}{\sinh 2n\pi} = \frac{20(1-\cos\frac{n\pi}{2})}{n\pi\sinh 2n\pi}$$

and

$$U(x,y) = \sum_{n=1}^{\infty} A_n^* \sin nx \sinh ny = \sum_{n=1}^{\infty} \frac{20(1-\cos\frac{n\pi}{2})}{n\pi\sinh 2n\pi}\sin nx\sinh ny.$$

(b) Right side insulated means the normal derivative $\partial u/\partial x = 0$ along the right side, so $F'(\pi) = 0$. Temperature held at 0 on the left side means $F(0) = 0$. Then $0 = F(0) = B$, so $F(x) = A\sin px$, and then $0 = F'(\pi) = Ap\cos p\pi$. Therefore $p = n + \frac{1}{2}$ for some integer n, and the eigenfunctions are $F_n(x) = \sin(n+\frac{1}{2})x$. A is arbitrary.

(3)(a) The Laplacian in polar coordinates in the radially symmetric case (no θ derivatives) is $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$, so

$$V(r)G'(t) = c^2\left(V''(r) + \frac{1}{r}V'(r)\right)G(t).$$

Then

$$\frac{G'(t)}{c^2G(t)} = \frac{V''(r)+\frac{1}{r}V'(r)}{V(r)} = p,$$

so

$$V''(r) + \frac{1}{r}V'(r) - pV(r) = 0.$$

(b) From (a), we can use an integrating factor as follows:

$$\begin{aligned} G'(t) - c^2pG(t) &= 0 \\ \frac{d}{dt}\left(e^{-c^2pt}G(t)\right) &= 0 \\ G(t) &= Ae^{c^2pt}, \end{aligned}$$

where A is an arbitrary constant.

(c) With the boundary condition $u(R,t) = 0$ and with $p = -k^2$, we know from 12.9 that the solutions of the equation in (a) are $V(r) = J_0(\alpha_m r/R)$, where J_0 is a Bessel function and α_m is the mth of its roots, in order. Since $p < 0$, from (b) we see that $G(t) \to 0$ so the eigenfunctions do decay to 0.

(d) Plugging $V(r) = r^\alpha$ into the equation gives

$$\alpha(\alpha-1)r^{\alpha-2} + \alpha r^{\alpha-2} - pr^{\alpha-2} = 0,$$

which after dividing out the $r^{\alpha-2}$ becomes $\alpha^2 = p$ or $\alpha = \sqrt{p}$. From (b), the corresponding eigenfunction does not decay to 0, because $p > 0$.