MATH 425b SAMPLE MIDTERM EXAM 2
Spring 2009
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The midterm will again be open book. You can use Rudin, your lecture notes, your homework and solutions, but no other books or published materials.

(1)(25 points) Suppose $f:\mathbb{R}^n \to \mathbb{R}^m$, f is differentiable at a point $x \in \mathbb{R}^n$, $h_n \to 0$ and $k_n \to 0$. Show that

$$f(x+h_n+k_n)-f(x) = \big[f(x+h_n)-f(x)\big] + \big[f(x+k_n)-f(x)\big] + o(|h_n|+|k_n|).$$

HINT: $f(x+h)-f(x)$ can be approximated—by what? Also, any quantity expressed as $o(|h_n+k_n|)$ can also be expressed as $o(|h_n|+|k_n|)$–why?

(2)(25 points) Suppose A is an $n \times n$ matrix with $||A|| < 1$.

(a) Show that $\sum_{n=0}^{\infty} A^n$ converges. (Here the convergence of the series is in the matrix norm $||B|| = \sup_{x \neq 0} \frac{|Bx|}{|x|}$.) HINT: Show that the partial sums $S_N = \sum_{n=0}^{N} A^n$ form a Cauchy sequence.

(b) Show that $I - A$ is invertible and $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$. HINT: Given two matrices, what calculation do you do to show they are inverses of each other?

(3)(25 points) Suppose $E \subset \mathbb{R}^{n+m}$, $(a,b) \in E$, and $f : E \to \mathbb{R}^n$ is a $\mathcal{C}'$ map which satisfies the hypotheses of the Implicit Function Theorem (9.28 p. 224) with $a = 0$: $f(0,b) = 0$ and $A = f'(0,b)$ has A_x invertible. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Show that provided $||T||$ is sufficiently small (say, $0 < ||T|| < \epsilon_0$), there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$, with $(0,b) \in U$ and $b \in W$, having the following property:

For each $y \in U$ there is a unique x such that $(x,y) \in U$ and $f(x,y) = T(x)$.

HINT: For $T \equiv 0$ this is just the Implicit Function Theorem. The proof is quite short–you don't need to do a proof like the proof of the Implicit Function Theorem. Instead, find a way to USE that theorem, that is, figure out what function to apply it to.

(4)(30 points) Let us associate a 2×2 matrix to each vector$(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$ by putting the coordinates in the arrangement $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. For example, $(1,0,0,1)$ corresponds to the identity matrix.

We can then define a function $f : \mathbb{R}^4 \to \mathbb{R}^4$ by specifying that the coordinates of $f(\mathbf{x})$

are the entries of the matrix $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^2$, for $i = 1, 2, 3, 4$, in the order shown above. (Don't overlook the square on the matrix here!) For example, the first coordinate of $f(x)$ is $x_1^2 + x_2 x_3$.

Use this f to show that there are neighborhoods U, V of the identity matrix I such that for each matrix $B \in V$ there is a unique matrix $C \in U$ which is a square root of B, that is, $C^2 = B$, and this C is a continuously differentiable function of B.